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QUASI-PERIODIC SOLUTIONS OF NONLINEAR BEAM EQUATIONS WITH QUINTIC QUASI-PERIODIC NONLINEARITIES

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Abstract. In this article, we consider the one-dimensional nonlinear beam equations with quasi-periodic quintic nonlinearities

$$u_{tt} + u_{xxxx} + (B + \varepsilon\phi(t))u^5 = 0$$

under periodic boundary conditions, where B is a positive constant, ε is a small positive parameter, $\phi(t)$ is a real analytic quasi-periodic function in t with frequency vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)$. It is proved that the above equation admits many quasi-periodic solutions by KAM theory and partial Birkhoff normal form.

1. Introduction and statement of main results

Recently, there has been a lot of publications concerning the dynamic behavior for nonlinear beam equations by using different methods, see for instance, [1, 3, 8, 9, 11, 20, 27]. In these works, little is considered about quasi-periodic solutions of this kind of equations. Significant results have been obtained with respect to quasi-periodic solutions of autonomous beam equations by KAM theory, see [13, 14, 15]. In particular, Liang and Geng [21] considered the existence of the quasi-periodic solutions of completely resonant beam equations

$$u_{tt} + u_{xxxx} + Bu^3 = 0 \tag{1.1}$$

with hinged boundary conditions with $B = 1$.

The works mentioned above do not non-autonomous include the case. More recently, Wang [26] obtained the existence of quasi-periodic solutions for the non-autonomous beam equation

$$u_{tt} + u_{xxxx} + \mu u + \varepsilon g(\omega t, x)u^3 = 0 \tag{1.2}$$

under hinged boundary conditions

$$u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0.$$

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In this paper, we consider the existence of quasi-periodic solutions for nonlinear beam equations with quintic quasi-periodic nonlinearities

$$u_{tt} + u_{xxxx} + (B + \varepsilon\phi(t))u^5 = 0, \tag{1.3}$$

subject to periodic boundary conditions

$$u(t, x) = u(t, x + 2\pi), \tag{1.4}$$

where B is a positive constant, ε is a small positive parameter, $\phi(t)$ is a real analytic quasi-periodic function in t with frequency vector $\omega = (\omega_1, \omega_2 \dots, \omega_m)$. Equation (1.3) can be regarded as a quasi-periodic perturbation (with perturbation term $\varepsilon\phi(t)u^5$) of the completely resonant nonlinear beam equation

$$u_{tt} + u_{xxxx} + Bu^5 = 0.$$

Firstly, we consider the existence quasi-periodic solutions of an ordinary differential equation with respect to the unknown function $x(t)$,

$$\ddot{x} + (B + \varepsilon\phi(t))x^5 = 0. \tag{1.5}$$

Secondly, we obtain the nonlinear beam equation

$$v_{tt} + v_{xxxx} + V_1(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon)v + \sum_{k=1}^{4} \varepsilon^k V_{k+1}(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon)v^{k+1} = 0 \tag{1.6}$$

by letting $u = u_0(t) + \varepsilon v(x, t)$ in (1.3), here $u_0(t)$ is a nonzero quasi-periodic solution of (1.5) and V_k $(k = 1, 2, \dots, 4)$ defined as in Section 3.

Thirdly, we construct the invariant tori or quasi-periodic solutions of (1.6) with (1.4) by means of KAM theory. Finally, we prove that (1.3) with (1.4) have many quasi-periodic solutions in the neighborhood of the quasi-periodic solutions to (1.5).

As in our previous works [25, 28, 30], the method used in this paper is based on the infinite-dimensional KAM theory as developed by Kuksin [18] and Pöchel [23]. Thus the main step is to reduce the equation to a setting where KAM theory for PDE can be applied. We note that (1.6) is a nonlinear beam equation with quasi-periodic potential and quasi-periodic nonlinearities, which needs to reduce the linearized system of (1.6) to constant coefficients by a linear quasi-periodic change of variables with the same basic frequencies as the initial system. However, we cannot guarantee in general such reducibility. The strategy of the proof in this paper is similar to the one in [25]. However, the details are quite different.

Now the reducibility problems of infinite-dimensional linear quasi-periodic systems, by KAM techniques, has become an active field of research. The first result was obtained by Bambusi and Graffi [2], after that, Eliasson and Kuksin [10], Yuan [29], Liu and Yuan [22], and Grébert and Thomann [16]. However, in general, the reducibility of infinite-dimensional linear quasi-periodic systems remains open and is very attracting.

For our purpose, we first introduce a hypothesis.

(H1) $B > 0$, $\phi(t)$ is a real analytic quasi-periodic function in t with frequency vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)$, where $\omega \in D_\Lambda$,

$$D_\Lambda := \{\omega \in \mathbb{R}^m : |\langle k, \omega\rangle| \ge \Lambda |k|^{-(m+1)},\ 0 \ne k \in \mathbb{Z}^m\}$$

with $\Lambda > 0$.

For $\gamma > 0$ we define the set

$$A_\gamma = \{\alpha \in \mathbb{R} : |\langle k, \omega\rangle + l\alpha| > \gamma(|k| + |l|)^{-(m+1)}, \text{ for all } 0 \neq (k, l) \in \mathbb{Z}^m \times \mathbb{Z}\}$$

with its complex neighborhood $A_\gamma + h$ of radius h. The following theorem is the main result of this article.

Theorem 1.1. *Assume that* (H1) *is satisfied. For an arbitrary index set* $\mathcal{N}_d = \{n_1, n_2, \dots, n_d\} \subset \mathbb{N}$*, there is a small enough positive* ε^{**} *such that for any* $0 < \varepsilon < \varepsilon^{**}$*, there are the sets* $\mathscr{J} \subset \hat{J} \subset [\pi/T, 3\pi/T]$ *and* $\Sigma_\varepsilon \subset \Sigma := D_\Lambda \times A_\gamma \times [0,1]^{d+1}$ *with* $\operatorname{meas} \hat{J} > 0, \operatorname{meas} \mathscr{J} > 0$ *and* $\operatorname{meas}(\Sigma \setminus \Sigma_\varepsilon) \leq \varepsilon$*, such that for any* $\bar{\xi} \in \mathscr{J}$ *and* $(\omega, \alpha(\bar{\xi}), \tilde{\xi}_0 \bigoplus (\tilde{\xi}_j)_{j \in \mathcal{N}_d}) \in \Sigma_\varepsilon$*, the nonlinear beam equation* (1.3)-(1.4) *possess a solution of the form*

$$u(t,x) = u_0(t) + \varepsilon u_1(t,x) + o(\varepsilon),$$

$$u_1(t,x) = \sum_{j \in \{0\} \cup \mathcal{N}_d} \frac{\sqrt{2\tilde{\xi}_j/\pi}}{\sqrt[4]{j^4 + \varepsilon[\hat{V}]}} \cos\sqrt{j^4 + \varepsilon[\hat{V}]}t \, \cos(jx),$$

with frequency vector

$$\widehat{\hat{\omega}} = (\omega, \alpha(\bar{\xi}), (\hat{\omega}_j)_{\{0\} \cup \mathcal{N}_d}) \in \mathbb{R}^{m+d+2},$$

and $\hat{V}$ *as defined in Section 3.*

(i) ω *is the frequency vector of* ϕ *, while* α*,* $\hat{\omega}_j$ *are constructed in the proof, and are functions of* ε *and of parameters* $\bar{\xi}$*,* $\tilde{\xi} = (\tilde{\xi}_0, (\tilde{\xi}_j)_{j \in \mathcal{N}_d}) \in \mathbb{R}^{d+1}$*. In particular,*

$$\hat{\omega}_j = \mu_j(\varepsilon) + \varepsilon^2 a(\bar{\xi}, \varepsilon), \quad \mu_j(\varepsilon) = \sqrt{j^4 + \varepsilon[\hat{V}]} + \mathcal{O}(\varepsilon^{5.5}), \quad j \in \{0\} \cup \mathcal{N}_d,$$

where $|a_j(\tilde{\xi}, \varepsilon)| \leq C$*;*

(ii) $u_0(t)$ *is a non-trivial solution of* (1.5) *depending on the parameters* $(\bar{\xi}, \varepsilon)$*, it is of size* $\mathcal{O}(\varepsilon^{1/4})$ *and of quasi-periodic with frequency* (ω, α)*, and* $u_1(t,x)$ *is a solution of the linear equation*

$$\partial_{tt} u_1 + \partial_{xxxx} u_1 + \varepsilon[\hat{V}] u_1 = 0, \tag{1.7}$$

and is quasi-periodic with frequencies $\sqrt{j^4 + \varepsilon[\hat{V}]}$*,* $j \in \{0\} \cup \mathcal{N}_d$*.*

Remark 1.2. Using the method of this paper we cannot expect an existence result for quasi-periodic solutions $u(t,x)$ of (1.3) with the same frequency ω as the data of the problem. Actually, the quasi-periodic solutions obtained in Theorem 1.1 bifurcate from quasi-periodic solutions of the nonlinear ODE (1.5), and the solutions have more frequencies than the data of the problem. That is because we need to add extra parameters while one considers the existence of quasi-periodic solutions for ODE (1.5) and PDE (1.6) by means of KAM theory respectively. The addition of the extra parameters will cause solutions with additional frequencies. Thus, the main aim of the work is the construction of solutions with also additional frequencies, and the solutions could be viewed as the interaction between $u_0(t)$, which is the (ω, α)-quasi-periodic solutions of (1.5), and $u_1(t,x)$, which is a solution of the linear equation (1.7), and is quasi-periodic with frequencies $\left(j^4 + \varepsilon[\hat{V}]\right)^{1/2}$. The result is still new even if the equation seems to be quite studied and well-understood.

Remark 1.3. *To avoid the double eigenvalues we restrict ourselves to choose an even complete orthogonal basis* $\phi_j(x) = \frac{1}{\sqrt{\pi}}\cos(jx)$, $j \geq 0$. *The solutions* $u(t,x)$ *constructed in Theorem 1.1 are even in space variable* x.

2. Quasi-periodic solutions of a nonlinear ODE

In 2000, Bibikov [5] developed a KAM theorem for nearly integrable Hamiltonian systems with one degree of freedom under the quasi-periodic perturbation:

$$
\begin{aligned}
\dot{r} &= -\frac{\partial H(r,\varphi,\omega t,a)}{\partial \varphi},\\
\dot{\varphi} &= a + \frac{\partial H(r,\varphi,\omega t,a)}{\partial r}
\end{aligned}
\tag{2.1}
$$

with a one-dimensional parameter a. In this section, we apply the results in [5] (See also [17]) to show the existence of the quasi-periodic solutions for the following nonlinear ordinary differential equation with quasi-periodic coefficient,

$$
\ddot{x} + (B + \varepsilon\phi(t))x^5 = 0. \tag{2.2}
$$

First we introduce the Bibikov's lemma.

