Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 113, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NONCONTINUOUS SOLUTIONS TO DEGENERATE PARABOLIC INEQUALITIES

KRZYSZTOF A. TOPOLSKI

Abstract. We consider the initial value problem for degenerate parabolic equations. We prove theorems on differential inequalities and comparison theorems in unbounded domain. As a solution of differential inequality we consider upper absolutely (lower absolutely) continuous in t function (we admit discontinuity in time variable). In the last section we compare our notion of subsolutions to the notion of viscosity subsolutions smooth in space variable. By giving a counterexample we show that upper absolutcontinuity plays crucial role in the equivalence of the two notions.

1. Introduction

The aim of this paper is to investigate generalized inequalities and comparison problems for degenerate parabolic equations with nonlinear comparison function. As it is well known they found numerous applications in differential problems. The basic examples are estimates of solutions, estimates of the domain of existence of solutions, uniqueness and error estimates for approximate solutions. Upper and lower functions are important in existence results. In the paper we admit noncontinuous in time solutions for differential inequalities (semiabsolutely continuous in time variable) and noncontinuous comparison functions.

In our opinion this paper is probably the first where semiabsolutely in time variable solutions of partial differential inequalities are considered. In the case of ordinary differential inequalities non-continuous solutions where considered in [14] where some class of piecewise continuous solutions (lower and upper functions) were defined and in [16] where some special functions of bounded variation were investigated. Both cases are covered by our definition.

First order partial differential inequalities were first treated in [7, 11]. Second order inequalities of parabolic type were first treated in [12, 13, 20] where the first step involves a strict inequality, and the result for weak inequality is then obtained by introduction of a suitable perturbation. For the first time the classical theory of parabolic differential inequalities was widely described in [18, 19]. Comparison theorems for viscosity solutions of first order were first investigated in [5]. Uniqueness results were only obtainable at the time. The best general reference for second order viscosity solutions is [4].

2010 *Mathematics Subject Classification.* 35D30, 35K51, 35R45.
Key words and phrases. Parabolic equations; Cauchy problem; generalized solution.

Submitted October 2, 2014. Published April 28, 2015.

Our paper is divided into three main parts. In Section 2 we present a definition and properties of semiabsolutely continuous functions of one variable. In Section 3, using this definition we consider noncontinuous solutions of parabolic inequalities in unbounded domain. We prove comparison theorems for generalized subsolutions (supersolutions) and solutions. In the last section we compare our notions of subsolutions to the notions of viscosity subsolutions smooth in space variable. We give sufficient conditions under which these two types of solutions are equivalent. We show that these conditions are optimal and that upper absolutcontinuity plays crucial role in the equivalence.

2. Semiabsolutely continuous functions

The genesis of the notion of semiabsolutely continuous functions dates to [17] (see also [10] and [15] where the term absolute upper (lower) semicontinuous was used). In this paper we base on the definition which is the most convenient in our investigation. We follow some of the notation proposed in [10]. In our notation the word semiabsolutly continiuous means that a function is upper absolutely continuous or lower absolutely continuous.

Definition 2.1. Let $a, b \in \mathbb{R}$, $a < b$, $z : [a, b] \to \mathbb{R}$. We write $z \in UAC([a, b], \mathbb{R})$ (upper absolutely continuous) if $z'(t)$ exists a.e. in $[a, b]$ is integrable and for all $s, t \in [a, b]$,

$$s \le t \Rightarrow z(t) - z(s) \le \int_s^t z'(\tau) d\tau. \tag{2.1}$$

Similarly, we write $z \in LAC([a, b], \mathbb{R})$ (lower absolutely continuous) if $z'(t)$ exists a.e. in $[a, b]$ is integrable and for all $s, t \in [a, b]$,

$$s \le t \Rightarrow z(t) - z(s) \ge \int_s^t z'(\tau) d\tau. \tag{2.2}$$

It is clear that $AC([a, b], \mathbb{R}) = UAC([a, b], \mathbb{R}) \cap LAC([a, b], \mathbb{R})$, where $AC([a, b], \mathbb{R})$ is the set of all absolutely continuous scalar function in $[a, b]$. Moreover, $z \in UAC([a, b], \mathbb{R})$ if and only if $-z \in LAC([a, b], \mathbb{R})$.

Remark 2.2. If $z \in UAC([a, b], \mathbb{R})$ ($z \in LAC([a, b], \mathbb{R})$), then z is left-hand side lower (upper) semicontinuous and right-hand side upper (lower) semicontinuous.

Remark 2.3. Notice that if $z \in UAC([a, b], \mathbb{R})$ then $z(t) - \int_a^t z'(\tau) d\tau$ is nonincreasing. From the property of monotone functions and by the continuity and a.e. differentiability of $\int_a^t z'(\tau) d\tau$ we see that, upper absolutely continuous function has at most countably many points of discontinuity and one-sided limits in every point of $[a, b]$.

