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A SIGN-CHANGING SOLUTION FOR NONLINEAR PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. In this article, we establish the existence of a least energy sign-changing solution for nonlinear problems involving the fractional Laplacian. Our main tool is constrained minimization in a closed subset containing all the sign-changing solutions of the equation.

1. Introduction

In this article, we establish the existence of least energy sign-changing solutions for the nonlinear problem involving the fractional Laplacian,

$$\begin{gathered} (-\Delta)^s u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $s \in (0,1)$, $N > 2s$, $(-\Delta)^s$ stands for the fractional Laplacian, which is defined by

$$(-\Delta)^s u(x) = C(N,s) \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $C(N,s)$ is a suitable positive normalization constant. The nonlinearity $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 and satisfies the growth conditions:

- (F1) $\lim_{t\to 0} f(x,t)/t = 0$ uniformly for a.e. $x \in \Omega$;
- (F2) $\lim_{|t|\to +\infty} f(x,t)/|t|^{2_s^*-1} = 0$ uniformly for a.e. $x \in \Omega$;
- (F3) There exist $R > 0$ and $\mu > 2$ such that
$$0 < \mu F(t,x) \le t f(x,t), \quad \forall x \in \Omega, \forall |t| \ge R;$$
- (F4) $f(x,t)/t$ is increasing in $|t| > 0$, for every $x \in \Omega$.

Here $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent, and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s)$.

Remark 1.1. The conditions (F1) and (F4) imply that $H(x,t) = tf(x,t) - 2F(x,t)$ is a nonnegative function, increasing in $|s|$ with

$$tH'(x,t) = t^2 f'(x,t) - f(x,t)t > 0 \quad \text{for any } |t| > 0.$$

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Recently, a great attention has been focused on the study nonlinear problems involving fractional Laplacian, because of its importance in the field of physics, engineering, finance and so on. For example, it arises in phase transition, quantum mechanics, flame propagation, chemical reaction liquids American options in finance, crystal dislocation, see [2, 11, 21] and so on. Owing to the pioneer work Caffarelli and Silvestre [6], who developed the technique of harmonic extension to define the fractional Laplace operator, many well-known results corresponding to classical elliptic problems have been obtained, see [3, 5, 7, 20] and the references therein. On the other hand, Servadei and Valdinoci [16, 17] who takes the fractional Laplace operator as singular integral operator, introduced a suitable Sobolev space which can be to develop the variational formula, some existence and multiple results for problem (1.1) have been obtained, one can see [9, 10, 14, 15, 18, 19] and the references therein, they deal with another nonlocal operator, the spectral fractional Laplacian, which is different from the operator considered in the present paper defined by the singular integral kernel.

As we know, in the past decades, the existence and multiplicity of sign-changing solutions for nonlinear elliptic problems have been intensively studied. There are some powerful methods which have been developed, such as the descended flow methods [12], constrained minimization methods [4], super and sub solution combining with truncation techniques [8] and so on. The existence and multiplicity of sign-changing solutions for problem (1.1) has been investigated by the recent paper of Chang and Wang [7], through using the descended flow methods and harmonic extension techniques. In a recent paper [1], the authors proved the existence of a least energy solution for Schrödinger-Poisson equations by the minimization methods, the problem is similar to ours, because of appearing the "nonlocal" term. In this paper, we borrow some ideas from [1], we use the constrained minimization methods to prove the existence of sign-changing solutions for problem (1.1).

For any measurable function $u : \mathbb{R}^N \to \mathbb{R}$ we define the Gagliadro seminorm by setting

$$[u]_s^2 = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y,$$

and we introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\},$$

which is a Hilbert space. We also define a closed subspace

$$X(\Omega) = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

Because of the fractional Sobolev inequality, $X(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y,$$

which induces a norm $\|\cdot\|_X = [\cdot]_s$. For $u \in X(\Omega)$, set

$$J(u) = \frac{1}{2}\|u\|_X - \int_\Omega F(x,u)\,\mathrm{d}x.$$

Then, $J \in C^1(X(\Omega))$ and

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f(x,u)v\,\mathrm{d}x, \quad \forall v \in X(\Omega).$$

Moreover, all its critical points are (up to a normalization constant depending on s and N, which we will neglect henceforth) weak solutions of (1.1), namely they satisfy

$$\langle u, v\rangle = \int_{\Omega} f(x,u)v \,\mathrm{d}x, \quad \forall v \in X(\Omega).$$

Our main result is the following.