Lemma 2.1 ([5]). *Suppose that* $H(r,\varphi,\theta,a)$ *is real analytic on*

$$
D = \{(r,\varphi,\omega t,a) : |r| < \delta_0,\ |\operatorname{Im}\varphi| < p_0,\ |\operatorname{Im}\theta| < p_0,\ a \in A_\gamma + \frac{1}{2}\gamma\delta_0\}
$$

and satisfies

$$
|H| < \gamma p_0^{m+3}\delta_0^2. \tag{2.3}
$$

Then

(i) There exists a δ_0^* *such that if* $\delta_0 < \delta_0^*$, *then there exists a function* $a_0 :, A_\gamma \to \mathbb{R}$ *and a change of variables*

$$
r = \rho + v(\rho,\psi,\theta,\alpha), \quad \varphi = \psi + u(\psi,\theta,\alpha), \quad \alpha \in A_\gamma,
$$

that transforms system (2.1) *with* $a = a_0(\alpha)$ *into a system*

$$
\dot{\rho} = B(\rho,\psi,\theta,\alpha), \quad \dot{\psi} = \Psi(\rho,\psi,\theta,\alpha)
$$

that satisfies the condition

$$
B(0,\psi,\theta,\alpha) = \frac{\partial B(0,\psi,\theta,\alpha)}{\partial\rho} = \Psi(0,\psi,\theta,\alpha) = 0.
$$

(ii) $\operatorname{meas}(A_{K\mu}) \to \mu$, *as* $K \to 0^+$, *where* $I \subset \mathbb{R}$ *is a unit interval,* $\mu > 0$, *and*

$$
A_{K\mu} = \{\alpha \in \mu I : |\langle k,\omega\rangle + l\alpha| \geq K\mu(|k| + |l|)^{-(m+1)},\ \text{for all } 0 \neq (k,l) \in \mathbb{Z}^m \times \mathbb{Z}\}.
$$

Equation (2.2) is equivalent to the system

$$
\dot{x} = -y, \quad \dot{y} = Bx^5 + \varepsilon\phi(t)x^5. \tag{2.4}
$$

Using Bibikov's theorem, we have the following lemma.

Lemma 2.2. *For any* $\omega \in D_\Lambda$, *there exists an* ε^* *such that for any positive* $0 < \varepsilon < \varepsilon^*$ *and sufficiently small* $\gamma > 0$ *there exist a real analytic function* $a_0(\alpha) : A_\gamma \to \mathbb{R}$ *and a set* $\hat{J} \subset [\pi/T, 3\pi/T]$ *with* $\operatorname{meas}\hat{J} > 0$, *such that for* $\alpha \in A_\gamma$, $\bar{\xi} \in \hat{J}$ *and some* $\sigma > 0$, *Equation* (2.2) *has a quasi-periodic solution* $x(t,\bar{\xi},\varepsilon) \in Q_\sigma(\tilde{\omega})$ *with* $\tilde{\omega}(\bar{\xi}) = (\omega_1,\omega_2,\dots,\omega_m,\alpha(\bar{\xi}))$ *satisfying* $x(t,\bar{\xi},\varepsilon) = \mathcal{O}(\varepsilon^{\frac{1}{4}})$.

Remark 2.3. $\omega = (\omega_1, \omega_2 \dots, \omega_m)$ is the frequency of $\phi(t)$ and $\alpha(\bar{\xi})$ is obtained from Lemma 2.2, then the frequency of solution for (2.2) is dimension $m+1$.

3. Hamiltonian formalism

From Lemma 2.2 we know that for every $\varepsilon \in (0, \varepsilon^*)$ equation (2.2) has a non-trivial quasi-periodic solution $u_0(t, \bar{\xi}, \varepsilon)$ with frequency vector $\tilde{\omega}(\bar{\xi})$. Taking $u = u_0(t, \bar{\xi}, \varepsilon) + \varepsilon v(t, x)$ in (1.3), we get the equation

$$v_{tt} + v_{xxxx} + V_1(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon)v + \sum_{k=1}^{4} \varepsilon^k V_{k+1}(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon) v^{k+1} = 0, \tag{3.1}$$

where

$$\begin{gathered} V_1(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon) := 5\big(B + \varepsilon\widehat{\phi}(\omega t)\big)\bar{u}_0^4(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon), \\ V_{k+1}(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon) := C_5^{k+1}\big(B + \varepsilon\widehat{\phi}(\omega t)\big)\bar{u}_0^{4-k}(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon), \quad k = 1, 2, 3, 4 \end{gathered}$$

are quasi-periodic in time t with frequency vector $\tilde{\omega}(\bar{\xi})$, and $\widehat{\phi}$ and $\bar{u}_0$ are the shell functions of ϕ and u_0 respectively. Let us write

$$\begin{aligned} &\hat{V}(\theta, \bar{\xi}, \varepsilon) \\ &= 5c\big(B + \varepsilon\widehat{\phi}(\omega t)\big)\big[(\bar{\xi} + \sqrt{\varepsilon} I)^{1/4} C(\varphi T)\big]^4 \end{aligned} \tag{3.2}$$

with $\theta = \tilde{\omega}(\bar{\xi})t \in \mathbb{T}^{m+1}$, and

$$\begin{aligned} \frac{d}{d\bar{\xi}}\hat{V}(\theta, \bar{\xi}, \varepsilon) = 5c\big(B + \varepsilon\widehat{\phi}(\omega t)\big)\big[&(1 + \sqrt{\varepsilon}\frac{dI}{d\bar{\xi}})C^4(\varphi T) \\ &+ (\bar{\xi} + \sqrt{\varepsilon} I)4C^3(\varphi T)C'(\varphi T)\frac{d\varphi}{d\bar{\xi}}\big]. \end{aligned}$$

Therefore,

$$\begin{gathered} \lim_{\varepsilon\to 0}[\hat{V}(\theta, \bar{\xi}, \varepsilon)] = \frac{1}{(2\pi)^{m+1}} \int_{\mathbb{T}^{m+1}} \lim_{\varepsilon\to 0} \hat{V}(\theta, \bar{\xi}, \varepsilon) d\theta = 5cB\bar{\xi}C^4(\varphi_0 T), \\ \lim_{\varepsilon\to 0} \frac{d}{d\bar{\xi}}[\hat{V}(\theta, \bar{\xi}, \varepsilon)] = \frac{1}{(2\pi)^{m+1}} \int_{\mathbb{T}^{m+1}} \lim_{\varepsilon\to 0} \frac{d}{d\bar{\xi}}\hat{V}(\theta, \bar{\xi}, \varepsilon) d\theta = 5cBC^4(\varphi_0 T). \end{gathered}$$

Thus, there exists $0 < \varepsilon_1 < \varepsilon^*$ such that for any $\varepsilon \in (0, \varepsilon_1)$,

$$[\hat{V}(\theta, \bar{\xi}, \varepsilon)] > \frac{5}{2}cB\bar{\xi}C^4(\varphi_0 T) := I_1 > 0, \tag{3.3}$$

$$\frac{\partial}{\partial\bar{\xi}}[\hat{V}(\theta, \bar{\xi}, \varepsilon)] > \frac{5}{2}cBC^4(\varphi_0 T) := I_2 > 0. \tag{3.4}$$

Let us write

$$\hat{V}(\theta, \bar{\xi}, \varepsilon) := [\hat{V}(\theta, \bar{\xi}, \varepsilon)] + \widetilde{V}(\theta, \bar{\xi}, \varepsilon)$$

with $[\widetilde{V}(\theta, \bar{\xi}, \varepsilon)] = 0$, and $m_\varepsilon := \varepsilon[\hat{V}(\theta, \bar{\xi}, \varepsilon)] > 0$. Then

$$V_1(\theta, \bar{\xi}, \varepsilon) = \varepsilon\hat{V}(\theta, \bar{\xi}, \varepsilon) = m_\varepsilon + \varepsilon\widetilde{V}(\theta, \bar{\xi}, \varepsilon).$$

Furthermore, we can show that

$$\widetilde{V}(\theta, \bar{\xi}, \varepsilon) = \mathcal{O}(\varepsilon) \tag{3.5}$$

as $\varepsilon \ll 1$. In fact, from (3.2), we have

$$\hat{V}(\theta, \bar{\xi}, \varepsilon) = 5c\Big(B + \varepsilon\widehat{\phi}(\omega t)\Big)(\bar{\xi} + \sqrt{\varepsilon} I)C^4(\varphi T),$$

$$[\hat{V}(\theta,\bar{\xi},\varepsilon)] = 5c(B+\varepsilon[\widehat{\phi}])(\bar{\xi}+\sqrt{\varepsilon}I)C^4(\varphi T).$$

Therefore,

$$\begin{aligned}\widetilde{V}(\theta,\bar{\xi},\varepsilon) &= \hat{V}(\theta,\bar{\xi},\varepsilon) - [\hat{V}(\theta,\bar{\xi},\varepsilon)]\\ &= 5c\bar{\xi}C^4(\varphi T)(\varepsilon(\widehat{\phi}-[\widehat{\phi}])) + \mathcal{O}(\varepsilon^{3/2}) = \mathcal{O}(\varepsilon).\end{aligned}$$

We can rewrite the equation (3.1) as follows

$$\dot{v} = w, \quad \dot{w} + Av = -\varepsilon\widetilde{V}(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon)v - \sum_{k=1}^{4}\varepsilon^k V_{k+1}(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon)v^{k+1}, \tag{3.6}$$

where $A = d^4/dx^4 + m_\varepsilon, t \in \mathbb{R}$. As it is well known, the equation (3.6) can be studied as an infinite dimensional Hamiltonian system by taking the phase space to be the product of the Sobolev spaces $H_0^1([0,2\pi]) \times L^2([0,2\pi])$ with coordinates v and $w = \partial_t v$. The Hamiltonian for (3.6) is then

$$\begin{aligned}H = \frac{1}{2}\langle w,w\rangle + \frac{1}{2}\langle Av,v\rangle + \frac{1}{2}\varepsilon\widetilde{V}(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon)\int_0^{2\pi} v^2dx\\ + \sum_{k=1}^{4}\varepsilon^k\frac{V_{k+1}(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon)}{k+2}\int_0^{2\pi} v^{k+2}dx,\end{aligned} \tag{3.7}$$

where $\angle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2([0,2\pi])$. We will find the solutions $v(t,x)$ of (3.6) which satisfy

$$v(t,-x) = v(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{T}.$$

It is easy to see that $\lambda_j = j^4 + m_\varepsilon$ $(j = 0,1,\dots)$ and $\phi_j(x) = \frac{1}{\sqrt{\pi}}\cos(jx)$ $(j = 1,2,\dots)$, $\phi_0(x) = \frac{1}{\sqrt{2\pi}}$ are, respectively, the eigenvalues and eigenfunctions of Sturm-Liouville problems

$$\begin{gathered}Ay = \lambda y,\\ x\in\mathbb{T}, \quad y(-x) = y(x),\end{gathered} \tag{3.8}$$

and the eigenfunctions $\phi_j(x)'s$ with $j \geq 0$ form a complete orthogonal basis of the subspace consisting of all even functions of $L^2(0,2\pi)$.