Proposition 2.4. *$z \in UAC([a, b], \mathbb{R})$ if and only if there exists an integrable function $l : [a, b] \to \mathbb{R}$ such that for all $s, t \in [a, b]$,*

$$s \le t \Rightarrow z(t) - z(s) \le \int_s^t l(\tau) d\tau. \tag{2.3}$$

Proof. The proof (only "⇐" is not obvious) follows from [15, Theorem 1]. □

Corollary 2.5. *A function $z : [a, b] \to \mathbb{R}$ is non-increasing if and only if $z \in UAC([a, b], \mathbb{R})$ and $z' \le 0$ a.e. in $[a, b]$.*

Proposition 2.6. *Suppose that $z \in UAC([a,b],\mathbb{R})$ has a left-hand side local maximum in $\hat{t} \in (a,b]$ and A is a full measure subset of $[a,b]$. Then there exists a sequence $t_m \to \hat{t}^-$ such that $t_m \in A$ for every m, and $z'(t_m) \geq 0$.*

Proof. There exists $\delta_1 > 0$ such that $z(t) \leq z(\hat{t})$ in $[\hat{t}-\delta_1, \hat{t}]$. The conclusion follows from the fact that the following sentence is false: there exists $0 < \delta < \delta_1$ such that the set $\{t \in [\hat{t}-\delta, \hat{t}] : z'(t) < 0\}$ has Lebesgue measure δ. Of course, it follows from Corollary 2.5 that z is non-increasing in $[\hat{t}-\delta, \hat{t}]$ for such δ. This implies that z is constant and $z'(t) = 0$ in $[\hat{t}-\delta, \hat{t}]$, a contradiction. □

We write $c^+ = \max\{c, 0\}$ and $c^- = \max\{-c, 0\}$ for $c \in \mathbb{R}$. For a given scalar function z we define functions z^+, z^- in an obvious way.

Proposition 2.7. *If $z \in UAC([a,b],\mathbb{R})$, then $z^+ \in UAC([a,b],\mathbb{R})$ and $(z^+)' = (\operatorname{sgn} z^+)z'$ a.e.*

Proof. Since z satisfies (2.1), for $t \geq s$, $z^+(t) - z^+(s) \leq (z(t)-z(s))^+ \leq \int_s^t l^+(\tau)d\tau$ and $z^+ \in UAC([a,b])$. Consider the set $B \subseteq (a,b)$ of all t such that $z'(t)$ and $(z^+)'(t)$ exist. Of course, B has Lebesgue measure $b-a$. It is not difficult to show that for $t \in B$ we have $(z^+)'(t) = z'(t)$ if $z(t) > 0$, and $(z^+)'(t) = 0$ if $z(t) < 0$. Moreover, the set $\{t \in B : z(t) = 0, z'(t) \neq 0\}$ contains only isolated points, hence it has at most countable many elements. On the other hand, if $t \in B$ is such that $z(t) = 0$, $z'(t) = 0$, then

$$(z^+)'(t) = \lim_{h\to 0} \frac{z^+(t+h)}{h} = \lim_{h\to 0} \frac{(\operatorname{sgn} z^+(t+h))z(t+h)}{h} = 0.$$

□

Similarly, if $z \in LAC([a,b])$, then $z^- \in UAC([a,b])$ and $(z^-)' = -(\mathrm{sgnz}^-)z'$ a.e.

3. Comparison theorems

Define $E_T = [0,T] \times \mathbb{R}^n$, $T > 0$, $E_0 = \{0\} \times \mathbb{R}^n$, $\Theta_T = E_T \setminus E_0$. Let $S[n]$ be the set of all symmetric $n\times n$ real matrices. For X, $Y \in S[n]$, $X \leq Y$ means that $Y - X$ is a positive semidefinite matrix. For $X \in S[n]$, $|X|$ is any matrix norm of X and for $p \in \mathbb{R}^n$, $|p|$ is any vector norm of p. Suppose that $g : \Theta_T \times \mathbb{R} \times \mathbb{R}^n \times S[n] \to \mathbb{R}$ is monotone in matrix variable i.e. if $X \leq Y$, then $g(t,x,z,p,X) \leq g(t,x,z,p,Y)$ and $\psi : E_0 \to \mathbb{R}$. Consider problem

$$D_t v = g(t,x,v,Dv,D^2v) \quad \text{in } \Theta_T, \tag{3.1}$$

$$v = \psi \quad \text{in } E_0 \tag{3.2}$$

(we write $Dv = D_x v$, $D^2 v = D_x^2 v$ and $D_t v = \frac{\partial}{\partial t} v$).

Notice that our formulation includes as a particular case the first order equation (g does not depend on a matrix argument).

