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NONEXISTENCE RESULTS FOR A PSEUDO-HYPERBOLIC EQUATION IN THE HEISENBERG GROUP

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Abstract. Sufficient conditions are obtained for the nonexistence of solutions to the nonlinear pseudo-hyperbolic equation

$$u_{tt} - \Delta_{\mathbb{H}} u_{tt} - \Delta_{\mathbb{H}} u = |u|^p, \quad (\eta, t) \in \mathbb{H} \times (0, \infty),\ p > 1,$$

where $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$. Then, this result is extended to the case of a 2×2-system of the same type. Our technique of proof is based on judicious choices of the test functions in the weak formulation of the sought solutions.

1. Introduction

In this article, we are concerned with the nonexistence of weak solutions to the nonlinear pseudo-hyperbolic equation

$$u_{tt} - \Delta_{\mathbb{H}} u_{tt} - \Delta_{\mathbb{H}} u = |u|^p, \quad (\eta, t) \in \mathbb{H} \times (0, \infty),\ p > 1, \tag{1.1}$$

under the initial conditions

$$u(\eta, 0) = u_0(\eta), \quad u_t(\eta, 0) = u_1(\eta), \quad \eta \in \mathbb{H}, \tag{1.2}$$

where $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$. In the Euclidean case, pseudo-hyperbolic equations served as models for the unidirectional propagation of nonlinear dispersive long waves [2], creep buckling [5] for example. For further applications, one is referred to the valuable book [1] where a sizeable number of pseudo-hyperbolic equations are studied. Our proofs rely on the test function method [8, 12]. For the reader convenience, some background facts used in the sequel are recalled.

The $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$ is the space $\mathbb{R}^{2N+1}$ equipped with the group operation

$$\eta \circ \eta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y)),$$

for all $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

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On $\mathbb{H}$ it is natural to define a distance from $\eta = (x, y, \tau) =: (z, \tau)$ to the origin by

$$|\eta|_{\mathbb{H}} = \Big(\tau^2 + \Big(\sum_{i=1}^{N}(x_i^2 + y_i^2)\Big)^2\Big)^{1/4} = \big(\tau^2 + |z|^4\big)^{1/4},$$

where $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N)$.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ can be defined from the vectors fields

$$X_i = \partial_{x_i} + 2y_i\partial_\tau \quad \text{and} \quad Y_i = \partial_{y_i} - 2x_i\partial_\tau,$$

for $i = 1, \cdots, N$, as follows

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N}(X_i^2 + Y_i^2).$$

A simple computation gives the expression

$$\Delta_{\mathbb{H}} u = \sum_{i=1}^{N} \big(\partial^2_{x_i x_i} u + \partial^2_{y_i y_i} u + 4y_i\partial^2_{x_i\tau} u - 4x_i\partial^2_{y_i\tau} u + 4(x_i^2 + y_i)^2\partial^2_{\tau\tau} u\big).$$

The operator $\Delta_{\mathbb{H}}$ satisfies the following properties:

- It is invariant with respect to the left multiplication in the group, i.e., for all $\eta, \eta' \in \mathbb{H}$, we have
$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = \Delta_{\mathbb{H}} u(\eta \circ \eta');$$
- It is homogeneous with respect to a dilatation. More precisely, for $\lambda \in \mathbb{R}$ and $(x, y, \tau) \in \mathbb{H}$, we have
$$\Delta_{\mathbb{H}}(u(\lambda x, \lambda y, \lambda^2\tau)) = \lambda^2(\Delta_{\mathbb{H}} u)(\lambda x, \lambda y, \lambda^2\tau);$$
- If $u(\eta) = v(|\eta|_{\mathbb{H}})$, then
$$\Delta_{\mathbb{H}} v(\rho) = a(\eta)\Big(\frac{d^2v}{d\rho^2} + \frac{Q-1}{\rho}\frac{dv}{d\rho}\Big),$$
where $\rho = |\eta|_{\mathbb{H}}$, $a(\eta) = \rho^{-2}\sum_{i=1}^{N}(x_i^2 + y_i^2)$ and $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}$.

For more details on Heisenberg groups, we refer to [4, 7].