We introduce coordinates $q = (q_0,q_1,q_2,\dots)$, $p = (p_0,p_1,p_2,\dots)$ through the relations

$$v(t,x) = \sum_{j\geq 0}\frac{q_j(t)}{\sqrt[4]{\lambda_j}}\phi_j(x), \quad \partial_t v(t,x) = \sum_{j\geq 0}\sqrt[4]{\lambda_j}p_j(t)\phi_j(x). \tag{3.9}$$

The coordinates are taken from some real Hilbert space:

$$\begin{aligned}l^{a,s} = l^{a,s}(\mathbb{R}) := \Big\{&q = (q_0,q_1,q_2,\dots), q_i\in\mathbb{R}, i\geq 0 \text{ such that}\\ &\|q\|_{a,s}^2 = |q_0|^2 + \sum_{i\geq 1}|q_i|^2 i^{2s}e^{2ai} < \infty\Big\}.\end{aligned}$$

Next we assume that $a \geq 0$ and $s > 1/2$. One rewrites the Hamiltonian (3.7) in the coordinates (q,p),

$$H = \Lambda + G, \tag{3.10}$$

where

$$\Lambda = \frac{1}{2}\sum_{j\geq 0}\sqrt{\lambda_j}(p_j^2+q_j^2) + \varepsilon\frac{\widetilde{V}(\tilde\omega(\bar\xi)t,\bar\xi,\varepsilon)}{2\sqrt{\lambda_j}}q_j^2,$$

$$G = \sum_{k=1}^{4}\varepsilon^k\frac{V_{k+1}(\tilde\omega(\bar\xi)t,\bar\xi,\varepsilon)}{k+2}\int_0^{2\pi}\Big(\sum_{j\geq 0}\frac{q_j(t)}{\sqrt[4]{\lambda_j}}\phi_j(x)\Big)^{k+2}dx.$$

The equations of motion are

$$\dot q_j = \frac{\partial H}{\partial p_j} = \sqrt{\lambda_j}p_j, \quad \dot p_j = -\frac{\partial H}{\partial q_j} = -\sqrt{\lambda_j}q_j - \varepsilon\frac{\widetilde{V}(\theta,\bar\xi,\varepsilon)}{\sqrt{\lambda_j}}q_j - \frac{\partial G}{\partial q_j}, \quad j\geq 0 \tag{3.11}$$

with respect to the symplectic structure $\sum dq_i\wedge dp_i$ on $l^{a,s}\times l^{a,s}$.

Lemma 3.1. *Let I be an interval and let*

$$t\in I\to (q(t),p(t))\equiv(\{q_j(t)\}_{j\geq 0},\{p_j(t)\}_{j\geq 0})$$

be a real analytic solution of (3.11) *for $a>0$. Then*

$$v(t,x) = \sum_{j\geq 0}\frac{q_j(t)}{\sqrt[4]{\lambda_j}}\phi_j(x)$$

is a classical solution of (3.1) *that is real analytic on $I\times[0,2\pi]$.*

One introduces a pair of action-angle variables (J,θ), where $J\in\mathbb{R}^{m+1}$ is canonically conjugate to $\theta=\tilde\omega(\bar\xi)t\in\mathbb{T}^{m+1}$. Then (3.6) can be written as a Hamiltonian system

$$\dot q_j = \frac{\partial H}{\partial p_j}, \quad \dot p_j = -\frac{\partial H}{\partial q_j}, \quad j\geq 0, \quad \dot\theta = \tilde\omega(\bar\xi), \quad \dot J = -\frac{\partial H}{\partial\theta} \tag{3.12}$$

with the Hamiltonian

$$\begin{aligned} H = \langle\tilde\omega(\bar\xi),J\rangle + \frac{1}{2}\sum_{j\geq 0}\sqrt{\lambda_j}(p_j^2+q_j^2) + \varepsilon\frac{\widetilde{V}(\theta,\bar\xi,\varepsilon)}{2\sqrt{\lambda_j}}q_j^2 \\ + \varepsilon G^3(q,\theta,\bar\xi,\varepsilon) + \varepsilon^2G^4(q,\theta,\bar\xi,\varepsilon) + \varepsilon^3G^5(q,\theta,\bar\xi,\varepsilon) + \varepsilon^4G^6(q,\vartheta), \end{aligned} \tag{3.13}$$

where $\vartheta=\omega t$,

$$G^3(q,\theta,\bar\xi,\varepsilon) = \sum_{i,j,d\geq 0}G^3_{i,j,d}(\theta,\bar\xi,\varepsilon)q_iq_jq_d \tag{3.14}$$

with

$$G^3_{i,j,d}(\theta,\bar\xi,\varepsilon) = \frac{1}{3}\frac{V_2(\theta,\bar\xi,\varepsilon)}{\sqrt[4]{\lambda_i\lambda_j\lambda_d}}\int_{\mathbb{T}}\phi_i(x)\phi_j(x)\phi_d(x)dx. \tag{3.15}$$

It is not difficult to verify that, from the definition of eigenfunctions,

$$G^3_{i,j,d}(\theta,\bar\xi,\varepsilon)=0, \quad \text{unless } i\pm j\pm d=0\,.$$

Expressions G^4, G^5 and G^6 are similar. We introduce complex coordinate

$$z_j = \frac{1}{\sqrt{2}}(q_j-\mathrm{i}p_j), \quad \bar z_j = \frac{1}{\sqrt{2}}(q_j+\mathrm{i}p_j), \quad j\geq 0,$$

that live in the now complex Hilbert space:

$$\begin{aligned} l^{a,s} &= l^{a,s}(\mathbb{C}) \\ &:= \Big\{z=(z_0,z_1,z_2,\dots), z_j\in\mathbb{C}, j\geq 0\text{such that} \end{aligned}$$

$$\|z\|_{a,s}^2 = |z_0|^2 + \sum_{j\geq 1} |z_j|^2 j^{2s} e^{2aj} < \infty \Big\}.$$

This transformation is symplectic with $dq \wedge dp = \sqrt{-1} dz \wedge d\bar{z}$. Then (3.13) is changed into

$$H = \tilde{H} + \varepsilon G^3(z,\theta,\bar{\xi},\varepsilon) + \varepsilon^2 G^4(z,\theta,\bar{\xi},\varepsilon) + \varepsilon^3 G^5(z,\theta,\bar{\xi},\varepsilon) + \varepsilon^4 G^6(z,\theta,\varepsilon), \quad (3.16)$$

where

$$\tilde{H} = \langle \tilde{\omega}(\bar{\xi}), J\rangle + \sum_{j\geq 0} \sqrt{\lambda_j} z_j \bar{z}_j + \frac{\varepsilon \widetilde{V}(\theta,\bar{\xi},\varepsilon)}{4\sqrt{\lambda_j}} (z_j + \bar{z}_j)^2, \quad (3.17)$$

$$\begin{aligned} G^3(z,\theta,\bar{\xi},\varepsilon) &= G^3_{0,0,0}(\theta,\bar{\xi},\varepsilon)\big(\frac{z_0+\bar{z}_0}{\sqrt{2}}\big)^3 \\ &\quad + 3\sum_{j,d\neq 0} G^3_{0,j,d}(\theta,\bar{\xi},\varepsilon)\big(\frac{z_j+\bar{z}_j}{\sqrt{2}}\big)\big(\frac{z_d+\bar{z}_d}{\sqrt{2}}\big)\big(\frac{z_0+\bar{z}_0}{\sqrt{2}}\big) \\ &\quad + \sum_{i,j,d\neq 0} G^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)\big(\frac{z_i+\bar{z}_i}{\sqrt{2}}\big)\big(\frac{z_j+\bar{z}_j}{\sqrt{2}}\big)\big(\frac{z_d+\bar{z}_d}{\sqrt{2}}\big). \end{aligned} \quad (3.18)$$

Expressions G^4, G^5, G^6 are defined similarly to G^3.

4. Reducibility of linear Hamiltonian system

In this section, we are concerned with the reducibility of linear quasi-periodic Hamiltonian system (3.17). Our result shows that system (3.17) can be reduced to constant coefficients for any fixed $\omega \in D_\Lambda$. Let us rewrite Hamiltonian (3.17) as

$$\tilde{H} = H_0 + \varepsilon H_1, \quad (4.1)$$

where

$$H_0 = \langle \tilde{\omega}(\bar{\xi}), J\rangle + \sum_{j\geq 0} \sqrt{\lambda_j} z_j \bar{z}_j, \quad H_1 = \sum_{j\geq 0} \frac{\widetilde{V}(\theta,\bar{\xi},\varepsilon)}{4\sqrt{\lambda_j}} (z_j + \bar{z}_j)^2.$$

4.1. Reducibility theorem.

Theorem 4.1. *Consider the Hamiltonian $\tilde{H}$ given by equation (4.1). Then there is a $0 < \varepsilon^{**} < \varepsilon^*$, $0 < \varrho < 1$ and a set $\overline{J} \subset \hat{J}$ with $\operatorname{meas} \overline{J} \geq \operatorname{meas} \hat{J}\big(1 - \mathcal{O}(\varrho)\big)$ such that for any $0 < \varepsilon < \varepsilon^{**}, \bar{\xi} \in \overline{J}$ and $\alpha(\bar{\xi}) \in A_\gamma$ there is a linear symplectic transformation*

$$\Sigma^\infty : \mathcal{D}^{a,s}(\sigma/2, r/2) \times \overline{J} \to \mathcal{D}^{a,s}(\sigma, r)$$

such that the following statements hold:

(i) *There is some absolute constant $C > 0$ such that*

$$|\Sigma^\infty - \mathrm{id}\,|^*_{a,s+1,\,\mathcal{D}^{a,s}(\sigma/2,r/2)\times\overline{J}} \leq C\varepsilon,$$

where id *is identity mapping.*

(ii) *The transformation Σ^∞ changes Hamiltonian (4.1) into*

$$\tilde{H} \circ \Sigma^\infty = \langle \tilde{\omega}(\bar{\xi}), J\rangle + \sum_{j\geq 0} \mu_j z_j \bar{z}_j,$$

where

$$\mu_j = \sqrt{\lambda_j} + \sum_{k=1}^{\infty} \varepsilon^k \tilde{\lambda}_{j,k}(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon), \quad (4.2)$$

$$\tilde{\lambda}_{j,1}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)=\frac{[\widetilde{V}(\theta,\bar{\xi},\varepsilon)]}{2\sqrt{\lambda_j}}=0,$$
$$\tilde{\lambda}_{j,k}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)=[\zeta_{j,k-1,1,1}],|\tilde{\lambda}_{j,k}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)|\le C,\quad k=2,3,\dots.$$