Definition 3.1. We say that $v : E_T \to \mathbb{R}$ is a subsolution (supersolution, solution) of (3.1) if

(i) for every $x \in \mathbb{R}^n$, $c \in (0,T)$, $v(\cdot,x) \in UAC([c,T],\mathbb{R})$ ($LAC([c,T],\mathbb{R})$, $AC([c,T],\mathbb{R})$),
(ii) there exist Dv, D^2v in Θ_T,
(iii) for every $x \in \mathbb{R}^n$, $Dv(\cdot,x), D^2v(\cdot,x)$ are left-hand side continuous in $(0,T]$,

(iv) for every $x \in \mathbb{R}^n$ v satisfies

$$D_t v \le g(t,x,v,Dv,D^2v) \quad \text{a.e in } t \in (0,T] \quad (\text{“} \ge \text{”}, \text{ “} = \text{”}). \tag{3.3}$$

(In case of first order equations (g does not depend on X) we assume that (ii) and (iii) are satisfied only for Dv.)

We write $v \in \mathrm{Sub}(g,\psi)$ $(\mathrm{Sup}(g,\psi),\ \mathrm{Sol}(g,\psi))$ if v is a subsolution (supersolution, solution) of (3.1), and in addition $v \le \psi$ (“ $\ge$ ”, “ $=$ ”) in E_0. Of course, $\mathrm{Sol}(g,\psi) = \mathrm{Sub}(g,\psi) \cap \mathrm{Sup}(g,\psi)$.

We say that v satisfies (3.3) ((3.3) with a reversed inequality) in a generalized sense if it is a subsolution (supersolution) of (3.1).

Condition (iii) in Definition 3.1 has a technical meaning. We need it in the proof of the maximum principle (see Theorem 3.1). It can be relaxed (Remark 3.4). What is interesting is that it will be necessary also in the last section where viscosity solutions are considered (see Theorem 4.4). We give a simple example of the problem such that (iii) in Definition 3.1 is not satisfied. Consider $D_t u + Du = 0$, $u(0,x) = \psi(x)$ in $[0,2] \times \mathbb{R}$ where ψ' exists is bounded but ψ' is not right hand continuous at $x = 0$. It is easy to verify that $u(t,x) = \psi(x-t)$ satisfies all condition of Definition 3.1 except for (iii) at point $(1,1)$.

The notion of subsolutions (supersolutions, solutions) given in Definition 3.1 extends the definition of classical subsolutions (supersolutions, solutions) (Remark 3.4). Moreover, in the case of first order equations it covers the definition of $CC-$ solutions considered by Cinquni-Cibrario (see [2,3] for existence results). In [1] close but not the same extension of $CC-$ solutions is considered and some comparison results are proved under the assumption that solutions exist.

The reason why we introduce Definition 3.1 is the theorem on differential inequalities (see Proposition 3.13 and Corollary 3.15). In the definition we require as little as is needed in the proof. Our notion of subsolutions (supersolutions, solutions) is placed between classical definition and more generalized definitions a.e. “almost everywhere” in the case of first order equations and viscosity subsolutions (supersolutions, solutions) in the case of first and second order equations. It is, however, still close to the classical meaning and we need relatively simple assumptions on g to obtain the theorem on differential inequalities and comparison results. We cannot say the same when we consider “almost everywhere” solutions (first order equations) where convexity in p is required and uniqueness is proved under additional “entropy condition” (see [9]). In the case of viscosity solutions even more complicated assertions are needed to obtain comparison theorems (see [4,5]).

Let $C(E_T,\mathbb{R})$, $LSC(E_T,\mathbb{R})$, $USC(E_T,\mathbb{R})$, be sets of scalar functions which are resp. continuous, lower semicontinuous and upper semicontinuous in E_T. Let $C_b(E_T,\mathbb{R})$, $LSC_b(E_T,\mathbb{R})$, $USC_b(E_T,\mathbb{R})$ be sets of such functions which are in addition resp. bounded, bounded from below, and bounded from above.

We write $\omega \in \mathcal{M}$ if $\omega : [0,\infty) \to \mathbb{R} \cup \{\infty\}$ and $\lim_{r \to 0^+} \omega(r) = \omega(0) = 0$.

Theorem 3.2. *Suppose that*

(i) $v \in USC_b(E_T,\mathbb{R})$ *is a subsolution of* (3.1),

(i) *for every* $R > 0$ *there exists* $\omega_R \in \mathcal{M}$ *such that* $g(t,x,z,p,X) \le \omega_R(|p| + |X|)$ *for* $z \in [0,R]$.

Then

$$\sup_{E_T} v^+ = \sup_{E_0} v^+. \tag{3.4}$$

Proof. It is sufficient to prove that $\sup_{E_T} v \le \sup_{E_0} v^+$. Define $w(t,x) = v(t,x) - \eta t$ for $\eta > 0$. We will show

$$\sup_{E_T} w \le \sup_{E_0} w^+ = \sup_{E_0} v^+. \tag{3.5}$$

The proof will be completed by letting $\eta \to 0$.

Set $M = \sup_{E_T} w$. We only need to consider the case $M > 0$ for some η. For every $M/2 > \delta > 0$ there exists $(\bar t_\delta, \bar x_\delta) \in E_T$ such that

$$M \ge w(\bar t_\delta, \bar x_\delta) > M - \delta > \frac{M}{2}.$$

Define

$$\Phi_\delta(t,x) = w(t,x) + 2\delta\xi_\delta(x),$$

where $\xi_\delta \in C_0^2(\mathbb{R}^n),\ \xi_\delta(\bar x_\delta) = 1,\ 0 \le \xi_\delta \le 1,\ |D\xi_\delta|,\ |D^2\xi_\delta| \le 2$.