In this work, we first provide a sufficient condition for the nonexistence of weak solutions to the nonlinear problem (1.1)-(1.2), then we extend the result to the case of the 2×2 system

$$\begin{gathered}
u_{tt} - \Delta_{\mathbb{H}} u_{tt} - \Delta_{\mathbb{H}} u = |v|^q, \quad (\eta, t) \in \mathbb{H} \times (0, \infty), \\
v_{tt} - \Delta_{\mathbb{H}} v_{tt} - \Delta_{\mathbb{H}} v = |u|^p, \quad (\eta, t) \in \mathbb{H} \times (0, \infty), \\
u(\eta, 0) = u_0(\eta),\ u_1(\eta, 0) = u_1(\eta), \quad \eta \in \mathbb{H} \\
v(\eta, 0) = v_0(\eta),\ v(\eta, 0) = v_1(\eta), \quad \eta \in \mathbb{H},
\end{gathered} \tag{1.3}$$

where $p, q > 1$ are real numbers, for which we provide a sufficient condition for the nonexistence of weak solutions.

2. Results and proofs

Let $\mathcal{H}_T = \mathbb{H} \times (0, T)$, $\mathcal{H} = \mathbb{H} \times (0, \infty)$. For $R > 0$, let

$$\mathcal{U}_R = \{(x, y, \tau, t) \in \mathcal{H} : 0 \le t^4 + |x|^4 + |y|^4 + \tau^2 \le 2R^4\}.$$

2.1. **Case of a single equation.** The definition of solutions we adopt for (1.1)-(1.2) is:

We say that u is a local weak solution to (1.1)-(1.2) on $\mathcal{H}$ with initial data $u(0,\cdot)=u_0\in L^1_{\rm loc}(\mathbb{H})$, if $u\in L^p_{\rm loc}(\mathcal{H})$ and satisfies

$$\begin{aligned}&\int_{\mathcal{H}}|u|^p\varphi\,d\vartheta\,dt+\int_{\mathbb{H}}u_1(\vartheta)\varphi(\vartheta,0)\,d\vartheta+\int_{\mathbb{H}}u_1(\vartheta)\Delta_{\mathbb{H}}\varphi(\vartheta,0)\,d\vartheta\\&=\int_{\mathcal{H}}u\varphi_{tt}\,d\vartheta\,dt+\int_{\mathcal{H}}u\Delta_{\mathbb{H}}\varphi_{tt}\,dt\,d\vartheta-\int_{\mathcal{H}}u\Delta_{\mathbb{H}}\varphi\,dt\,d\vartheta,\end{aligned}$$

for any test function φ, $\varphi(\cdot,t)=0$, $\varphi_t(\cdot,t)=0, t\geq T$. The solution u is said global if it exists on $(0,\infty)$.

Our first main result is given by the following theorem.

Theorem 2.1. *Let $u_1\in L^1(\mathbb{H})$. Suppose that*

$$\int_{\mathbb{H}}u_0\,d\vartheta>0. \tag{2.1}$$

If

$$1<p\leq 1+\frac{2}{Q-1},$$

then any weak solution to (1.1)-(1.2) *blows-up in a finite time.*

Proof. Suppose that u is a weak solution to (1.1)-(1.2). Then for any regular test function φ, we have

$$\begin{aligned}&\int_{\mathcal{H}}|u|^p\varphi\,d\vartheta\,dt+\int_{\mathbb{H}}u_1(\vartheta)\varphi(\vartheta,0)\,d\vartheta\\&\leq\int_{\mathcal{H}}|u||\varphi_{tt}|\,d\vartheta\,dt+\int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}\varphi_{tt}|\,dt\,d\vartheta\\&\quad+\int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}\varphi|\,dt\,d\vartheta+\int_{\mathbb{H}}|u_1(\vartheta)||\Delta_{\mathbb{H}}\varphi(\vartheta,0)|\,d\vartheta.\end{aligned}\tag{2.2}$$