4.2. **Regularity of the perturbation term.** Let

$$G^3_{i,j,d}=G^3_{|i|,|j|,|d|},\quad G^4_{ijdl}=G^4_{|i||j||d||l|},G^5_{ijdlm}=G^5_{|i||j||d||l||m|},$$
$$G^6_{ijdlmn}=G^6_{|i||j||d||l||m||n|}.$$

Noting that the transformation Σ^∞ is linear, and from (i) of Theorem 4.1 we get for $j=0,1,2,\dots$

$$z_j\circ\Sigma^\infty=z_j+\varepsilon\tilde{f}^*_{j,\infty}(\theta;\bar{\xi},\varepsilon)z_j+\varepsilon\tilde{f}^*_{\infty,j}(\theta;\bar{\xi},\varepsilon)\bar{z}_j,$$

where

$$\|\tilde{f}^*_{j,\infty}(\theta;\bar{\xi},\varepsilon)\|^*_{\Theta(\sigma/2)\times\overline{J}},\quad \|\tilde{f}^*_{\infty,j}(\theta;\bar{\xi},\varepsilon)\|^*_{\Theta(\sigma/2)\times\overline{J}}\le C.$$

For convenience we introduce another coordinates $(\dots,w_{-2},w_{-1},w_0,w_1,w_2,\dots)$ in l^s_b by letting $z_0=w_0,\bar{z}_0=w_{-0},z_j=w_j,\bar{z}_j=w_{-j}$ where l^s_b consists of all bi-infinite sequence with finite norm

$$\|w\|^2_{a,s}=|w_0|^2+|w_{-0}|^2+\sum_{|j|\ge1}^{\infty}|w_j|^2|j|^{2s}e^{2a|j|}.$$

Hamiltonian (3.17) is changed into

$$\widehat{H}:=\tilde{H}\circ\Sigma^\infty=\langle\tilde{\omega}(\bar{\xi}),J\rangle+\sum_{j\ge0}\mu_jw_jw_{-j},\tag{4.3}$$
$$(z_0+\bar{z_0})\circ\Sigma^\infty=S_{11}(\theta,\bar{\xi},\varepsilon)w_0+S_{12}(\theta,\bar{\xi},\varepsilon)w_{-0},$$

where

$$S_{11}(\theta,\bar{\xi},\varepsilon):=1+\varepsilon\tilde{f}^1_{0,\infty}(\theta;\bar{\xi},\varepsilon),\quad S_{12}(\theta,\bar{\xi},\varepsilon):=1+\varepsilon\tilde{f}^2_{0,\infty}(\theta;\bar{\xi},\varepsilon)$$

with

$$\|\tilde{f}^1_{0,\infty}(\theta;\bar{\xi},\varepsilon)\|^*_{\Theta(\sigma/2)\times\overline{J}},\quad \|\tilde{f}^2_{0,\infty}(\theta;\bar{\xi},\varepsilon)\|^*_{\Theta(\sigma/2)\times\overline{J}}\le C.$$

This implies the Hamiltonian (3.16) is changed by the transformation Σ^∞ into

$$H=\widehat{H}+\varepsilon\tilde{G}^3+\varepsilon^2\tilde{G}^4+\varepsilon^3\tilde{G}^5+\varepsilon^4\tilde{G}^6.\tag{4.4}$$

Next we consider the regularity of the gradient of $\tilde{G}^3,\dots,\tilde{G}^6$.

Lemma 4.2. *For $a\ge0$ and $s>1/2$, the space $l^{a,s}$ is a Hilbert algebra with respect to convolution of the sequences, $(q*p)_j:=\sum_k q_{j-k}p_k$, and*

$$\|q*p\|_{a,s}\le C\|q\|_{a,s}\|p\|_{a,s}$$

with a constant C depending only on s.

The proof for the above lemma is similar to that of [23, Lemma A]. Using the above lemma, we can prove the following Lemma.

Lemma 4.3. *For $a\ge0$ and $s>1$, the gradient $\tilde{G}^3_w,\tilde{G}^4_w,\tilde{G}^5_w$ and $\tilde{G}^6_w$ are real analytic for real argument as a map from some neighborhood of origin in $l^{a,s}$ into $l^{a,s+1/2}$, with*

$$\|\tilde{G}^3_w\|_{a,s+1/2}\le C\|w\|^2_{a,s},\quad \|\tilde{G}^4_w\|_{a,s+1/2}\le C\|w\|^3_{a,s},\ \dots,\ |\tilde{G}^6_w\|_{a,s+1/2}\le C\|w\|^5_{a,s}$$

uniformly for $(\theta,\bar{\xi}) \in \Theta(\sigma/2) \times \hat{J}$, *where* C *is a constant large enough as* ε *small enough. The Hamiltonian* $\tilde{G}^3$ *till* $\tilde{G}^5$ *depend on the "time"* $\theta = (\theta_1, \dots, \theta_m, \theta_{m+1}) = (\omega_1 t, \dots, \omega_m t, \alpha(\bar{\xi})t)$.

5. The Birkhoff normal form

In this section, we transform the hamiltonian (4.4) into some partial Birkhoff normal form of order six so that it appears, in a sufficiently small neighbourhood of the origin, as a small perturbation of some nonlinear integrable system. To this end we have to kill the perturbation $\tilde{G}^3, \tilde{G}^4, \tilde{G}^5$ and the non-resonant part of the perturbation $\tilde{G}^6$ by Birkhoff normal form.

By $X^1_{F_3}$ denote the time-1 map of the vector field of the Hamiltonian εF_3, Then letting

$$\tilde{G}^3 + \{\widehat{H}, F_3\} = 0,$$

we have found a quasi-periodic function F_3 such that $\tilde{G}^3 + \{\widehat{H}, F_3\} = 0$. Therefore, we get the new Hamiltonian

$$H = \widehat{H} + \varepsilon^2 \mathcal{G}^4 + \varepsilon^3 \mathcal{G}^5 + \varepsilon^4 \mathcal{G}^6 + \varepsilon^5 \mathcal{R}_1, \tag{5.1}$$

Repeating this process, we eliminate all terms in $\mathcal{G}^4$ and $\mathcal{G}^5$. Hamiltonian (5.1) are changed into

$$H = \widehat{H} + \varepsilon^4 \mathbb{G}^6 + \varepsilon^5 R_{11} + \varepsilon^6 R_{22} + \varepsilon^7 R_{33} + \varepsilon^8 R_{44} + \varepsilon^9 R_{55} + \varepsilon^{10} R_{66}, \tag{5.2}$$

and we can write

$$\mathbb{G}^6 = \sum_{|\alpha|+|\beta|+r+s=6} \mathbb{G}^6_{rs\alpha\beta} w_0^r w_{-0}^s w_j^\alpha w_{-j}^\beta. \tag{5.3}$$

Let $\mathcal{L}_n = \{(i,j,d,l,m,n) \in \mathbb{Z}^6 : 0 \neq \min(|i|,|j|,|d|,|l|,|m|,|n|) \leq n\}$, and $\mathcal{N}_n \subset \mathcal{L}_n$ be the subset of all $(i,j,d,l,m,n) \equiv (i,-i,j,-j,l,-l)$. That is, they are of the form $(i,-i,j,-j,l,-l)$ or some permutation of it.

We define the indices set $\Delta_*, * = 0,1,2,3$. For each $* = 0,1,2, \Delta_*$ is the set of indices $\{i,j,d,l,m,n\}$ which have exactly $*$ components not in $\mathcal{L}_n, \Delta_3$ is the set which has at least three components not in $\mathcal{L}_n$. then we split (5.3) into three parts:

$$\mathbb{G}^6 = \overline{\mathbb{G}}^6 + \widehat{\mathbb{G}}^6 + \widetilde{\mathbb{G}}^6,$$

where $\overline{\mathbb{G}}^6$ is the normal form part of $\mathbb{G}^6$, with $(i,\dots,n) \in (\Delta_0 \cup \Delta_1 \cup \Delta_2) \cap \mathcal{N}_n$:

$$\overline{\mathbb{G}}^6 = \sum_{i\in\mathcal{N}_n,\, j,l>1} \mathbb{G}^6_{iijjll} |w_i|^2 |w_j|^2 |w_l|^2,$$

$\widehat{\mathbb{G}}^6$ is the normal form part of $\mathbb{G}^6$, with $(i,\dots,n) \in (\Delta_0 \cup \Delta_1 \cup \Delta_2) \backslash \mathcal{N}_n$:

$$\widehat{\mathbb{G}}^6 = \sum_{(i,\dots,n)\in(\Delta_0\cup\Delta_1\cup\Delta_2)\backslash\mathcal{N}_n} \mathbb{G}^6_{ijdlmn} w_i w_j w_d w_l w_m w_n,$$

$\widetilde{\mathbb{G}}^6$ is the normal form part of $\mathbb{G}^6$, with $(i,\dots,n) \in \triangle_3$:

$$\widetilde{\mathbb{G}}^6 = \sum_{(i,\dots,n)\in\triangle_3} \mathbb{G}^6_{ijdlmn} w_i w_j w_d w_l w_m w_n.$$

Using the same methods as in Section 5.1, there is a hamiltonian F_6 which has the same form as that of $\mathbb{G}^6$, and will eliminate $\widehat{\mathbb{G}}^6$ by a symplectic transformation $X^1_{F_6}$, which is the time-1-map of the flow of a hamiltonian vector X_{F_6} with hamiltonian $\varepsilon^4 F_6$, let

$$\{\widehat{H}, F_6\} + \mathbb{G}^6 = [\chi_{33}(\theta,\bar{\xi},\varepsilon)]w_0^3 w_{-0}^3 + w_0^2 w_{-0}^2 \sum_{j\geq 1} [G^6_{22jj}](\theta,\bar{\xi},\varepsilon) w_j w_{-j} + w_0 w_{-0} \sum_{i\in\mathcal{N}_n, j\geq 1} [\mathbb{G}^6_{00iijj}](\theta,\bar{\xi},\varepsilon) w_i w_{-i} w_j w_{-j} + \overline{\mathbb{G}}^6 + \widetilde{\mathbb{G}}^6.$$

By a direct calculation, we obtain the following lemma.