Since $\Phi_\delta(t,x) = w(t,x) \le M$ when $x \notin \operatorname{supp}\xi_\delta$ and

$$\Phi_\delta(\bar t_\delta, \bar x_\delta) = w(\bar t_\delta, \bar x_\delta) + 2\delta\xi_\delta(\bar x_\delta) > M - \delta + 2\delta = M + \delta > M \tag{3.6}$$

Φ_δ attains its supremum in some $(t_\delta, x_\delta) \in E_T$, where $x_\delta \in \operatorname{supp}\xi_\delta$.

It follows from (3.6) that

$$w(t_\delta, x_\delta) + 2\delta \ge \Phi_\delta(t_\delta, x_\delta) \ge \Phi_\delta(\bar t_\delta, \bar x_\delta) > \frac{M}{2} + 2\delta \tag{3.7}$$

and consequently

$$v(t_\delta, x_\delta) \ge w(t_\delta, x_\delta) > \frac{M}{2}. \tag{3.8}$$

Define A_δ as the full measure set of all $t \in [0,T]$ such that $D_t v(t, x_\delta)$ exists and (3.3) is satisfied at point (t, x_δ). Let us fix a sequence $\delta_m \in (0, \frac{M}{2}),\ m \in \mathbb{N}$ approaching zero. We consider two cases (taking a subsequence if necessary).

(i) If $t_{\delta_m} = 0$ for $m \in \mathbb{N}$ then $\sup_{E_T} \Phi_{\delta_m} \le \sup_{E_0} \Phi_{\delta_m},\ m \in \mathbb{N}$, which implies that (3.5) holds true.

(ii) Suppose now that $t_{\delta_m} > 0,\ m \in \mathbb{N}$. Note that $D_t w = D_t \Phi_{\delta_m}$ (in the set of existence). It follows from the fact that $\Phi_{\delta_m}(\cdot, x_{\delta_m})$ has a local maximum in t_{δ_m} (left-hand side if $t_{\delta_m} = T$) that for every m there exists $t_{k,m} \in A_{\delta_m}$ such that $D_t w(t_{k,m}, x_{\delta_m}) \ge 0$ and $t_{k,m} \to t_{\delta_m}^-$ when $k \to \infty$ (see Proposition 2.6). Moreover, by the left-hand side continuity of v in t and by (3.8) we can assume that $v(t_{k,m}, x_{\delta_m}) > 0$ and $Dv(t_{k,m}, x_{\delta_m}),\ D^2 v(t_{k,m}, x_{\delta_m})$ exist.

Since (3.1) is satisfied in $(t_{k,m}, x_{\delta_m})$ and $D_t v(t_{k,m}, x_{\delta_m}) = D_t w(t_{k,m}, x_{\delta_m}) + \eta \ge \eta$ we obtain

$$\eta \le g(t_{k,m}, x_{\delta_m}, v(t_{k,m}, x_{\delta_m}), Dv(t_{k,m}, x_{\delta_m}), D^2 v(t_{k,m}, x_{\delta_m})) \tag{3.9}$$

where

$$Dv(t_{k,m}, x_{\delta_m}) = Dv(t_{k,m}, x_{\delta_m}) - Dv(t_{\delta_m}, x_{\delta_m}) - 2\delta_m D\xi_\delta(x_\delta) \tag{3.10}$$

$(D\Phi(t_{\delta_m}, x_{\delta_m}) = Dv(t_{\delta_m}, x_{\delta_m}) + 2\delta_m D\xi_{\delta_m}(x_{\delta_m}) = 0)$ and

$$D^2 v(t_{k,m}, x_{\delta_m}) \le D^2 v(t_{k,m}, x_{\delta_m}) - D^2 v(t_{\delta_m}, x_{\delta_m}) - 2\delta_m D^2\xi_{\delta_m}(x_{\delta_m}) = B_{k,m} \tag{3.11}$$

$(D^2\Phi(t_{\delta_m}, x_{\delta_m}) = D^2 w(t_{\delta_m}, x_{\delta_m}) + 2\delta D^2\xi_\delta(x_{\delta_m}) \le 0)$.

We will show that the above estimation leads to a contradiction. It follows from (ii) that for every $\varepsilon > 0$ and $R > 0$ there exists $\rho > 0$ such that $g(t,x,z,p,X) \le \varepsilon$ if $|p|,\ |X| < \rho$ and $z \in [0,R]$. Let $\rho > 0$ be such that $g(t,x,z,p,X) \le \eta/2$ for

$|p|, |X| \le \rho$, $0 \le z \le R = \sup_{E_T} |v^+|$. Fix m such that $\delta_m < \frac{\rho}{8}$. This gives $2\delta_m |D^2\xi_{\delta_m}(x_{\delta_m})| \le \frac{\rho}{2}$ and $2\delta_m |D\xi_{\delta_m}(x_{\delta_m})| \le \rho/2$.