Theorem 1.2. *Suppose that f satisfies* (F1)–(F4). *Then problem* (1.1) *possesses a least energy sign-changing solution.*

Remark 1.3. We observe that (F2) is weaker than the usual subcritical condition, such as the (F2) of [7, Theorem 1.1].

Remark 1.4. Under the conditions (F1)–(F3), using the usual mountain pass theorem, we can obtain two solutions: a positive solution and a negative solution.

Remark 1.5. *In this framework, because there is a nonlocal term*

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y,$$

we can not prove the exactly nodal domains corresponding to the sign-changing solution obtained in Theorem 1.2.

In the proof of Theorem 1.2, we will prove that the functional J achieves a minimum value on the nodal set

$$\mathcal{M} = \{u \in \mathcal{N} : \langle J'(u), u^+\rangle = 0,\ \langle J'(u), u^-\rangle = 0 \text{ and } u^{\pm} \neq 0\}$$

where $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \min\{u(x), 0\}$ and

$$\mathcal{N} = \{u \in X(\Omega)\backslash\{0\} : \langle J'(u), u\rangle = 0\}.$$

More precisely, we prove that there is $w \in \mathcal{M}$ such that

$$J(w) = \inf_{u\in\mathcal{M}} J(u).$$

2. Preliminary lemmas

Lemma 2.1. *For any $a, b \in \mathbb{R}$, we have:*

(i) $(ka)^{\pm} = ka^{\pm}$, *for all* $k \geq 0$, $(a+b)^{\pm} \leq a^{\pm} + b^{\pm}$;
(ii) $(a-b)(a^+ - b^+) \geq (a^+ - b^+)^2$;
(iii) $(a-b)(a^- - b^-) \geq (a^- - b^-)^2$.

Remark 2.2. The conclusion (ii) of Lemma 2.1 implies that $(a^+-b^+)(a^--b^-) \geq 0$, for any $a, b \in \mathbb{R}$.

From Lemma 2.1, by simple computations, we deduce the following Lemma.

Lemma 2.3. *For any $u \in X(\Omega)$, the following element facts hold.*

(i) $\|u^{\pm}\|_X \leq \|u\|_X$;
(ii)

$$\langle u, u^{\pm}\rangle = \langle u^{\pm}, u^{\pm}\rangle - \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y - \int_{\mathbb{R}^{2N}} \frac{u^-(x)u^+(y)}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y;$$

(iii)

$$\langle J'(u), u^{\pm}\rangle = \langle J'(u^{\pm}), u^{\pm}\rangle - \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y - \int_{\mathbb{R}^{2N}} \frac{u^-(x)u^+(y)}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y.$$

Lemma 2.4. *Let $\{u_n\}$ be a bounded sequence in $X(\Omega)$. Then up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in X(\Omega)$ such that*

(i)
$$\lim_{n\to\infty}\int_\Omega |u_n^\pm|^q\,\mathrm{d}x = \int_\Omega |u^\pm|^q\,\mathrm{d}x;$$
(ii)
$$\lim_{n\to\infty}\int_\Omega u_n^\pm f(x,u_n^\pm)\,\mathrm{d}x = \int_\Omega u^\pm f(x,u^\pm)\,\mathrm{d}x;$$
(iii)
$$\lim_{n\to\infty}\int_\Omega F(x,u_n^\pm)\,\mathrm{d}x = \int_\Omega F(x,u^\pm)\,\mathrm{d}x;$$
(iv)
$$\liminf_{n\to\infty}\Big[-\int_{\mathbb{R}^{2N}}\frac{u_n^+(x)u_n^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big] \ge -\int_{\mathbb{R}^{2N}}\frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. From the boundedness of sequence $\{u_n\}$ in $X(\Omega)$, it follows that up to a subsequence, we may assume that there exists $u \in X(\Omega)$ such that
$$\begin{gathered} u_n \rightharpoonup u, \quad \text{weakly in } X(\Omega),\\ u_n \to u, \quad \text{strongly in } L^q(\mathbb{R}^N) \text{ with } q \in [2,2_s^*),\\ u_n(x) \to u(x), \quad \text{a.e. in } \mathbb{R}^N. \end{gathered} \tag{2.1}$$

(i) From Lemma 2.1, one has
$$\begin{aligned} \Big|\int_\Omega |u_n^\pm|^q\,\mathrm{d}x - \int_\Omega |u^\pm|^q\,\mathrm{d}x\Big| &\le \int_\Omega |u_n^\pm - u^\pm|^q\,\mathrm{d}x\\ &\le \int_\Omega |(u_n-u)^\pm|^q\,\mathrm{d}x\\ &\le \int_\Omega |u_n-u|^q\,\mathrm{d}x. \end{aligned}$$
Hence, the conclusion (i) can be deduced from (2.1).