Using the ε-Young inequality

$$ab\leq\varepsilon a^p+C_\varepsilon b^{p'},\quad a,b,\varepsilon,C_\varepsilon>0,\ 1<p,p',\ p+p'=pp',$$

with parameters p and $p'=p/(p-1)$, we obtain

$$\begin{aligned}\int_{\mathcal{H}}|u||\varphi_{tt}|\,d\vartheta\,dt&=\int_{\mathcal{H}}|u|\varphi^{1/p}\varphi^{-\frac{1}{p}}|\varphi_{tt}|\,d\vartheta\,dt\\&\leq\varepsilon\int_{\mathcal{H}}|u|^p\varphi\,d\vartheta\,dt+c_\varepsilon\int_{\mathcal{H}}\varphi^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,d\vartheta\,dt,\end{aligned}\tag{2.3}$$

for some positive constant c_ε.

Similarly, we have

$$\int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}\varphi_{tt}|\,dt\,d\vartheta\leq\varepsilon\int_{\mathcal{H}}|u|^p\varphi\,d\vartheta\,dt+c_\varepsilon\int_{\mathcal{H}}\varphi^{-\frac{1}{p-1}}|\Delta_{\mathbb{H}}\varphi_{tt}|^{\frac{p}{p-1}}\,dt\,d\vartheta,\tag{2.4}$$

$$\int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}\varphi|\,dt\,d\vartheta\leq\varepsilon\int_{\mathcal{H}}|u|^p\varphi\,d\vartheta\,dt+c_\varepsilon\int_{\mathcal{H}}\varphi^{-\frac{1}{p-1}}|\Delta_{\mathbb{H}}\varphi|^{\frac{P}{P-1}}\,dt\,d\vartheta.\tag{2.5}$$

Using (2.2), (2.3), (2.4) and (2.5), for $\varepsilon > 0$ small enough, we obtain

$$\begin{aligned}&\int_{\mathcal{H}} |u|^p \varphi \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi(\vartheta, 0)\, d\vartheta \\ &\le C\Big(A_p(\varphi) + B_p(\varphi) + C_p(\varphi) + \int_{\mathbb{H}} |u_1(\vartheta)||\Delta_{\mathbb{H}}\varphi(\vartheta,0)|\, d\vartheta\Big),\end{aligned} \tag{2.6}$$

where

$$A_p(\varphi) = \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}} \, d\vartheta \, dt, \tag{2.7}$$

$$B_p(\varphi) = \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi_{tt}|^{\frac{p}{p-1}} \, d\vartheta \, dt, \tag{2.8}$$

$$C_p(\varphi) = \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}} \, d\vartheta \, dt. \tag{2.9}$$

Now, let us consider the test function

$$\varphi_R(t,\vartheta) = \phi^{\omega}\Big(\frac{t^4 + |x|^4 + |y|^4 + \tau^2}{R^4}\Big), \quad R > 0, \ \omega \gg 1, \tag{2.10}$$

where $\phi \in C_0^{\infty}(\mathbb{R}^+)$ is a decreasing function satisfying

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2. \end{cases}$$

Observe that $\operatorname{supp}(\varphi_R)$ is a subset of $\mathcal{U}_R$, while $\operatorname{supp}(\varphi_{R tt})$, $\operatorname{supp}(\Delta_{\mathbb{H}}\varphi_R)$ and $\operatorname{supp}(\Delta_{\mathbb{H}}(\varphi_R)_{tt})$ are subsets of

$$\Theta_R = \{(t,x,y,\tau) \in \mathcal{H} : \ R^4 \le t^4 + |x|^4 + |y|^4 + \tau^2 \le 2R^4\}.$$

Let

$$\rho = \frac{t^4 + |x|^4 + |y|^4 + \tau^2}{R^4}.$$

Then we have

$$\begin{aligned}&\Delta_{\mathbb{H}}\varphi_R(t,\vartheta)\\ &= \frac{4\omega(N+4)}{R^4}\Big(|x|^2 + |y|^2\Big)\phi'(\rho)\phi^{\omega-1}(\rho)\\ &\quad + \frac{16\omega(\omega-1)}{R^8}\Big((|x|^6 + |y|^6) + 2\tau(|x|^2 - |y|^2)x\cdot y + \tau^2(|x|^2 + |y|^2)\Big)\phi'^2(\rho)\phi^{\omega-2}(\rho)\\ &\quad + \frac{16\omega}{R^8}\Big((|x|^6 + |y|^6) + 2\tau(|x|^2 - |y|^2)x\cdot y + \tau^2(|x|^2 + |y|^2)\Big)\phi''(\rho)\phi^{\omega-1}(\rho)\end{aligned}$$

for example.