On the other hand, (see Definition 3.1 (iii)) there exists k such that

$$|Dv(t_{k,m}, x_{\delta_m}) - Dv(t_{\delta_m}, x_{\delta_m})| < \frac{\rho}{2},$$
$$|D^2v(t_{k,m}, x_{\delta_m}) - D^2v(t_{\delta_m}, x_{\delta_m})| \le \frac{\rho}{2}.$$

Applying (3.10), (3.11) we have $|Dv(t_{k,m}, x_{\delta_m})|$, $|B_{k,m}| < \rho$. Finally, in view of (3.9) and (3.11) we obtain

$$\eta \le g(t_{k,m}, x_{\delta_m}, v(t_{k,m}, x_{\delta_m}), Dv(t_{k,m}, x_{\delta_m}), B_{k,m}) \le \frac{\eta}{2}$$

a contradiction. □

Remark 3.3. The statement of Theorem 3.1 holds if we assume that (ii)-(iv) in Definition 3.1 hold only in the set $v^{-1}((0,\infty)) \cap \Theta_T$. Note that if v satisfies (i) of Definition 3.1 then $v(t,x) > 0,\ t > 0$ implies that $v(s,x) > 0,\ s \in (t-\varepsilon, t]$ for some $\varepsilon > 0$ (see Remark 2.2).

Remark 3.4. The statement of Theorem 3.1 holds if in place of Definition 3.1 (iii) we assume that $Dv(\cdot, x), D^2v(\cdot, x)$ are left-hand side continuous in every point $(t,x) \in (0,T] \times \mathbb{R}^n$ such that $D_t v(t,x)$ does not exist.

Proposition 3.5. *Suppose that*

(i) $v \in USC_b(E_T, \mathbb{R})$ *is a subsolution of* (3.1),

(ii) *there exists integrable function* $h : [0,T] \to \mathbb{R}_+$ *and for every* $R > 0$ *there exists* $\omega_R \in \mathcal{M}$ *such that* $g(t,x,z,p,X) \le h(t) + \omega_R(|p| + |X|)$ *if* $z \in [0,R]$.

Then

$$\sup_{E_t} v^+ \le \sup_{E_0} v^+ + \int_0^t h(s)ds \quad \text{for } t \in [0,T].$$

Proof. Set $\bar{v}(t,x) = v(t,x) - \int_0^t h(s)ds$ and

$$\bar{g}(t,x,z,p,X) = g(t,x,z + \int_0^t h(s)ds, p, X) - h(t).$$

For $z \in [0,R]$ we have

$$\bar{g}(t,x,z,p,X) = g(t,x,z + \int_0^t h(s)ds, p, X) - h(t) \le \omega_{R+R_1}(|p| + |X|)$$

where $R_1 = \int_0^T h(s)ds$.

It follows easily that $\bar{v}$ is a subsolution of (3.1) with g repleaded by $\bar{g}$. Thus we can apply Theorem 3.2 to $\bar{v}$ and $\bar{g}$ in the set $E_t = \{(s,x) \in E_T : s \le t\}$. This gives:

$$\sup_{E_t} v^+ - \int_0^t h(s)ds \le \sup_{(\tau,x)\in E_t} \{(v(\tau,x) - \int_0^\tau h(s)ds)^+\} = \sup_{E_0} v^+$$

which completes the proof. □

Analogically to Remark 3.3 we can formulate

Remark 3.6. The statement of Proposition 3.5 holds true if we assume that (ii)-(iv) in Definition 3.1 hold only in the set $Z = \{(t,x) \in \Theta_T : v^+(t,x) > \int_0^t h(s)ds\}$.

Remark 3.7. It follows from Proposition 3.5 (ii) that $g(t,x,z,0,0) \le h(t)$ in $\Theta_T \times \mathbb{R}_+$. Moreover, if we assume this and the following: for every $R > 0$ there exists $\omega_R \in \mathcal{M}$ such that $g(t,x,z,p,X) - g(t,x,z,0,0) \le \omega_R(|p|+|X|)$ for $z \in [0,R]$, then (ii) is satisfied.

Definition 3.8. For $M \in \mathbb{R}_+$ and $\sigma : [0,T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ we write $\sigma \in O_M$, if

(i) $\sigma = \sigma(t,z)$ is nondecreasing in z,
(ii) $\sigma(t,z(t))$ is integrable for every nondecreasing $z : [0,T] \to \mathbb{R}$,
(iii) there exists $\mu(t) = \mu_\sigma(t,M) \in AC([0,T],\mathbb{R}_+)$ such that $\mu(t)$ is a solution of the problem

$$z'(t) = \sigma(t,z(t)), \quad \text{a.e. in } [0,T], \quad z(0) = M \tag{3.12}$$

and

$$\mu(t) = \max\{z \in AC([0,T],\mathbb{R}) : z'(t) \le \sigma(t,z(t)) \text{ a.e in } [0,T],\ z(0) = M\}.$$

A typical example of $\sigma \in O_M$ (for every $M \ge 0$) is $\sigma(t,z) = l(t)z + m(t)$ where $l : [0,T] \to \mathbb{R}_+$ integrable. It is not difficult to show that we can also take σ which is nondecreasing in both variables and sublinear or if it satisfies the well known Carathéodory conditions (see [8] for even more general conditions). Note that σ is not supposed to be continuous.