(ii) By hypotheses (F1) and (F2), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon|t|^{q-1} + \varepsilon|t|^{2_s^*-1} \quad \text{for all } t \in \mathbb{R} \text{ and almost all } x \in \Omega. \tag{2.2}$$
Thus, $\{f(x,u_n^\pm)\}$ is bounded in $L^{2_s^*-1}(\Omega)$, and so there exists $v \in L^{2_s^*-1}(\Omega)$ such that $f(x,u_n^\pm) \rightharpoonup v$ in $L^{2_s^*-1}(\Omega)$. Because $f(x,u_n^\pm) \to f(x,u^\pm)$ a.e. $x \in \Omega$, we obtain $f(x,u_n^\pm) \rightharpoonup f(x,u^\pm)$ in $L^{2_s^*-1}(\Omega)$. As a result, we deduce that
$$\int_\Omega u^\pm(f(x,u_n^\pm) - f(x,u^\pm))\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$
By Hölder's inequality and (2.2), we have
$$\begin{aligned} &\Big|\int_\Omega [u_n^\pm f(x,u_n^\pm) - u^\pm f(x,u^\pm)]dx\Big|\\ &\le \int_\Omega |u_n^\pm - u^\pm||f(x,u_n^\pm)|\,\mathrm{d}x + |\int_\Omega u^\pm(f(x,u_n^\pm) - f(x,u^\pm))\,\mathrm{d}x|\\ &\le \varepsilon\|u_n^\pm - u^\pm\|_2\|u_n^\pm\|_2 + C_\varepsilon\|u_n^\pm - u^\pm\|_q\|u_n^\pm\|_q + \varepsilon\|u_n^\pm - u^\pm\|_{2_s^*}\|u_n^\pm\|_{2_s^*}\\ &\quad + |\int_\Omega u^\pm(f(x,u_n^\pm) - f(x,u^\pm))\,\mathrm{d}x| \end{aligned}$$

$$\le C_0[\|u_n^\pm - u^\pm\|_2 + \|u_n^\pm - u^\pm\|_q] + C_0\varepsilon + |\int_\Omega u^\pm(f(x,u_n^\pm) - f(x,u^\pm))\,\mathrm{d}x|.$$

Taking the limit in the above inequality and using the arbitrariness of ε and (2.3), conclusion (ii) follows.

(iii) By Hölder's inequality and (2.3), we have

$$\begin{aligned}
\Big|\int_\Omega [F(x,u_n^\pm) - F(x,u^\pm)]\,\mathrm{d}x\Big| &= \int_\Omega\int_0^1 |u_n^\pm - u^\pm||f(x,\theta u_n^\pm + (1-\theta)u^\pm)|\,\mathrm{d}\theta\,\mathrm{d}x \\
&\le \varepsilon\|u_n^\pm - u^\pm\|_2\|u_n^\pm\|_2 + C_\varepsilon\|u_n^\pm - u^\pm\|_q\|u_n^\pm\|_q \\
&\quad + \varepsilon\|u_n^\pm - u^\pm\|_{2_s^*}\|u_n^\pm\|_{2_s^*} \\
&\le C_0[\|u_n^\pm - u^\pm\|_2 + \|u_n^\pm - u^\pm\|_q] + C_0\varepsilon.
\end{aligned}$$

Hence, taking the limit in the above inequality and using the arbitrariness of ε, conclusion (iii) follows.

(iv) By Fatou's Lemma, conclusion (iv) follows is trivially. □

Lemma 2.5. *There exists $C_1 > 0$ and $\alpha > 0$ such that*

(i) $J(u) \ge (\frac{\mu}{2} - 1)\|u\|_X^2 - C_1$ *and* $\|u\|_X \ge \alpha$, $\forall u \in \mathcal{N}$;
(ii) $\|u^\pm\|_X \ge \alpha$, $\forall u \in \mathcal{M}$.