Observe first that $(\varphi_R)_t(\vartheta, 0) = 0$ as required in the definition. It follows that there is a positive constant $C > 0$, independent of R, such that for all $(t,\vartheta) \in \times_R$, we have

$$|\Delta_{\mathbb{H}}\varphi_R(t,\vartheta)| \le CR^{-2}\phi^{\omega-2}(\rho)\chi(\rho), \tag{2.11}$$

where

$$\chi(\rho) = |\phi'(\rho)|\phi(\rho) + \phi'^2(\rho) + |\phi''(\rho)|\phi(\rho),$$

and

$$|(\Delta_{\mathbb{H}}\varphi_R)_t(t,\vartheta)| \le CR^{-3}, \tag{2.12}$$

$$|(\varphi_R)_{tt}(t,\vartheta)| \le CR^{-4}. \tag{2.13}$$

Using (2.11) and (2.12), we obtain

$$A_p(\varphi_R) \le C R^{Q+1-\frac{2p}{p-1}}, \tag{2.14}$$

$$B_p(\varphi_R) \le C R^{Q+1-\frac{4p}{p-1}}, \tag{2.15}$$

$$C_p(\varphi_R) \le C R^{Q+1-\frac{2p}{p-1}}. \tag{2.16}$$

Let us consider now the change of variables

$$(t,x,y,\tau) = (t,\vartheta) \mapsto (\widetilde{t},\widetilde{v}) = (\widetilde{t},\widetilde{x},\widetilde{y},\widetilde{\tau}), \tag{2.17}$$

where

$$\widetilde{t} = R^{-1}t, \quad \widetilde{\tau} = R^{-2}\tau, \widetilde{x} = R^{-1}x, \quad \widetilde{y} = R^{-1}y.$$

Let

$$\widetilde{\rho} = \widetilde{t}^4 + |\widetilde{x}|^4 + |\widetilde{y}|^4 + \widetilde{\tau}^2,$$
$$\widetilde{\mathcal{C}_R} = \{(\widetilde{t},\widetilde{x},\widetilde{y},\widetilde{\tau}) \in \mathcal{H} : 1 \le \widetilde{\rho} \le 2\},$$
$$\mathcal{C}_R = \{(x,y,\tau) \in \mathbb{H} : R^4 \le |x|^4 + |y|^4 + \tau^2 \le 2R^4\}.$$

Using (2.6), (2.15) and (2.16), we obtain

$$\begin{aligned} &\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta \\ &\le C\Big(R^{\vartheta_1} + R^{\vartheta_2} + \int_{\mathcal{C}_R} |u_1(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(\vartheta,0)|\, d\vartheta\Big), \end{aligned} \tag{2.18}$$

where

$$\vartheta_1 = Q+1-\frac{2p}{p-1} \quad \text{and} \quad \vartheta_2 = Q+1-\frac{4p}{p-1}.$$

On the other hand, we have

$$\begin{aligned} &\liminf_{R\to\infty} \int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta \\ &\ge \liminf_{R\to\infty} \int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \liminf_{R\to\infty} \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta. \end{aligned}$$

Using the monotone convergence theorem, we obtain

$$\liminf_{R\to\infty} \int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt = \int_{\mathcal{H}} |u|^p \, d\vartheta \, dt.$$

Since $u_1 \in L^1(\mathbb{H})$, by the dominated convergence theorem, we have

$$\liminf_{R\to\infty} \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta = \int_{\mathbb{H}} u_1(\vartheta)\, d\vartheta.$$