For $v : E_T \to \mathbb{R}$ bounded and $0 \le t \le T$ we define $\|v\|_t = \sup_{E_t} |v|$.

Proposition 3.9. *Suppose that*

(i) $v \in \mathrm{Sub}(g,\psi) \cap USC_b(E_T,\mathbb{R})$, $M^+ = \sup_{E_0} \psi^+$,
(ii) *there exists* $\sigma \in O_{M^+}$ *and for every* $R > 0$ *there exists* $\omega_R \in \mathcal{M}$ *such that* $g(t,x,z,p,X) \le \sigma(t,z) + \omega_R(|p| + |X|)$ *if* $z \in [0,R]$.

Then

$$\|v^+\|_t \le \mu_\sigma(t,M^+) \quad \textit{for } t \in [0,T].$$

Proof. Define $\bar g(t,x,z,p,X) = g(t,x,v^+(t,x),p,X)$. Notice that $\bar g$ satisfies assumption (ii) of Proposition 3.5 with $h(t) = \sigma(t,\|v^+\|_t) \ge 0$ independently of R ($\bar g$ does not depend on z). Indeed,

$$\begin{aligned}
\bar g(t,x,z,p,X) = g(t,x,v^+(t,x),p,X) &\le \sigma(t,v^+(t,x)) + \omega_{\tilde R}(|p| + |X|) \\
&\le \sigma(t,\|v^+\|_t) + \omega_{\tilde R}(|p| + |X|) = h(t) + \omega_{\tilde R}(|p| + |X|)
\end{aligned}$$

where $\tilde R = \|v^+\|_T$. On the other hand v^+ satisfies

$$D_t v \le \bar g(t,x,v,D_x v,D_x^2 v)$$

in the set $Z = \{(t,x) \in \Theta_T : v^+(t,x) > \int_0^t h(s)ds\}$. Indeed since $v^+ = v$, $D_t v^+ = D_t v$, $Dv^+ = Dv$, $D^2 v^+ = D^2 v$ in Z, we obtain

$$\begin{aligned}
D_t v^+ = D_t v &\le g(t,x,v,Dv,D^2 v) = g(t,x,v^+,Dv^+,D^2 v^+) \\
&= \bar g(t,x,v^+,Dv^+,D^2 v^+)
\end{aligned}$$

in Z a.e in t.

This gives (see Proposition 3.5 and Remark 3.6)

$$\|v^+\|_t \le \|v^+\|_0 + \int_0^t \sigma(\tau,\|v^+\|_\tau)d\tau.$$

Similarly, for $t \geq s$ we have

$$\|v^+\|_t \leq \|v^+\|_s + \int_s^t \sigma(\tau, \|v^+\|_\tau) d\tau.$$

This implies that $\alpha(t) = \|v^+\|_t$ is in $UAC([0,T])$ and $\alpha'(t) \leq \sigma(\tau, \alpha(t))$ a.e in $[0,T]$. Since α is nondecreasing, it belongs to $AC([0,T])$. This completes the proof in view of Definition 3.8 (iii). □

Proposition 3.10. *Suppose that*

(i) $v \in \mathrm{Sup}(g, \psi) \cap LSC_b(E_T, \mathbb{R})$, $M^- = \sup_{E_0} \psi^-$,
(ii) *there exists* $\sigma \in O_{M^-}$ *and for every* $R > 0$ *there exists* $\omega_R \in \mathcal{M}$ *such that* $g(t,x,z,p,X) \geq -\sigma(t,-z) - \omega_R(|p| + |X|)$ *if* $z \in [-R, 0]$.

Then

$$\|v^-\|_t \leq \mu_\sigma(t, M^-) \quad \textit{for } t \in [0,T].$$

Proof. Notice that $v^- = (-v)^+$, $\psi^- = (-\psi)^+$, $-v \in \mathrm{Sub}(\tilde{g}, -\psi)$ where

$$\tilde{g}(t,x,z,p,X) = -g(t,x,-z,-p,-X)$$

and $\tilde{g}$ satisfies all assumptions of Proposition 3.9. □

Corollary 3.11. *Let assumptions of Propositions 3.10 hold, and* $\mu_\sigma(\cdot, 0) \equiv 0$. *Then* $\psi \geq 0$ *implies* $v \geq 0$.