Proof. (i) For any $u \in \mathcal{N}$, by (F3) and (2.2), we have

$$\begin{aligned}
\mu J(u) = \mu J(u) - \langle J'(u), u\rangle &= (\frac{\mu}{2} - 1)\|u\|_X^2 + \int_\Omega [uf(x,u) - \mu F(x,u)]\,\mathrm{d}x \\
&= (\frac{\mu}{2} - 1)\|u\|_X^2 + \int_{\{|u|\le R\}} [uf(x,u) - \mu F(x,u)]\,\mathrm{d}x \\
&\quad + \int_{\{|u|\ge R\}} [uf(x,u) - \mu F(x,u)]\,\mathrm{d}x \\
&\ge (\frac{\mu}{2} - 1)\|u\|_X^2 - C_1.
\end{aligned}$$

From (F1) and (F2), for any $\varepsilon > 0$, there exists $C_2 > 0$ such that

$$f(x,t)t \le \varepsilon t^2 + C_2|t|^{2_s^*}, \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{2.4}$$

Since $\langle J'(u), u\rangle = 0$, by Sobolev embedding and (2.4), we have

$$\|u\|_X^2 = \int_\Omega uf(x,u)\,\mathrm{d}x \le \varepsilon\int_\Omega u^2\,\mathrm{d}x + C_2\int_\Omega |u|^{2_s^*}\,\mathrm{d}x \le C_3[\varepsilon\|u\|_X^2 + \|u\|_X^{2_s^*}],$$

from this inequality, taking $\varepsilon = 1/(2C_3)$, $\alpha = (1/(2C_3))^{2_s^*-2}$, it follows that

$$\|u\|_X \ge \alpha, \quad \forall u \in \mathcal{N}.$$

(ii) Assume that $u \in \mathcal{M}$, we have that $\langle J'(u), u^\pm\rangle = 0$. From (iii) of Lemma 2.3, we have

$$\langle J'(u^\pm), u^\pm\rangle = \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^{2N}} \frac{u^-(x)u^+(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y < 0.$$

Thus,

$$\|u^\pm\|_X^2 < \int_\Omega u^\pm f(x,u^\pm)\,\mathrm{d}x.$$

As in the proof of (i), we deduce that $\|u^\pm\|_X \ge \alpha$. □

Lemma 2.6. *If $\{u_n\}$ is a bounded sequence in $\mathcal{M}$ and $p \in (2, 2_s^*)$, we have*

$$\liminf_{n\to\infty} \int_\Omega |u_n^\pm|^p \,\mathrm{d}x > 0.$$

Proof. By (F1) and (F2), for any $\varepsilon > 0$, there exists $C_4 > 0$ such that

$$tf(x,t) \le \varepsilon t^2 + C_4|t|^p + \varepsilon |t|^{2_s^*}, \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{2.5}$$

Since $u_n \in \mathcal{M}$, by Lemma 2.5 and (2.5), we have

$$\alpha^2 \le \|u^\pm\|_X^2 < \int_\Omega u_n^\pm f(x, u_n^\pm)\,\mathrm{d}x \le \varepsilon \int_\Omega |u_n^\pm|^2 \,\mathrm{d}x + C_4 \int_\Omega |u_n^\pm|^p \,\mathrm{d}x + \varepsilon \int_\Omega |u_n^\pm|^{2_s^*} \,\mathrm{d}x.$$

From the boundedness of $\{u_n\}$, there is $C_5 > 0$ such that

$$\alpha^2 \le \varepsilon C_5 + C_4 \int_\Omega |u_n^\pm|^p \,\mathrm{d}x,$$

which implies

$$\liminf_{n\to\infty} \int_\Omega |u_n^\pm|^p \,\mathrm{d}x > 0.$$

□

3. Proof of main result

In this section, we devote to proving Theorem 1.2. We denote by c_0 the infimum of J on $\mathcal{M}$; that is,

$$c_0 = \inf_{u\in\mathcal{M}} J(u).$$

To prove that this infimum can be attained, moreover, it is a critical point of functional J, we need the following Lemmas.