Lemma 5.1. *For each finite $n \geq 1$, there exists a real analytic, symplectic change of coordinates $X^1_{F_6}$ in some neighborhood of the origin on the complex Hilbert space $l^{a,s}$ such that the hamiltonian (5.2) is changed into*

$$H \circ X^1_{F_6} = \widehat{H} + c_0\varepsilon^4 z_0^3 \bar{z}_0^3 + \varepsilon^4 z_0^2 \bar{z}_0^2 \sum_{j\geq 1} c_j z_j \bar{z}_j + \varepsilon^4 z_0 \bar{z}_0 \sum_{i,j\geq 1} [\mathbb{G}^6_{00iijj}] z_i \bar{z}_i z_j \bar{z}_j + \varepsilon^4 K,$$

where

$$K = \overline{\mathbb{G}}^6 + \hat{\mathbb{G}}^6 + \varepsilon R_{11} + \varepsilon R_{22} + \dots + \varepsilon^9 \int_0^1 \{R_{66}, F_6\} \circ X^s_{F_6} ds,$$

$$c_0 = \frac{[\phi]}{24\pi^2 [\hat{V}]^{3/2}} \varepsilon^{-3/4}(1 + \mathcal{O}(\varepsilon^{\frac{1}{4}})),$$

$$c_j = \frac{[\phi]}{24\pi^2 [\hat{V}]\sqrt{\lambda_j}} \varepsilon^{-1/2}(1 + \mathcal{O}(\varepsilon^{1/2})),$$

and $[\mathbb{G}^6_{00iijj}]$ denotes the 0-Fourier coefficient of $\mathbb{G}^6_{00iijj}$ with

$$[\mathbb{G}^6_{00iijj}] = \begin{cases} \frac{6!}{48}[\mathbb{G}^6_{00iijj}] = \frac{15[\phi]}{\pi^2\sqrt{\lambda_0\lambda_i\lambda_j}}(1+\mathcal{O}(\varepsilon)) + \varpi_{ij}(\bar{\xi},\varepsilon)), & i \neq j, \\ \frac{6!}{48\times 2}[\mathbb{G}^6_{00iiii}] = \frac{15[\phi]}{4\pi^2\lambda_i\sqrt{\lambda_0}}(1+\mathcal{O}(\varepsilon)) + \varpi_{ij}(\bar{\xi},\varepsilon)), & i = j, \end{cases} \tag{5.4}$$

and

$$\mathbb{G}^6_{iijjll} = \frac{[\phi]}{48\pi^2\sqrt{\lambda_i\lambda_j\lambda_l}}(4 + 2\delta_{ij} + 2\delta_{jl} + 2\delta_{jk} + 2\delta_{i+j,l} + 2\delta_{i+l,j} + 2\delta_{l+j,i}). \tag{5.5}$$

Here

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$\varpi_{ij}(\bar{\xi},\varepsilon)$ depends smoothly on $\bar{\xi}$ and ε and there is an absolute constant C such that $\|\varpi_{ij}(\bar{\xi},\varepsilon)\|^*_{\mathscr{J}} \leq C\varepsilon$ for ε small enough, while $\hat{\mathcal{G}}$ is only dependent on the coordinates $\hat{z}$ and we have

$$|\hat{\mathcal{G}}| = O(\|\hat{z}\|^6_{a,s}), \quad |K| = \mathcal{O}(\|z\|^7_{a,s})$$

uniformly for $|\operatorname{Im}\theta| < \sigma/5$, $\bar{\xi} \in \mathscr{J}$, $\hat{z} = (z_j)_{j\in\mathbb{N}\setminus\mathcal{N}_d}$.

We introduce the action-angle variable by setting

$$z_j = \begin{cases} \sqrt{I_j} e^{-\mathrm{i}\hat{\theta}_j}, & j \in \mathcal{N}_d \cup \{0\}, \\ z_j = z_j, & j \notin \mathcal{N}_d \cup \{0\}. \end{cases} \tag{5.6}$$

By the symplectic change (5.6), the normal form becomes

$$\widehat{H} + c_0\varepsilon^4 z_0^3 \bar{z}_0^3 + \varepsilon^4 z_0^2 \bar{z}_0^2 \sum_{j\in\mathbb{N}} c_j z_j \bar{z}_j + \varepsilon^4 z_0 \bar{z}_0 \sum_{i\in\mathcal{N}_n, j\geq 1} c_j |z_i|^2 |z_j|^2$$

$$
\begin{aligned}
&= \langle \tilde\omega(\bar\xi), J\rangle + \mu_0 I_0 + c_0\varepsilon^4 I_0^3 + \sum_{j\in\mathcal{N}_d} (\mu_j + \varepsilon^4 I_0^2 c_j) I_j + \sum_{j\notin\mathcal{N}_d} (\mu_j + \varepsilon^4 I_0^2 c_j) z_j \bar z_j \\
&\quad + I_0 \frac{\varepsilon^{3.5}}{\sqrt{[\widehat V]}} \big(\frac{1}{2}\langle AI, I\rangle + \langle BI, \hat Z\rangle\big)
\end{aligned}
$$

with $I = (I_1, \dots, I_d)$, $A = (a_{ij})_{i,j\in\mathcal{N}_d}$, $B = (a_{ij})_{j\in\mathbb{N}, i\in\mathcal{N}_d}$, $\widehat Z = (|z_{d+1}|^2, |z_{d+2}|^2, \dots)$, and

$$
a_{ij} = \begin{cases} \frac{15[\phi]}{\pi^2\sqrt{\lambda_i\lambda_j}} (1 + \mathcal{O}(\varepsilon)) + \varpi_{ij}(\bar\xi, \varepsilon)), & i \neq j, \\ \frac{15[\phi]}{4\pi^2\lambda_i} (1 + \mathcal{O}(\varepsilon)) + \varpi_{ij}(\bar\xi, \varepsilon)), & i = j. \end{cases}
$$

Now let us introduce the parameter vector $\tilde\xi = (\tilde\xi_j)_{j\in\mathcal{N}_d\cup\{0\}}$ and the new action variable and $\tilde\rho = (\tilde\rho_j)_{j\in\mathcal{N}_d\cup\{0\}}$ as follows

$$
I_j = \varepsilon\tilde\xi_j + \tilde\rho_j, \quad \tilde\xi_j \in [1, 2], \quad |\tilde\rho_j| \langle \varepsilon^4, \quad j = \{0\} \cup \mathcal{N}_d.
$$

Clearly, $d\hat\theta_j \wedge dI_j = d\hat\theta_j \wedge d\tilde\rho_j$. So the transformation is symplectic. Then the normal form is changed into

$$
\begin{aligned}
&\langle \tilde\omega(\bar\xi), J\rangle + \Big(\mu_0 + 3c_0\varepsilon^6\tilde\xi_0^2 + 2\varepsilon^6\tilde\xi_0 \sum_{j\in\mathcal{N}_d} c_j\tilde\xi_j\Big)\tilde\rho_0 + \sum_{j\in\mathcal{N}_d} (\mu_j + \varepsilon^6\tilde\xi_0^2 c_j)\tilde\rho_j \\
+&\quad (3c_0\varepsilon^5\tilde\xi_0 + \varepsilon^5 \sum_{j\in\mathcal{N}_d} c_j\tilde\xi_j)\tilde\rho_0^2 + 2\varepsilon^5\tilde\xi_0\tilde\rho_0 \sum_{j\in\mathcal{N}_d} c_j\tilde\rho_j + c_0\varepsilon^4\tilde\rho_0^3 + \\
+&\quad \sum_{j\notin\mathcal{N}_d} (\mu_j + \varepsilon^4(\varepsilon\tilde\xi_0 + \tilde\rho_0)^2 c_j) z_j\bar z_j + \varepsilon^5\tilde\xi_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde\rho_i\tilde\rho_j \\
&\quad + \varepsilon^6\tilde\xi_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde\rho_i\tilde\xi_j + \varepsilon^6\tilde\xi_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde\xi_i\tilde\rho_j + \varepsilon^6\tilde\xi_0 \sum_{i\in\mathcal{N}_d, j\notin\mathcal{N}_d} [G^6_{00iiii}]\tilde\xi_i |z_j|^2 \\
&\quad + \varepsilon^5\tilde\rho_0 \sum_{i\in\mathcal{N}_d, j\notin\mathcal{N}_d} [G^6_{00iiii}]\tilde\xi_j\tilde\rho_i + \varepsilon^5\tilde\rho_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde\xi_i\tilde\rho_j + \varepsilon^5\tilde\rho_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde\rho_i\tilde\rho_j \\
&\quad + \varepsilon^4(\tilde\xi_0 + \tilde\rho_0)^2 \sum_{i\in\mathcal{N}_d, j\notin\mathcal{N}_d} [G^6_{00iiii}]\tilde\rho_i |z_j|^2.
\end{aligned}
$$

Hence, the total Hamiltonian is

$$
\begin{aligned}
H &= \langle \tilde\omega(\bar\xi), J\rangle + \breve\omega_0\tilde\rho_0 + \sum_{j\in\mathcal{N}_d} \breve\omega_j\tilde\rho_j + \sum_{j\in\mathbb{N}} \check\lambda_j z_j\bar z_j \\
&\quad + \varepsilon^6\tilde\xi_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iijj}]\tilde\rho_i\tilde\xi_j + \varepsilon^6\tilde\xi_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iijj}]\tilde\xi_i\tilde\rho_j \\
&\quad + \varepsilon^6\tilde\xi_0 \sum_{i\in\mathcal{N}_d, j\notin\mathcal{N}_d} [G^6_{00iijj}]\tilde\xi_i |z_j|^2 + P,
\end{aligned} \tag{5.7}
$$

where

$$
\begin{gathered}
\breve\omega_0 = \mu_0 + 3c_0\varepsilon^6\tilde\xi_0^2 + 2\varepsilon^6\tilde\xi_0 \sum_{j\in\mathcal{N}_d} c_j\tilde\xi_j, \\
\breve\omega_j = \mu_j + \varepsilon^6\tilde\xi_0^2 c_j + \varepsilon^6\tilde\xi_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde\xi_i, \quad j \in \mathcal{N}_d \\
\check\lambda_j = \mu_j + \varepsilon^6\tilde\xi_0^2 c_j + \varepsilon^6\tilde\xi_0 \sum_{i\in\mathcal{N}_d,} [G^6_{00iiii}]\tilde\xi_i, \quad j \notin \mathcal{N}_d,
\end{gathered}
$$