Proposition 3.12. *Suppose that*

(i) $v \in \mathrm{Sol}(g, \psi) \cap C_b(E_T, \mathbb{R})$, $M = \sup_{E_0} |\psi|$,
(ii) *there exists* $\sigma \in O_M$ *and for every* $R > 0$ *there exists* $\omega_R \in \mathcal{M}$ *such that*

$$(\operatorname{sgn} z) g(t,x,z,p,X) \leq \sigma(t, |z|) + \omega_R(|p| + |X|) \quad \text{for } z \in [-R, R].$$

Then

$$\|v\|_t \leq \mu_\sigma(t, M) \quad \textit{for } t \in [0,T].$$

Proof. Since $|v| = \max\{v^+, v^-\}$, $M = \max\{M^+, M^-\}$ and $\mu(t, M^-), \mu(t, M^+) \leq \mu(t, M)$ the conclusion follows from Propositions 3.9 and 3.10. □

Proposition 3.13. *Suppose that*

(i) $v \in \mathrm{Sub}(g, \psi) \cap USC_b(E_T, \mathbb{R})$, $\bar{v} \in \mathrm{Sup}(\bar{g}, \bar{\psi})$, $v - \bar{v} \in USC_b(E_T, \mathbb{R})$ $M^+ = \sup_{E_0}(\psi - \bar{\psi})^+$,
(ii) *there exists* $\sigma \in O_{M^+}$ *and for every* $R > 0$ *there exists* $\omega_R \in \mathcal{M}$ *such that*

$$g(t,x,z,p,X) - \bar{g}(t,x,\bar{z},\bar{p},\bar{X}) \leq \sigma(t, z - \bar{z}) + \omega_R(|p - \bar{p}| + |X - \bar{X}|)$$

for $z - \bar{z} \in [0, R]$.

Then

$$\|(v - \bar{v})^+\|_t \leq \mu_\sigma(t, M^+) \quad \textit{for } t \in [0,T].$$

Proof. It is easy to check that $v - \bar{v} \in \mathrm{Sub}(G, \psi - \bar{\psi})$ where

$$G(t,x,z,p,X) = g(t,x,z+\bar{v}, p + D\bar{v}, X + D^2\bar{v}) - \bar{g}(t,x,\bar{v}, D\bar{v}, D^2\bar{v}). \tag{3.13}$$

The conclusion follows from Proposition 3.9. □

In a similar way Proposition 3.12 yields the following result.

Proposition 3.14. *Suppose that*

(i) $v \in \mathrm{Sol}(g, \psi)$, $\bar{v} \in \mathrm{Sol}(\bar{g}, \bar{\psi})$, $v - \bar{v} \in C_b(E_T, \mathbb{R})$ *and* $M = \sup_{E_0} |\psi - \bar{\psi}|$,

(ii) *there exists* $\sigma \in O_M$ *and for every* $R > 0$ *there exists* $\omega_R \in \mathcal{M}$ *such that*

$$\operatorname{sgn}(z-\bar z)[g(t,x,z,p,X)-\bar g(t,x,\bar z,\bar p,\bar X)] \le \sigma(t,|z-\bar z|)+\omega_R(|p-\bar p|+|X-\bar X|)$$

for $|z-\bar z| \le R$.

Then

$$\|v-\bar v\|_t \le \mu_\sigma(t,M) \quad \text{for } t \in [0,T].$$

Corollary 3.15. *If* $\mu_\sigma(\cdot,0) \equiv 0$, $g=\bar g$, $\psi=\bar\psi$, *then Proposition 3.13 implies the theorem on differential inequalities and Proposition 3.14 implies the theorem on the uniqueness for problem* (3.1) (3.2).

4. Viscosity solutions

Definition 4.1. We say that $u : E_T \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (3.1) if $u \in USC(E_T,\mathbb{R})$ (resp. $u \in LSC(E_T,\mathbb{R})$) and for each $\phi \in C^{1,2}(\Theta_T)$ if $u-\phi$ attains a local maximum (resp. local minimum) at $(\tilde t,\bar x) \in \Theta_T$, then

$$D_t\phi(\tilde t,\tilde x) \le g(\tilde t,\tilde x,u(\tilde t,\tilde x),D\phi(\tilde t,\tilde x),D^2\phi(\tilde t,\tilde x)) \quad (\text{resp. } \ge)$$

We say that u is a viscosity solution of (3.1) if it is both viscosity subsolution and supersolution.

It follows easily that if $u \in C^{1,2}(\Theta_T)$, then u is a viscosity subsolution (supersolution, solution) of (3.1) if and only if it is a classical subsolution (supersolution, solution). We will extend this result to subsolutions (supersolutions, solutions) given by Definition 3.1

First we present rather known result which generalize this find in [6] (for first order equations).

Lemma 4.2. *Suppose that* $u : E_T \to \mathbb{R}$ *and* D_tu, Du, D^2u *exist at* $(\bar t,\bar x) \in \Theta_T$ *and in some neighborhood of* $(\bar t,\bar x)$ *we have*

$$\begin{aligned} u(t,x) &= u(\bar t,\bar x)+D_tu(\bar t,\bar x)(t-\bar t)+\langle Du(\bar t,\bar x),(x-\bar x)\rangle \\ &\quad + \langle D^2u(\bar t,\bar x)(x-\bar x),x-\bar x\rangle + o(|t-\bar t|+|x-\bar x|^2). \end{aligned} \tag{4.1}$$

Then there exists $\phi \in C^{1,2}(\Theta_T)$ *such that* $u-\phi$ *has a maximum at* $(\bar t,\bar x)$ *and* $D_tu(\bar t,\bar x)=D_t\phi(\bar t,\bar x)$, $Du(\bar t,\bar x)=D\phi(\bar t,\bar x)$, $D^2u(\bar t,\bar x)=D^2\phi(\bar t,\bar x)$.