Now, we have

$$\liminf_{R\to\infty} \Big(\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta\Big) \ge \int_{\mathcal{H}} |u|^p \, d\vartheta \, dt + \ell,$$

where from (2.1),

$$\ell = \int_{\mathbb{H}} u_1(\vartheta)\, d\vartheta > 0.$$

By the definition of the limit inferior, for every $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta$$

$$
\begin{aligned}
&> \liminf_{R\to\infty} \Big(\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta \Big) - \varepsilon \\
&\geq \int_{\mathcal{H}} |u|^p \, d\vartheta \, dt + \ell - \varepsilon,
\end{aligned}
$$

for every $R \geq R_0$. Taking $\varepsilon = \ell/2$, we obtain

$$
\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi_R(\vartheta,0)\, d\vartheta \geq \int_{\mathcal{H}} |u|^p \, d\vartheta \, dt + \frac{\ell}{2},
$$

for every $R \geq R_0$. Then from (2.18), we have

$$
\int_{\mathcal{H}} |u|^p \, d\vartheta \, dt + \frac{\ell}{2} \leq C\Big(R^{\vartheta_1} + R^{\vartheta_2} + \int_{\mathcal{C}_R} |u_0(\vartheta)| |\Delta_{\mathbb{H}} \varphi_R(\vartheta,0)| \, d\vartheta \Big), \tag{2.19}
$$

for R large enough.

Now, we require that $\vartheta_1 = \max\{\vartheta_1, \vartheta_2\} \leq 0$, which is equivalent to $1 < p \leq 1 + \frac{2}{Q-1}$. We distinguish two cases.

Case 1: $1 < p < 1 + \frac{2}{Q-1}$. In this case, letting $R \to \infty$ in (2.19) and using the dominated convergence theorem, we obtain

$$
\int_{\mathcal{H}} |u|^p \, d\vartheta \, dt + \frac{\ell}{2} \leq 0,
$$

which is a contradiction as $\ell > 0$.

Case 2: $p = 1 + \frac{2}{Q-1}$. From (2.19), we obtain

$$
\int_{\mathcal{H}} |u|^p \, d\vartheta \, dt \leq C < \infty \quad \Rightarrow \quad \lim_{R\to\infty} \int_{\mathcal{C}_R} |u|^p \varphi_R \, d\vartheta \, dt = 0. \tag{2.20}
$$

Using the Hölder inequality with parameters p and $p/(p-1)$, from (2.2), we obtain

$$
\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt + \frac{\ell}{2} \leq C\Big(\int_{\Theta_R} |u|^p \varphi_R \, d\vartheta \, dt \Big)^{1/p}.
$$

Letting $R \to \infty$ in the above inequality and using (2.20), we obtain

$$
\int_{\mathcal{H}} |u|^p \, d\vartheta \, dt + \frac{\ell}{2} = 0.
$$

A contradiction; the proof of the theorem is complete. □

2.1.1. *The case of system* (1.3). The definition of solutions we adopt for (1.3) is:

We say that the pair (u, v) is a local weak solution to (1.3) on $\mathcal{H}$ with initial data $(u(0,\cdot), v(0,\cdot)) = (u_0, v_0) \in L^1_{\rm loc}(\mathbb{H}) \times L^1_{\rm loc}(\mathbb{H})$, if $(u, v) \in L^p_{\rm loc}(\mathcal{H}) \times L^q_{\rm loc}(\mathcal{H})$ and satisfies

$$
\begin{aligned}
&\int_{\mathcal{H}} |v|^q \varphi \, d\vartheta \, dt + \int_{\mathbb{H}} u_1(\vartheta)\varphi(\vartheta,0)\, d\vartheta \\
&= \int_{\mathcal{H}} u\varphi_{tt} \, d\vartheta \, dt + \int_{\mathcal{H}} u(\Delta_{\mathbb{H}}\varphi)_{tt} \, dt \, d\vartheta - \int_{\mathcal{H}} u\Delta_{\mathbb{H}}\varphi \, dt \, d\vartheta + \int_{\mathbb{H}} u_1(\vartheta)\Delta_{\mathbb{H}}\varphi(\vartheta,0)\, d\vartheta
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathcal{H}} |u|^p \varphi \, d\vartheta \, dt + \int_{\mathbb{H}} v_1(\vartheta)\varphi(\vartheta,0)\, d\vartheta \\
&= \int_{\mathcal{H}} v\varphi_{tt} \, d\vartheta \, dt + \int_{\mathcal{H}} v(\Delta_{\mathbb{H}}\varphi)_{tt} \, dt \, d\vartheta - \int_{\mathcal{H}} v\Delta_{\mathbb{H}}\varphi \, dt \, d\vartheta + \int_{\mathbb{H}} v_1(\vartheta)\Delta_{\mathbb{H}}\varphi(\vartheta,0)\, d\vartheta,
\end{aligned}
$$