$$
\begin{aligned}
P = {} & (3c_0\varepsilon^5\tilde{\xi}_0 + \varepsilon^5 \sum_{j\in\mathcal{N}_d} c_j\tilde{\xi}_j)\tilde{\rho}_0^2 + 2\varepsilon^5\tilde{\xi}_0\tilde{\rho}_0 \sum_{j\in\mathcal{N}_d} c_j\tilde{\rho}_j + c_0\varepsilon^4\tilde{\rho}_0^3 \\
& + \varepsilon^5\tilde{\rho}_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde{\xi}_j\tilde{\rho}_i + \varepsilon^5\tilde{\rho}_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde{\xi}_i\tilde{\rho}_j + \varepsilon^5\tilde{\rho}_0 \sum_{i,j\in\mathcal{N}_d} [G^6_{00iiii}]\tilde{\rho}_i\tilde{\rho}_j \\
& + \varepsilon^4(\tilde{\xi}_0 + \tilde{\rho}_0)^2 \sum_{i\in\mathcal{N}_d, j\notin\mathcal{N}_d} [G^6_{00iiii}]\tilde{\rho}_i|z_j|^2 + \varepsilon^4 \sum_{i,j,l\in\mathcal{N}_d} \mathbb{G}^6\tilde{\rho}_i\tilde{\rho}_j\tilde{\rho}_l \\
& + \varepsilon^5 \sum_{i,j,l\in\mathcal{N}} \mathbb{G}^6\tilde{\xi}_l\tilde{\rho}_i\tilde{\rho}_j + \varepsilon^5 \sum_{i,j,l\in\mathcal{N}_d} \mathbb{G}^6\tilde{\xi}_i\tilde{\rho}_l\tilde{\rho}_j + \varepsilon^5 \sum_{i,j,l\in\mathcal{N}_d} \mathbb{G}^6\tilde{\xi}_j\tilde{\rho}_i\tilde{\rho}_l \\
& + \varepsilon^5 \sum_{i,j\in\mathcal{N}_d, l\notin\mathcal{N}_d} \mathbb{G}^6\tilde{\xi}_i\tilde{\rho}_j z_l\overline{z}_l + \varepsilon^5 \sum_{i,j\in\mathcal{N}_d, l\notin\mathcal{N}_d} \mathbb{G}^6\tilde{\xi}_j\tilde{\rho}_i z_l\overline{z}_l \\
& + \varepsilon^4 \sum_{i,j\in\mathcal{N}_d, l\notin\mathcal{N}_d} \mathbb{G}^6\tilde{\rho}_i\tilde{\rho}_j z_l\overline{z}_l + \varepsilon^5 \sum_{i\in\mathcal{N}_d, j,l\notin\mathcal{N}_d} \tilde{\xi}_i z_j\overline{z}_j z_l\overline{z}_l \\
& + \varepsilon^4 \sum_{i\in\mathcal{N}_d, j,l\notin\mathcal{N}_d} \tilde{\rho}_i z_j\overline{z}_j z_l\overline{z}_l + \varepsilon^4 \sum_{(i,\dots,n)\in\triangle_3} \mathbb{G}^6_{ijdlmn} z_i z_j z_d z_l z_m z_n + \varepsilon^5 K \\
= {} & \varepsilon^4 Q + \varepsilon^4\hat{\mathbb{G}}^6 + \varepsilon^5\hat{K} + \varepsilon^5 K,
\end{aligned}
$$

with

$$
\begin{gathered}
Q = \mathcal{O}(|\tilde{\rho}|^3) + \mathcal{O}(|\tilde{\rho}|\|\hat{Z}\|^2), \quad \hat{\mathbb{G}}^6 = \sum_{(i,\dots,n)\in\triangle_3} \mathbb{G}^6_{ijdlmn} z_i z_j z_d z_l z_m z_n, \\
\hat{K} = \mathcal{O}(|\tilde{\rho}|^2) + \mathcal{O}(|\tilde{\rho}|\|\hat{Z}\|), \quad K = R_{11} + R_{22} + \dots + \varepsilon^9 \int_0^1 \{R_{66}, F_6\} \circ X^s_{F_6} ds.
\end{gathered}
$$

Next, we give the estimates of the perturbed term P. To this end we need some notation which is taken from [23]. Let $l^{a,s}$ is now the Hilbert space consisting of those vectors Z with $\|Z\|_{a,s} < \infty$. with

$$
\|Z\|^2_{a,s} = \sum_{j\notin\mathcal{N}_d} |z_j|^2|j|^{2s}e^{2aj} < \infty, \quad a, s > 0.
$$

Let $x = (\theta, \hat{\theta}_0) \oplus \hat{\theta}$, with $\hat{\theta} = (\hat{\theta}_j)_{j\in\mathcal{N}_d}$, $y = (J, \tilde{\rho}_0) \oplus \tilde{\rho}$, $\tilde{\rho} = (\tilde{\rho}_j)_{j\in\mathcal{N}_d}$, $Z = (z_j)_{j\notin\mathcal{N}_d}$, and let us introduce the phase space

$$
\mathcal{P}^{a,s} = \widehat{\mathbb{T}}^{m+d+2} \times \mathbb{C}^{m+d+2} \times l^{a,s} \times l^{a,s} \ni (x, y, Z, \bar{Z}),
$$

where $\widehat{\mathbb{T}}^{m+d+2}$ is the complexiation of the usual $(m+d+2)$-torus $\mathbb{T}^{m+d+2}$. Set

$$
D(s, r) := \{(x, y, Z, \bar{Z}) \in \mathcal{P}^{a,s} : |\operatorname{Im} x| < s, |y| < r^2, \|Z\|_{a,s} + \|\bar{Z}\|_{a,s} < r\}.
$$

We define the weighted phase norms

$$
|W|_r = |W|_{\bar{s},r} = |x| + \frac{1}{r^2}|y| + \frac{1}{r}\|Z\|_{a,\bar{s}} + \frac{1}{r}\|\bar{Z}\|_{a,\bar{s}}
$$

for $W = (x, y, Z, \bar{Z}) \in \mathcal{P}^{a,\bar{s}}$ with $\bar{s} = s + 1$. Denote by $\underline{\Sigma}$ the parameter set $\mathscr{J} \times [1, 2]^{m+d+1}$. For a map $U : D(s, r) \times \underline{\Sigma} \to \mathcal{P}^{a,\bar{s}}$, define its Lipschitz semi-norm $|U|^{\mathcal{L}}_r$:

$$
|U|^{\mathcal{L}}_r = \sup_{\hat{\xi}\neq\xi} \frac{|\Delta_{\hat{\xi}\xi} U|_r}{|\hat{\xi} - \xi|},
$$

where $\Delta_{\hat{\xi}\xi}U = U(\cdot,\hat{\xi}) - U(\cdot,\xi)$, and where the supremum is taken over $\underline{\Sigma}$. Denote by X_P the vector field corresponding the Hamiltonian P with respect to the symplectic structure $dx \wedge dy + \mathrm{i}dZ \wedge d\bar{Z}$, namely,

$$X_P = (\partial_y P, -\partial_x P, \nabla_{\bar{Z}} P, -\nabla_Z P).$$

Lemma 5.2. *The Perturbation $P(x,y,Z,\bar{Z};\zeta)$ is real analytic for real argument $(x,y,Z,\bar{Z}) \in D(s,r)$ for given $s,r>0$, and Lipschitz in the parameters $\xi \in \underline{\Sigma}$, and for each $\xi \in \underline{\Sigma}$ its gradients with respect to $Z,\bar{Z}$ satisfy*

$$\partial_Z P, \quad \partial_{\bar{Z}} P \in \mathcal{A}(l^{a,s}, l^{a,s+1/2}),$$

where $\mathcal{A}(l^{a,s}, l^{a,s+1/2})$ denotes the class of all maps from some neighborhood of the origin in $l^{a,s}$ into $l^{a,s+1/2}$, which is real analytic in the real and imaginary parts of the complex coordinate Z. In addition, for the perturbed term P we have the two estimates

$$\sup_{D(s,r)\times\underline{\Sigma}} |X_P|_r \le C\varepsilon^4, \qquad \sup_{D(s,r)\times\underline{\Sigma}} |\partial_\xi X_P|_r \le C\varepsilon^4,$$

where $s = \sigma/5$ and $r = \varepsilon$.

Proof. For $j \in \mathcal{N}_d$, from (5.6) it follows that $|\partial_{\tilde{\theta}_j} w_j| \le C\varepsilon^{1/2}$ and $|\partial_{\tilde{\rho}_j} w_j| \le C\varepsilon^{-1/2}$ where $w_j = z_j$ or $w_j = \bar{z}_j$. From (5.6) and $\|Z\|_{a,s} \le r = \varepsilon$, we obtain $\|z\|_{a,s} \le C\varepsilon^{1/2}$ where $z = (z_0, \underline{z}) \oplus Z$ with $\underline{z} = (z_j \in \mathbb{C} : j \in \mathcal{N}_d)$. In view of $|\overline{\mathbb{G}}^6| = \mathcal{O}(\varepsilon^8)$, $|\hat{\mathbb{G}}^6| = \mathcal{O}(\varepsilon^6)$ and $K = \mathcal{O}(\varepsilon^{17/2})$, it follows that $|P| = \mathcal{O}(\varepsilon^{10})$ on $D(s,2r)$. Using Cauchy estimates for $\partial_x P$, $\partial_y P$, $\partial_{\bar{Z}} P$ and $\partial_Z P$, we obtain $|\partial_x P| = \mathcal{O}(\varepsilon^{10})$, $|\partial_y P| = \mathcal{O}(\varepsilon^8)$, $|\partial_{\bar{Z}} P| = \mathcal{O}(\varepsilon^9)$, $|\partial_Z P| = \mathcal{O}(\varepsilon^9)$ on $D(s',r)$. Hence, we have $\sup_{D(s,r)\times\underline{\Sigma}} |X_P|_r \le C\varepsilon^4$. By a direct computation with respect to ξ, we also have $\sup_{D(s,r)\times\underline{\Sigma}} |\partial_\xi X_P|_r \le C\varepsilon^4$. $\square$

6. Proof of main theorem

To apply the infinite-dimensional KAM theorem which was first proved by Kuksin [18, 19] and Pöschel [23] to our problem, we need to introduce a new parameter $\bar{\omega}$ below.