Lemma 3.1. *Let $u \in X(\Omega)$ with $u^\pm \neq 0$. Then there are $t, \theta > 0$ such that $\langle J'(tu^+ + \theta u^-), u^+\rangle = 0$ and $\langle J'(tu^+ + \theta u^-), u^-\rangle = 0$.*

Proof. We define a function

$$\gamma(t,\theta) = (\langle J'(tu^+ + \theta u^-), tu^+\rangle, \langle J'(tu^+ + \theta u^-), \theta u^-\rangle),$$

obviously, $\gamma \in C([0,+\infty) \times [0,+\infty))$. Since

$$\langle J'(tu^+ + \theta u^-), tu^+\rangle = t^2\|u^+\|_X^2 - 2ts \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega tu^+ f(x, tu^+) dx$$

and

$$\langle J'(tu^+ + \theta u^-), \theta u^-\rangle = \theta^2\|u^-\|_X^2 - 2t\theta \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega \theta u^- f(x, \theta u^-) dx,$$

from (2.4), by Sobolev embedding, we obtain

$$\begin{aligned}\langle J'(tu^+ + \theta u^-), tu^+\rangle &\ge t^2\|u^+\|_X^2 - 2t\theta \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \varepsilon t^2 \|u^+\|_2^2 - C_2 t^{2_s^*} \|u^+\|_{2_s^*}^{2_s^*} \\ &\ge t^2(1 - C_6\varepsilon)\|u^+\|_X^2 - C_7 t^{2_s^*}\|u^+\|_X^{2_s^*}.\end{aligned}$$

Similarly,

$$\langle J'(tu^+ + \theta u^-), \theta u^-\rangle \ge \theta^2(1 - C_6\varepsilon)\|u^-\|_X^2 - C_7\theta^{2_s^*}\|u^-\|_X^{2_s^*}.$$

Hence, There is $r_1 > 0$ such that

$$\langle J'(r_1u^+ + \theta u^-), r_1u^+\rangle > 0,\ \forall s > 0, \quad \langle J'(tu^+ + r_1u^-), r_1u^-\rangle > 0,\ \forall t > 0. \tag{3.1}$$

From (F3), there are $C_8, C_9 > 0$ such that

$$tf(x,t) \geq C_8|t|^\mu - C_9, , \quad \text{for all } t \in \mathbb{R} \text{ and all } x \in \bar{\Omega}. \tag{3.2}$$

Indeed, from (F3), we obtain

$$tf(x,t) \geq \mu F(x,t) \geq C_8|t|^\mu, \quad \forall |t| \geq R \text{ and a.e. } x \in \Omega. \tag{3.3}$$

From the continuity of $tf(x,t)$, $tf(x,t)$ is bounded on $\bar{\Omega} \times [-R, R]$; that is, there exists a constant $k_0 > 0$ such that

$$tf(x,t) \geq -k_0 \geq C_8|t|^\mu - C_8R^\mu - k_0, \quad \forall |t| \leq R. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain that (3.2) holds. Hence, by 3.2), one has

$$\begin{aligned}\langle J'(tu^+ + \theta u^-), tu^+\rangle &\leq t^2\|u^+\|_X^2 - 2t\theta \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - C_8t^\mu \int_\Omega |u^+|^\mu dx + C_9|\Omega|\end{aligned}$$

and

$$\begin{aligned}\langle J'(tu^+ + \theta u^-), \theta u^-\rangle &\leq \theta^2\|u^-\|_X^2 - 2t\theta \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - C_8\theta^\mu \int_\Omega |u^-|^\mu dx + C_9|\Omega|,\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Hence, there is $r_2 > 0$ such that

$$\langle J'(r_2u^+ + \theta u^-), r_2u^+\rangle < 0, \quad \langle J'(tu^+ + r_2u^-), r_2u^-\rangle > 0, \quad \forall t, \theta \in [r_1, r_2]. \tag{3.5}$$

From (3.1) and (3.5), this Lemma follows applying Miranda's Theorem [13]. $\square$

For $u \in X(\Omega)$ with $u^\pm \neq 0$, we consider the function $h_u(t,\theta) : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ defined as

$$h_u(t,\theta) = J(tu^+ + \theta u^-)$$

and $g_u : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2$ given by

$$g_u(t,\theta) = \Big(\frac{\partial h_u}{\partial t}(t,\theta), \frac{\partial h_u}{\partial \theta}(t,\theta)\Big) = (\langle J'(tu^+ + \theta u^-), u^+\rangle, \langle J'(tu^+ + \theta u^-), u^-\rangle).$$

By the assumption $f \in C^1$, we can deduce that g_u is also a C^1 map.