for any test function φ, $\varphi(\cdot,t)=0$, $\varphi_t(\cdot,t)=0$, $t\geq T$. The solution is said global if it exists for $T=+\infty$.

Our second main result is given by the following theorem.

Theorem 2.2. *Let* $(u_1,v_1)\in L^1(\mathbb{H})\times L^1(\mathbb{H})$. *Suppose that*
$$\int_{\mathbb{H}} u_1\,d\vartheta>0 \quad \text{and} \quad \int_{\mathbb{H}} v_1\,d\vartheta>0.$$
If $1<pq\leq (pq)^*$, *where*
$$(pq)^*=1+\frac{2}{Q-1}\max\{p+1,q+1\},$$
then there exists no nontrivial weak solution to (1.3).

Proof. Suppose that (u,v) is a nontrivial weak solution to (1.3). Then for any regular test function φ, we have
$$\begin{aligned}
&\int_{\mathcal{H}}|v|^q\varphi\,d\vartheta\,dt+\int_{\mathbb{H}}u_1(\vartheta)\varphi(\vartheta,0)\,d\vartheta\\
&\leq \int_{\mathcal{H}}|u||\varphi_{tt}|\,d\vartheta\,dt+\int_{\mathcal{H}}|u||(\Delta_{\mathbb{H}}\varphi)_{tt}|\,dt\,d\vartheta\\
&\quad+\int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}\varphi|\,dt\,d\vartheta+\int_{\mathbb{H}}|u_1(\vartheta)||\Delta_{\mathbb{H}}\varphi(\vartheta,0)|\,d\vartheta
\end{aligned}$$
and
$$\begin{aligned}
&\int_{\mathcal{H}}|u|^p\varphi\,d\vartheta\,dt+\int_{\mathbb{H}}v_1(\vartheta)\varphi(\vartheta,0)\,d\vartheta\\
&\leq \int_{\mathcal{H}}|v||\varphi_{tt}|\,d\vartheta\,dt+\int_{\mathcal{H}}|v||(\Delta_{\mathbb{H}}\varphi)_{tt}|\,dt\,d\vartheta\\
&\quad+\int_{\mathcal{H}}|v||\Delta_{\mathbb{H}}\varphi|\,dt\,d\vartheta+\int_{\mathbb{H}}|v_1(\vartheta)||\Delta_{\mathbb{H}}\varphi(\vartheta,0)|\,d\vartheta.
\end{aligned}$$

Taking $\varphi=\varphi_R$, the test function given by (2.10), and using the Hölder inequality with parameters p and $p/(p-1)$, we obtain
$$\begin{aligned}
&\int_{\mathcal{H}}|v|^q\varphi_R\,d\vartheta\,dt+\int_{\mathbb{H}}u_1(\vartheta)\varphi_R(\vartheta,0)\,d\vartheta-\int_{\mathbb{H}}|u_1(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(\vartheta,0)|\,d\vartheta\\
&\leq \Big(A_p(\varphi_R)^{\frac{p-1}{p}}+B_p(\varphi_R)^{\frac{p-1}{p}}+C_p(\varphi_R)^{\frac{p-1}{p}}\Big)\Big(\int_{\mathcal{H}}|u|^p\varphi_R\,d\vartheta\,dt\Big)^{1/p},
\end{aligned}$$
where $A_p(\varphi)$, $B_p(\varphi)$ and $C_p(\varphi)$ are given respectively by (2.7), (2.8) and (2.9). Similarly, by the Hölder inequality with parameters q and $q/(q-1)$, we get
$$\begin{aligned}
&\int_{\mathcal{H}}|u|^p\varphi_R\,d\vartheta\,dt+\int_{\mathbb{H}}v_1(\vartheta)\varphi_R(\vartheta,0)\,d\vartheta-\int_{\mathbb{H}}|v_1(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(\vartheta,0)|\,d\vartheta\\
&\leq \Big(A_q(\varphi_R)^{\frac{q-1}{q}}+B_q(\varphi_R)^{\frac{q-1}{q}}+C_q(\varphi_R)^{\frac{q-1}{q}}\Big)\Big(\int_{\mathcal{H}}|v|^q\varphi_R\,d\vartheta\,dt\Big)^{1/q}.
\end{aligned}$$
Without restriction of the generality, we may assume that for R large enough, we have
$$\begin{gathered}
\int_{\mathbb{H}}u_1(\vartheta)\varphi_R(\vartheta,0)\,d\vartheta-\int_{\mathbb{H}}|u_1(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(\vartheta,0)|\,d\vartheta\geq 0,\\
\int_{\mathbb{H}}v_1(\vartheta)\varphi_R(\vartheta,0)\,d\vartheta-\int_{\mathbb{H}}|v_1(\vartheta)||\Delta_{\mathbb{H}}\varphi_R(\vartheta,0)|\,d\vartheta\geq 0.
\end{gathered}$$