For any $\bar{\xi} \in \mathscr{J}$, we have $\alpha(\bar{\xi}) \in A_\gamma$. Hence, for fixed $\omega_- = (\omega_-^1, \omega_-^2, \dots, \omega_-^m) \in D_\Lambda$ and $\omega_-^{m+1}(\bar{\xi}) \in A_\gamma$ arbitrarily. For

$$\begin{aligned}\tilde{\omega}(\bar{\xi}) \in \bar{\bar{\Omega}} := \Big\{ &\tilde{\omega}(\bar{\xi}) = (\omega_1, \dots, \omega_m, \alpha(\bar{\xi})) \in D_\Lambda \times A_\gamma : |\omega_i - \omega_-^i| \le \varepsilon, \\ &|\alpha(\bar{\xi}) - \omega_-^{m+1}(\bar{\xi})| \le \varepsilon \Big\},\end{aligned}$$

we can introduce new parameter $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_m, \bar{\omega}_{m+1})$ by the following

$$\begin{gathered}\omega_j = \omega_-^j + \varepsilon^6 \bar{\omega}_j, \quad \bar{\omega}_j \in [0,1], \quad j = 1,2,\dots,m, \\ \alpha(\bar{\xi}) = \omega_-^{m+1}(\bar{\xi}) + \varepsilon^6 \bar{\omega}_{m+1}, \quad \bar{\omega}_{m+1} \in [0,1].\end{gathered}$$

Hence, the Hamiltonian (5.7) becomes

$$H = \langle \widehat{\omega}(\xi), \hat{y} \rangle + \langle \widehat{\Omega}(\xi), \hat{Z} \rangle + P, \tag{6.1}$$

where $\widehat{\omega}(\xi) = \tilde{\omega}(\bar{\xi}) \oplus \breve{\omega}_0 \oplus \breve{\omega}$ with

$$\breve{\omega} = \tilde{\alpha} + \frac{\varepsilon^{5.5}\tilde{\xi}_0}{\sqrt{[\bar{V}]}} A\tilde{\xi}, \quad \widehat{\Omega}(\xi) = \tilde{\beta} + \varepsilon^{5.5} \frac{\tilde{\xi}_0}{\sqrt{[\bar{V}]}} B\tilde{\xi},$$

$$\xi = \bar{\omega} \oplus \tilde{\xi}_0 \oplus \tilde{\xi}, \quad \tilde{\xi} := (\tilde{\xi}_j)_{j\in\mathcal{N}_d},$$
$$\hat{y} = J \oplus \tilde{\rho}_0 \oplus \tilde{\rho}, \quad \tilde{\alpha} = (\check{\omega}_1, \dots, \check{\omega}_d), \quad \tilde{\beta} = (\check{\lambda}_{d+1}, \check{\lambda}_{d+2}, \dots).$$

Lemma 6.1. *Let $\Pi = [1,2]^{m+d+2}$. Then we have $X_P \in \mathcal{A}(l^{a,s}, l^{a,s+1/2})$ and*
$$\sup_{D(s,r)\times\Pi} |X_P|_r \le C\varepsilon^4, \quad \sup_{D(s,r)\times\Pi} |\partial_\zeta X_P|_r \le C\varepsilon^4.$$

The proof of the above lemma is the same as one of the lemma 5.2. In the following, we verify the assumptions A, B and C in [23], for the above Hamiltonian (6.1). Recalling (5.4), we have

$$\begin{aligned} A &= (a_{ij}) \\ &= \begin{pmatrix} \frac{b}{\lambda_{i_1}} + \varpi_{11}(\bar{\xi},\varepsilon) & \frac{a}{\sqrt{\lambda_{i_1}\lambda_{i2}}} + \varpi_{12}(\bar{\xi},\varepsilon) & \dots & \frac{a}{\sqrt{\lambda_{i1}\lambda_{id}}} + \varpi_{1d}(\bar{\xi},\varepsilon) \\ \frac{a}{\sqrt{\lambda_{i2}\lambda_{i1}}} + \varpi_{21}(\bar{\xi},\varepsilon) & \frac{b}{\lambda_{i2}} + \varpi_{22}(\bar{\xi},\varepsilon) & \dots & \frac{a}{\sqrt{\lambda_{i2}\lambda_{id}}} + \varpi_{4}(\bar{\xi},\varepsilon) \\ \dots & \dots & \dots & \dots \\ \frac{a}{\sqrt{\lambda_{id}\lambda_{i1}}} + \varpi_{d1}(\bar{\xi},\varepsilon) & \frac{a}{\sqrt{\lambda_{id}\lambda_{i2}}} + \varpi_{n2}(\bar{\xi},\varepsilon) & \dots & \frac{b}{\lambda_{id}} + \varpi_{dd}(\bar{\xi},\varepsilon) \end{pmatrix}_{d\times d}, \end{aligned}$$

$$B = \begin{pmatrix} \frac{a}{\sqrt{\lambda_{i(d+1)}\lambda_{i1}}} + \varpi_{d+1,1}(\bar{\xi},\varepsilon) & \dots & \frac{a}{\sqrt{\lambda_{i(d+1)}\lambda_{id}}} + \varpi_{d+1,d}(\bar{\xi},\varepsilon) \\ \frac{a}{\sqrt{\lambda_{i(d+2)}\lambda_{i1}}} + \varpi_{d+2,1}(\bar{\xi},\varepsilon) & \dots & \frac{a}{\sqrt{\lambda_{i(d+2)}\lambda_{id}}} + \varpi_{d+2,d}(\bar{\xi},\varepsilon) \\ \vdots & \vdots & \vdots \end{pmatrix}_{\infty\times d},$$

where
$$a = \frac{15[\phi]}{\pi^2}, \quad b = \frac{15[\phi]}{4\pi^2}. \tag{6.2}$$

$$\lim_{\varepsilon\to 0} A = \begin{pmatrix} \frac{b}{i_1^2\times i_1^2} & \frac{a}{i_1^2\times i_2^2} & \cdots & \frac{a}{i_1^2\times i_d^2} \\ \frac{a}{i_2^2\times i_1^2} & \frac{b}{i_2^2\times i_2^2} & \cdots & \frac{a}{i_2^2\times i_d^2} \\ \dots & \dots & \dots & \dots \\ \frac{a}{i_d^2\times i_1^2} & \frac{a}{i_d^2\times i_2^2} & \cdots & \frac{b}{i_d^2\times i_d^2} \end{pmatrix}_{d\times d} := D,$$
$$\lim_{\varepsilon\to 0} B = \begin{pmatrix} \frac{a}{i_{d+1}^2\times i_1^2} & \cdots & \frac{a}{i_{d+1}^2\times i_d^2} \\ \frac{a}{i_{d+2}^2\times i_1^2} & \cdots & \frac{a}{i_{d+2}^2\times i_d^2} \\ \vdots & \vdots & \vdots \end{pmatrix}_{\infty\times d} := \widetilde{D}.$$

Setting $\hat{u} = (i_1^2, i_2^2, \dots, i_d^2)$ and $\hat{v} = (i_{d+1}^2, i_{d+2}^2, \dots)$, and defining the matrices
$$\bar{E} := \operatorname{diag}[\hat{u}], \quad \bar{F} := \operatorname{diag}[\hat{v}],$$
we can rewrite D and $\widetilde{D}$ as
$$D = \bar{E}^{-1}\overline{A}\bar{E}^{-1}, \quad \widetilde{D} = \bar{F}^{-1}\overline{B}\bar{E}^{-1},$$
where
$$\overline{A} = \begin{pmatrix} b & a & \dots & a \\ a & b & \dots & a \\ \dots & \dots & \dots & \dots \\ a & a & \dots & b \end{pmatrix}_{d\times d}, \quad \overline{B} = \begin{pmatrix} b & \dots & b \\ b & \dots & b \\ \vdots & \vdots & \vdots \end{pmatrix}_{\infty\times d}.$$

We know that $\det D \neq 0$ since $\det \overline{A} = (b-a)^{d-1}(b+(d-1)a) \neq 0$ (a, b have the same sign). Therefore, we get $\det A \neq 0$ provided that $0 < \varepsilon \ll 1$. Moreover, by the definition of $\widehat{\omega}$, we get that

$$\frac{\partial \widehat{\omega}}{\partial \xi} = \frac{\partial(\widetilde{\omega}, \breve{\omega}_0, \breve{\omega})}{\partial(\overline{\omega}, \widetilde{\xi_0}, \widetilde{\xi})} = \varepsilon^6 \tilde{\xi}_0 \begin{pmatrix} I_{m+1} & 0 & 0 \\ 0 & 6c_0 + \sum_{j\in\mathcal{N}_d} c_j\xi_j & 12\mathbb{Y} \\ 0 & 12\tilde{\xi}_0\mathbb{Y}^T + A\xi & A \end{pmatrix}, \quad \text{for } \xi \in \Pi,$$

where I_{m+1} denotes the $(m+1)\times(m+1)$ unit matrix, $\mathbb{Y} = (c_1, c_2, \dots, c_d)$. Let $c_0' = 6c_0 + \sum_{j\in\mathcal{N}_d} c_j\xi_j$. In view of $c_0 = \mathcal{O}(\varepsilon^{-3/4})$, $c_j = \mathcal{O}(\varepsilon^{-1/2})$ and

$$\begin{aligned} &\begin{pmatrix} 1 & -\mathbb{Y}A^{-1} \\ 0 & I_d \end{pmatrix} \begin{pmatrix} c_0' & 12\mathbb{Y} \\ 12\mathbb{Y}^T + A\xi & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1}\mathbb{Y}^T & I_d \end{pmatrix} \\ &= \begin{pmatrix} c_0' - \mathbb{Y}A^{-1}\mathbb{Y}^T - \mathbb{Y}\xi & 0 \\ A\xi & A \end{pmatrix}, \end{aligned}$$

we get

$$\det \begin{pmatrix} c_0' - \mathbb{Y}A^{-1}\mathbb{Y}^T - \mathbb{Y}\xi & 0 \\ A\xi & A \end{pmatrix} \neq 0$$

provided that $0 < \varepsilon \ll 1$. Therefore, the real map $\xi \mapsto \widehat{\omega}(\xi)$ is a lipeomorphism between Π and its imagine.

For any $k \in \mathbb{Z}^{m+2+d}$, we write

$$k = (k_1, k_2, k_3), \quad k_1 \in \mathbb{Z}^{m+1},\ k_2 \in \mathbb{Z}, k_3 \in \mathbb{Z}^d.$$

Let

$$\begin{aligned} \mathcal{Y}(\xi) &= \langle k, \widehat{\omega}(\xi)\rangle + \langle l, \widehat{\Omega}(\xi)\rangle \\ &= \langle k_1, \tilde{\omega}(\bar{\xi})\rangle + k_2\breve{\omega}_0 + \langle k_3, \tilde{\alpha}\rangle + \langle k_3, \varepsilon^{5.5}\frac{\tilde{\xi}_0}{\sqrt{[\widehat{V}]}}A\tilde{\xi}\rangle \\ &\quad + \langle l, \tilde{\beta} + \varepsilon^{5.5}\frac{\tilde{\xi}_0}{\sqrt{[\widehat{V}]}}B\tilde{\xi}\rangle, \end{aligned}$$

$$\Delta := \{\xi \in \Pi : \mathcal{Y}(\xi) = 0\}.$$

We need to prove that $\operatorname{meas}\Delta = 0$. We discuss the following two cases.