Lemma 3.2. *Let $u \in \mathcal{M}$,*

(i) $h_u(t,\theta) < h_u(1,1)$, *for all* $\theta, t \geq 0$ *and* $(t,\theta) \neq (1,1)$;
(ii) $\det(g_u)'(1,1) > 0$.

Proof. (i) Since $u \in \mathcal{M}$, then $\langle J'(u), u^\pm\rangle = 0$; that is,

$$\begin{aligned}\|u^+\|_X^2 - 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega u^+ f(x,u^+)\,\mathrm{d}x, \\ \|u^-\|_X^2 - 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega u^- f(x,u^-)\,\mathrm{d}x\,.\end{aligned}$$

The above two equalities imply that $(1,1)$ is a critical point of h_u. By condition (F3), it is easy to prove that

$$\lim_{|(t,\theta)|\to\infty} h_u(t,\theta) = -\infty.$$

By the property of continuous function, we can get that h_u attains a global maximum in some $(a,b)\in[0,+\infty)\times[0,+\infty)$.

1. We claim that $a,b>0$. If $b=0$, obviously $a\neq 0$ and then $\langle J'(au^+),au^+\rangle=0$, i.e.,

$$a^2\|u^+\|_X^2 = \int_\Omega au^+ f(x,au^+)\,\mathrm{d}x. \tag{3.6}$$

Since

$$\begin{aligned}
\langle J'(u^+),u^+\rangle &= \langle J'(u),u^+\rangle + 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&= 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y < 0,
\end{aligned}$$

we obtain

$$\|u^+\|_X^2 < \int_\Omega u^+ f(x,u^+)\,\mathrm{d}x. \tag{3.7}$$

By (3.6) and (3.7), one has

$$(1-\frac{1}{a^2})\|u^+\|_X^2 < \int_\Omega \Big(\frac{u^+f(x,u^+)}{(u^+)^2} - \frac{au^+f(x,au^+)}{(au^+)^2}\Big)(u^+)^2\,\mathrm{d}x.$$

From (F4), we can infer that $a\le 1$. Thus, by Remark 1.1, we have

$$\begin{aligned}
h_u(a,0) = J(au^+) &= J(au^+) - \frac{1}{2}\langle J'(au^+),au^+\rangle\\
&= \frac{1}{2}\int_\Omega [au^+f(x,au^+) - 2F(x,au^+)]\,\mathrm{d}x\\
&\le \frac{1}{2}\int_\Omega [u^+f(x,u^+) - 2F(x,u^+)]\,\mathrm{d}x\\
&\le \frac{1}{2}\int_\Omega [u^+f(x,u^+) - 2F(x,u^+)]\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_\Omega [u^-f(x,u^-) - 2F(x,u^-)]\,\mathrm{d}x\\
&= \frac{1}{2}\int_\Omega [uf(x,u) - 2F(x,u)]\\
&= J(u) - \frac{1}{2}\langle J'(u),u\rangle = h_u(1,1).
\end{aligned}$$

This contradicts tht (a,b) is a global maximum point for h_u. Similarly, we can show that $a\neq 0$, and the proof of claim is complete.

2. We claim that $a,b\le 1$. Since $(h_u)'(a,b)=0$, we have

$$\begin{aligned}
a^2\|u^+\|_X^2 - 2ab\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega au^+f(x,au^+)\,\mathrm{d}x,\\
b^2\|u^-\|_X^2 - 2ab\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega bu^-f(x,bu^-)\,\mathrm{d}x.
\end{aligned}$$

Without loss of generality, we suppose that $a \geq b$. Hence,

$$a^2\|u^+\|_X^2 - 2a^2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \geq \int_\Omega au^+f(x,au^+)\,\mathrm{d}x.$$

Note that the assumption $\langle J'(u),u^+\rangle = 0$, we deduce that

$$0 \geq \int_\Omega \Big(\frac{au^+f(x,au^+)}{(au^+)^2} - \frac{u^+f(x,u^+)}{(u^+)^2}\Big)(u^+)^2\,\mathrm{d}x.$$

The condition (F4) implies that $a \leq 1$ and so the proof of the claim is complete.