Slight modifications yield the proof in the general case (see the proof of Theorem 2.1). Then for R large enough, we have

$$\begin{aligned}&\int_{\mathcal{H}} |v|^q \varphi_R \, d\vartheta \, dt \\ &\le \Big(A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}} + C_p(\varphi_R)^{\frac{p-1}{p}} \Big) \Big(\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt \Big)^{1/p}\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt \\ &\le \Big(A_q(\varphi_R)^{\frac{q-1}{q}} + B_q(\varphi_R)^{\frac{q-1}{q}} + C_q(\varphi_R)^{\frac{q-1}{q}} \Big) \Big(\int_{\mathcal{H}} |v|^q \varphi_R \, d\vartheta \, dt \Big)^{1/q}.\end{aligned} \tag{2.22}$$

Using the change of variables (2.17), from (2.21) and (2.22), we obtain

$$\int_{\mathcal{H}} |v|^q \varphi_R \, d\vartheta \, dt \le C R^{\frac{Q(p-1)-2}{p}} \Big(\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt \Big)^{1/p}, \tag{2.23}$$

$$\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt \le C R^{\frac{Q(q-1)-2}{q}} \Big(\int_{\mathcal{H}} |v|^q \varphi_R \, d\vartheta \, dt \Big)^{1/q}. \tag{2.24}$$

Combining (2.23) with (2.24), we obtain

$$\Big(\int_{\mathcal{H}} |u|^p \varphi_R \, d\vartheta \, dt \Big)^{1-\frac{1}{pq}} \le C R^{\upsilon_1}, \tag{2.25}$$

$$\Big(\int_{\mathcal{H}} |v|^q \varphi_R \, d\vartheta \, dt \Big)^{1-\frac{1}{pq}} \le C R^{\upsilon_2}, \tag{2.26}$$

where

$$\upsilon_1 = \frac{Q(pq-1) - 2(p+1)}{pq-1} \quad \text{and} \quad \upsilon_2 = \frac{Q(pq-1) - 2(q+1)}{pq-1}.$$

We require that $\upsilon_1 \le 0$ or $\upsilon_2 \le 0$ which is equivalent to $1 < pq \le 1 + \frac{2}{Q} \max\{p+1, q+1\}$. We distinguish two cases.

Case 1: $1 < pq < 1 + \frac{2}{Q} \max\{p+1, q+1\}$. Without loss of the generality, we may suppose that $0 < q \le p$. In this case, letting $R \to \infty$ in (2.25), we obtain

$$\int_{\mathcal{H}} |u|^p \, d\vartheta \, dt = 0,$$

which is a contradiction.

Case 2: $pq = 1 + \frac{2}{Q} \max\{p+1, q+1\}$. This case can be treated in the same way as in the proof of Theorem 2.1. □

Remark 2.3. If $p = q$ and $u = v$ in Theorem 2.2, we obtain the result for a single equation given by Theorem 2.1.

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