Case 1. Let $k_1 = (k^1, k^2, \dots, k^m, k^{m+1}) \neq 0$ and write

$$\langle k_1, \tilde{\omega}(\bar{\xi})\rangle = \sum_{i=1}^m k^i\omega_i + k^{m+1}\alpha(\bar{\xi}),$$

then there exists some $1 \le i_0 \le m$ such that $k^{i_0} \neq 0$. Observe that $\breve{\omega}_0$, $\tilde{\alpha}$ and $\tilde{\beta}$ do not involve the parameter $\bar{\omega}$. Then

$$\frac{\partial \mathcal{Y}(\xi)}{\partial \bar{\omega}_{i_0}} = k^{i_0}\varepsilon^6 \neq 0, \quad 0 < \varepsilon \ll 1,$$

which implies $\operatorname{meas}\Delta = 0$.

Case 2. Let $k_2 \neq 0$. Then

$$\frac{\partial \mathcal{Y}(\xi)}{\partial \tilde{\xi}_0} = 6c_0\varepsilon^6 + 2\varepsilon^6 \sum_{j\in\mathcal{N}_d} c_j\tilde{\xi}_j \neq 0,$$

which implies $\operatorname{meas}\Delta = 0$.

Case 3. Let $k_1 = k_2 = 0$, then

$$\begin{aligned}\mathcal{Y}(\xi) &= \langle k_1, \omega\rangle + k_2\breve{\omega}_0 + \langle k_3, \tilde{\alpha}\rangle + \langle k_3, \frac{\varepsilon^{5.5}}{\sqrt{[\hat{V}]}}A\tilde{\xi}\rangle + \langle l, \tilde{\beta} + \varepsilon^3 B\tilde{\xi}\rangle \\ &= \langle k_3, \tilde{\alpha}\rangle + \langle k_3, \frac{\varepsilon^{5.5}}{\sqrt{[\hat{V}]}}A\tilde{\xi}\rangle + \langle l, \tilde{\beta} + \frac{\varepsilon^{5.5}}{\sqrt{[\hat{V}]}}B\tilde{\xi}\rangle \\ &= \langle k_3, \tilde{\alpha}\rangle + \langle l, \tilde{\beta}\rangle + \frac{\varepsilon^{5.5}}{\sqrt{[\hat{V}]}}\langle Ak_3 + B^T l, \tilde{\xi}\rangle,\end{aligned}$$

where B^T is the transpose of B. (Note that A is symmetric.) We claim that either $\langle k_3, \tilde{\alpha}\rangle + \langle l, \tilde{\beta}\rangle \neq 0$ or $Ak_3 + B^T l \neq 0$.

Since

$$\lim_{\varepsilon\to 0}(Ak_3 + B^T l) = Dk_3 + \widetilde{D}^T l,$$

and

$$\lim_{\varepsilon\to 0}(\langle k_3, \tilde{\alpha}\rangle + \langle l, \tilde{\beta}\rangle) = \langle k_3, \hat{\alpha}\rangle + \langle l, \hat{\beta}\rangle,$$

with $\hat{\alpha} = (i_1^2, i_2^2, \dots, i_d^2)$ and $\hat{\beta} = (i_{d+1}^2, i_{d+2}^2, \dots)$, it suffices to show that $\langle k_3, \hat{\alpha}\rangle + \langle l, \hat{\beta}\rangle \neq 0$ or $Dk_3 + \widetilde{D}^T l \neq 0$. The result is proved in in[24, Lemma 6]. Hence, we get that $\langle k_3, \tilde{\alpha}\rangle + \langle l, \tilde{\beta}\rangle \neq 0$ or $Ak_3 + B^T l \neq 0$ as $0 < \varepsilon \ll 1$. Moreover, it is easy to that $\langle l, \widehat{\Omega}(\xi)\rangle \neq 0$ as $0 < \varepsilon \ll 1$, with $1 \leq |l| \leq 2$ and $\xi \in \Pi$. This completes the verification of Assumption A.

Noticing that

$$\lambda_j = \sqrt{j^4 + \varepsilon[\hat{V}]} = j^2 + \frac{\varepsilon[\hat{V}]}{2j^2} + O(j^{-4}),$$
$$\mu_j = \sqrt{\lambda_j} + \sum_{k=2}^{\infty}\varepsilon^k\tilde{\lambda}_{j,k}(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon),$$

we have that $\widehat{\Omega}_j = j^\varsigma + \dots$ with $\varsigma = 2$ and $\widehat{\Omega}_j - j^2$ is a Lipschitz map from Π to $l_\infty^{-\delta}$ with $\delta = -2$. Thus, Assumption B is fulfilled for $\widehat{\Omega}$ with $\delta = -2$, $\varsigma = 2$, and $\overline{\Omega} = \tilde{\beta}$.

Assumption C can be verified easily using Lemma 5.2, letting $\bar{p} = s + 1/2$, $p = s$.

Now let us verify the smallness condition in [23]. By letting $\alpha = \varepsilon^{8-\iota}$ with $0 < \iota < 8$ fixed and Lemma 6.1, we have

$$\sup_{D(s,r)\times\Pi}|X_P|_r + \sup_{D(s,r)\times\Pi}\frac{\alpha}{M}|X_P|_r^{\mathcal{L}} \leq \gamma\alpha,$$

if $0 < \varepsilon < \varepsilon^{**}$ with a constant $\varepsilon^{**} = \varepsilon^{**}(\gamma, C)$. This implies the smallness condition is satisfied. Next, Let us check the conditions of [23, Theorem D] for the Hamiltonian (6.1). First of all, we remark that $\widehat{\omega}(\xi)$ is affine function of the parameter ξ. and we can choose $\tilde{\mu} = 1$ in [23, Theorem D].

Let us run the infinite-dimensional KAM theorem for Hamiltonian (6.1). Then there is a subset $\Pi_\alpha \subset \Pi$ with

$$\operatorname{meas}(\Pi \setminus \Pi_\alpha) \leq \hat{c}L^{m+d+2}M^{m+d+2}(\operatorname{diam}\Pi)^{m+d+2}\alpha^{\tilde{\mu}} \leq C\varepsilon^{-1}\alpha^{\tilde{\mu}} < \varepsilon^{6-\iota},$$

and a Lipschitz continuous family of torus embedding $\Phi : \mathbb{T}^{m+d+2} \times \Pi_\alpha \to \mathcal{P}^{a,s+1/2}$, and a Lipschitz continuous map $\widehat{\widehat{\omega}} : \Pi_\alpha \to \mathbb{R}^{m+d+2}$, such that for each $\xi \in \Pi_\alpha$

the map Φ restricted to $\mathbb{T}^{m+d+2} \times \{\xi\}$ is a real analytic embedding of an elliptic rotational torus with frequencies $\widehat{\widehat{\omega}}(\xi)$ for the Hamiltonian H at ξ. Moreover, $|\widehat{\widehat{\omega}}(\xi) - \widehat{\omega}(\xi)| < c\varepsilon$. We return from the parameter set Π to

$$\Pi^*(\omega_-, \omega_-^{m+1}) = \bar{\bar{\Omega}} \times [0,1]^{d+1}.$$

Let

$$\Pi^* = \cup_{(\omega_-, \omega_-^{m+1}) \in D_\Lambda \times A_\gamma} \Pi^*(\omega_-, \omega_-^{m+1}),$$

where ω_-, ω_-^{m+1} are chosen such that $\Pi^*(\omega_-^*, \omega_-^{m+1^*}) \cap \Pi^*(\omega_-^{**}, \omega_-^{m+1^{**}}) = \emptyset$ if $(\omega_-^*, \omega_-^{m+1^*}) \neq (\omega_-^{**}, \omega_-^{m+1^{**}})$. Hence, we get a subset $\Pi_\alpha^* \subset \Pi^*$ such that

$$\Sigma_\alpha = \Pi_\alpha^* \subset D_\Lambda \times A_\gamma \times [0,1]^{d+1} \subset \Sigma$$

with

$$\operatorname{meas}(\Sigma \setminus \Sigma_\varepsilon) \leq \varepsilon.$$

Therefore, for the new parameter set, we have that there are a Lipschitz continuous family of torus embedding $\Phi : \mathbb{T}^{m+d+2} \times \Sigma_\varepsilon \to \mathcal{P}^{a,s+1}$, and a Lipschitz continuous map $\widehat{\widehat{\omega}} : \Sigma_\varepsilon \to \mathbb{R}^{m+d+2}$, such that for each $\xi \in \Sigma_\varepsilon$ the map Φ restricted to $\mathbb{T}^{m+d+2} \times \{\xi\}$ is a real analytic embedding of an elliptic rotational torus with frequencies $\widehat{\widehat{\omega}}(\xi) = (\tilde{\omega}(\bar{\xi}), (\hat{\omega}_j)_{j \in \mathcal{N}_d \cup \{0\}})$ for the Hamiltonian H at ξ. Also

$$|\Phi - \Phi_0|_r + \frac{\alpha}{M} |\Phi - \Phi_0|_r^{\mathcal{L}} \leq c\varepsilon^{3-(3-\iota)} = c\varepsilon^\iota, \tag{6.3}$$

$$|\widehat{\widehat{\omega}} - \omega^0|_r + \frac{\alpha}{M} |\widehat{\widehat{\omega}}(\xi) - \omega^0(\xi)|^{\mathcal{L}} \leq c\varepsilon^3, \tag{6.4}$$

where $\omega^0(\xi) = \widehat{\omega}(\xi)$ and $\xi = (\omega, \alpha(\bar{\xi}), \tilde{\xi}_0, \tilde{\xi}_1, \dots, \tilde{\xi}_d)$. Therefore, all motions starting from the torus $\Phi(\mathbb{T}^{m+d+2} \times \Sigma_\varepsilon)$ are quasi-periodic with frequencies $\widehat{\widehat{\omega}}(\xi)$. By (6.3) and (6.4), those motions can written as follows:

$$\begin{gathered}
\tilde{\rho}_0(t) = O(\varepsilon^3), \quad \hat{\theta}_0(t) = \hat{\omega}_0 t + O(\varepsilon^\iota), \\
\tilde{\rho}_j(t) = O(\varepsilon^3), \quad \hat{\theta}_j(t) = \hat{\omega}_j t + O(\varepsilon^\iota), \quad j \in \mathcal{N}_d, \\
\|Z(t)\|_{a,s+1} = O(\varepsilon), \quad \theta(t) = \tilde{\omega}(\bar{\xi}) t,
\end{gathered}$$

where $Z = (z_j)_{j \notin \mathcal{N}_d}$ and we have chosen the initial phase $\hat{\theta}_j(0) = 0$. Returning the original equation (1.3), we may get the solution described in Theorem 1.1.

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