To conclude the proof of (i), we only need to show that h_u does not have a global maximum in $[0,1]\times[0,1]\backslash\{(1,1)\}$. If not, we assume that (a,b) is a global maximum of h_u in $[0,1]\times[0,1]\backslash\{(1,1)\}$. From definition of h_u and Remark 1.1, we have

$$\begin{aligned}
h_u(a,b) &= J(au^+ + bu^-) - \frac{1}{2}((h_u)'(a,b),(a,b))\\
&= J(au^+ + bu^-) - \frac{1}{2}\langle J'(au^+ + bu^-),au^+\rangle - \frac{1}{2}\langle J'(au^+ + bu^-),bu^-\rangle\\
&= \frac{a^2}{2}\|u^+\|_X^2 + \frac{b^2}{2}\|u^-\|_X^2 - 2ab\int_{\mathbb{R}^{2N}}\frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \int_\Omega F(x,au^+)\,\mathrm{d}x - \int_\Omega F(x,bu^-)\,\mathrm{d}x - \frac{a^2}{2}\|u^+\|_X^2 - \frac{b^2}{2}\|u^-\|_X^2\\
&\quad + 2ab\int_{\mathbb{R}^{2N}}\frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_\Omega au^+f(x,au^+)\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_\Omega bu^-f(x,bu^-)\,\mathrm{d}x\\
&= \frac{1}{2}\int_\Omega [au^+f(x,au^+) - 2F(x,au^+)\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_\Omega [bu^-f(x,bu^-) - 2F(x,bu^-)\,\mathrm{d}x\\
&< \frac{1}{2}\int_\Omega [u^+f(x,u^+) - 2F(x,u^+)\,\mathrm{d}x + \frac{1}{2}\int_\Omega [u^-f(x,u^-) - 2F(x,u^-)\,\mathrm{d}x\\
&= \frac{1}{2}\int_\Omega [uf(x,u) - 2F(x,u)]\,\mathrm{d}x = h_u(1,1).
\end{aligned}$$

Hence, h_u does not have a global maximum in $[0,1]\times[0,1]\backslash\{(1,1)\}$. Therefore, the proof of (i) is complete.

(ii) Since

$$\begin{gathered}
\frac{\partial^2 h_u}{\partial t^2}(t,\theta) = \|u^+\|_X^2 - \int_\Omega f'(x,tu^+)(u^+)^2\,\mathrm{d}x,\\
\frac{\partial^2 h_u}{\partial \theta^2}(t,\theta) = \|u^-\|_X^2 - \int_\Omega f'(x,\theta u^-)(u^-)^2\,\mathrm{d}x,\\
\frac{\partial^2 h_u}{\partial t\partial\theta}(t,\theta) = -2\int_{\mathbb{R}^{2N}}\frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y,
\end{gathered}$$

by a simple computation, observing that $u\in\mathcal{M}$ and by Remark 1.1, we obtain

$$\det(g_u)'(1,1)$$

$$
\begin{aligned}
&= \Big[\|u^+\|_X^2 - \int_\Omega f'(x,u^+)(u^+)^2\,\mathrm{d}x\Big]\Big[\|u^-\|_X^2 - \int_\Omega f'(x,u^-)(u^-)^2\,\mathrm{d}x\Big] \\
&\quad - 4\Big(\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 \\
&= \Big[\int_\Omega \Big(u^+f(x,u^+) - f'(x,u^+)(u^+)^2\Big)\,\mathrm{d}x + 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big] \\
&\quad \times \Big[\int_\Omega \Big(u^-f(x,u^-) - f'(x,u^-)(u^-)^2\Big)\,\mathrm{d}x + 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big] \\
&\quad - 4\Big(\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big)^2 > 0.
\end{aligned}
$$

□

Corollary 3.3. *Suppose that $u \in X(\Omega)$ satisfies $u^\pm \neq 0$ and $\langle J'(u), u^\pm\rangle \le 0$. Then there are $t, s \in [0,1]$ such that $tu^+ + \theta u^- \in \mathcal{M}$.*

Proof. From (F3), it is easy to deduce that

$$\lim_{|(t,\theta)|\to\infty} h_u(t,\theta) = -\infty.$$

We assume that h_u attain a global maximum in some $(a,b) \in [0,+\infty)\times[0,+\infty)$ and then $(h_u)'(a,b) = 0$, we have

$$
\begin{aligned}
a^2\|u^+\|_X^2 - 2ab\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega au^+f(x,au^+)\,\mathrm{d}x, \\
b^2\|u^-\|_X^2 - 2ab\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega bu^-f(x,bu^-)\,\mathrm{d}x.
\end{aligned}
$$

If $a \ge b$, then

$$a^2\|u^+\|_X^2 - 2a^2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \ge \int_\Omega au^+f(x,au^+)\,\mathrm{d}x.$$

Note that by the assumption $\langle J'(u), u^+\rangle \le 0$, we deduce

$$0 \ge \int_\Omega \Big(\frac{au^+f(x,au^+)}{(au^+)^2} - \frac{u^+f(x,u^+)}{(u^+)^2}\Big)(u^+)^2\,\mathrm{d}x.$$

The condition (F4) implies that $a \le 1$ and so the proof is complete. □

Proof of Theorem 1.2. Let $\{u_n\}$ be a sequence in $\mathcal{M}$ such that

$$\lim_{n\to\infty} J(u_n) = c_0.$$

From Lemma 2.5 (i), $\{u_n\}$ is bounded in $X(\Omega)$. From Lemma 2.6, it follows that $u^\pm \neq 0$. Then, by Lemma 3.1, there are $t, \theta > 0$ such that

$$\langle J'(tu^+ + \theta u^-), u^+\rangle = 0, \quad \langle J'(tu^+ + \theta u^-), u^-\rangle = 0. \tag{3.8}$$

Next, we show that $t, \theta \le 1$. Since $\langle J'(u_n), u_n^\pm\rangle = 0$; that is,

$$
\begin{aligned}
\|u_n^+\|_X^2 - 2\int_{\mathbb{R}^{2N}} \frac{u_n^+(x)u_n^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega u_n^+f(x,u_n^+)\,\mathrm{d}x, \\
\|u_n^-\|_X^2 - 2\int_{\mathbb{R}^{2N}} \frac{u_n^+(x)u_n^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega u_n^-f(x,u_n^-)\,\mathrm{d}x\,.
\end{aligned}
$$

From Lemma 2.4, by the weak semi-continuity of norm function in Banach space, we obtain

$$\|u^+\|_X^2 - 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \int_\Omega u^+ f(x,u^+)\,\mathrm{d}x,$$

and

$$\|u^-\|_X^2 - 2\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \int_\Omega u^- f(x,u^-)\,\mathrm{d}x; \tag{3.9}$$

that is,

$$\langle J'(u), u^+\rangle \le 0, \quad \langle J'(u), u^-\rangle \le 0. \tag{3.10}$$

From (3.8), it follows that

$$t^2\|u^+\|_X^2 - 2t\theta\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega tu^+ f(x,tu^+)\,\mathrm{d}x$$

and

$$\theta^2\|u^-\|_X^2 - 2t\theta\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \theta u^- f(x,\theta u^-)\,\mathrm{d}x. \tag{3.11}$$

Without loss of generality, we can assume that $\theta \ge t$. From (3.9) and (3.11), we deduce that

$$0 \ge \int_\Omega \Big(\frac{\theta u^- f(x,\theta u^-)}{(\theta u^-)^2} - \frac{u^- f(x,u^-)}{(u^-)^2}\Big)(u^-)^2\Big)\,\mathrm{d}x$$

which implies that $\theta \le 1$ (using (F4)).

Next, we show that $J(tu^+ + \theta u^-) = c_0$. By (3.8) and $u^\pm \neq 0$, we see that $tu^+ + \theta u^- \in \mathcal{M}$. Thus, by Remark 1.1, one has

$$\begin{aligned}
c_0 \le J(tu^+ + \theta u^-) &= J(tu^+ + \theta u^-) - \frac{1}{2}\langle J'(tu^+ + \theta u^-), tu^+ + \theta u^-\rangle \\
&= J(tu^+) - \frac{1}{2}\langle J'(tu^+), tu^+\rangle + J(\theta u^-) - \frac{1}{2}\langle J'(\theta u^-), \theta u^-\rangle \\
&\le J(u^+) - \frac{1}{2}\langle J'(u^+), u^+\rangle + J(u^-) - \frac{1}{2}\langle J'(u^-), u^-\rangle \\
&= \lim_{n\to\infty}\Big(J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle\Big) \\
&= \lim_{n\to\infty} J(u_n) = c_0.
\end{aligned}$$

Consequently, we have proved that there exists $tu^+ + \theta u^- \in \mathcal{M}$ such that $J(tu^+ + \theta u^-) = c_0$. Hereafter, we denote $w = tu^+ + \theta u^-$.

As in the proof of [1, Theorem 1.3], by using Lemma 3.2, we conclude that w is a critical point of J on $X(\Omega)$. □

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