Proof. Without loss of generality we can assume that $u : \mathbb{R}^{1+n} \to \mathbb{R}$, D_tu, Du, D^2u exist at $(0,0)$ and $u(0,0)=D_tu(0,0)=Du(0,0)=D^2u(0,0)=0$.

Define $\alpha(t,x)=\frac{|x|^2+t^2}{\sqrt{|x|^4+t^2}}$ for $(t,x)\ne(0,0)$ and $\alpha(0,0)=0$. Set $\bar u=\alpha u$ $y=(t,x)$ and $\bar u=\bar u(y)$. Since $\frac12(|x|^2+|t|) \le \sqrt{|x|^4+t^2} \le |x|^2+|t|$ we easily see that $D_y\bar u$, $D_y^2\bar u$ exist at $0 \in \mathbb{R}^{1+n}$ and $\bar u(0)=D_y\bar u(0)=D_y^2\bar u(0)=0$. We define $\rho(y)=\frac{\bar u(y)}{|y|^2}, y\ne 0$ and $\rho(0)=0$. Set $\rho_1(r)=\max_{|y|\le r}|\rho(y)|$ and

$$\bar\phi(y)=\int_{|y|}^{2|y|}d\tau\int_\tau^{2\tau}\rho_1(r)dr.$$

Since $\bar\phi(y) \le 2|y|^2\rho_1(4|y|)$, we obtain $\bar\phi(0)=D\bar\phi(0)=D^2\bar\phi(0)=0$. Since $\bar\phi(y) \ge \rho_1(|y|)|y|^2 \ge \rho(|y|)|y|^2=\bar u(y)$, $\bar u-\bar\phi$ attains a maximum point at 0. It is easily seen that $\bar\phi \in C^1(\mathbb{R}^n)$. In order to have $\bar\phi \in C^2(\mathbb{R}^n)$ we repeat the procedure with $\bar u$ replaced by $\bar\phi$.

Define $\phi(t,x) = [\alpha(t,x)]^{-1}\bar{\phi}(t,x)$, $(t,x) \neq (0,0)$, $\phi(0,0) = 0$. It is not difficult to verify that ϕ is a desired function. $\square$

Proposition 4.3. *Suppose that u satisfies (i), (ii) in Definition 3.1 and for every $x \in \mathbb{R}^n$* (4.1) *is satisfied for a.e $t \in [0,T]$. If u is a viscosity subsolution (supersolution, solution) of* (3.1) *then u is a subsolution (supersolution, solution) of* (3.1).

Proof. Suppose that u satisfies (i), (ii) in Definition 3.1 and for every $x \in \mathbb{R}^n$ (4.1) is satisfied for a.e $t \in [0,T]$. Let $(\bar{t},\bar{x}) \in \Theta_T$ be such that $D_t u(\bar{t},\bar{x})$ exists and (4.1) is satisfied. Since $Du(\bar{t},\bar{x})$, $D^2u(\bar{t},\bar{x})$ exist by Lemma 4.2 we have $\phi \in C^{1,2}(\Theta_T)$ such that $u-\phi$ has a maximum at $(\bar{t},\bar{x})$ and $D_t u(\bar{t},\bar{x}) = D_t\phi(\bar{t},\bar{x})$, $Du(\bar{t},\bar{x}) = D\phi(\bar{t},\bar{x})$, $D^2u(\bar{t},\bar{x}) = D^2\phi(\bar{t},\bar{x})$. By Definition 4.1 this gives

$$D_t\phi(\bar{t},\bar{x}) \le g(\bar{t},\bar{x},u(\bar{t},\bar{x}),D\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x}))$$

and consequently u satisfies (3.3) in $(\bar{t},\bar{x})$. $\square$

Theorem 4.4. *Suppose that for $(\hat{t},\hat{x},\hat{u},\hat{p},\hat{X}) \in \Theta_T \times \mathbb{R} \times \mathbb{R}^n \times S[n]$*

$$\limsup_{(t,u,p,X)\to(\hat{t}^-,\hat{u},\hat{p},\hat{X})} g(t,\hat{x},u,p,X) \le g(\hat{t},\hat{x},\hat{u},\hat{p},\hat{X}). \tag{4.2}$$

Then if $u \in USC(E_T,\mathbb{R})$ is a subsolution of (3.1)*, then it is a viscosity subsolution of* (3.1).

Proof. Suppose that u is a subsolution of (3.1) and $\phi \in C^{1,2}(\Theta_T,\mathbb{R})$ such that $u-\phi$ has a local maximum point in $(\tilde{t},\tilde{x}) \in \Theta_T$. Let A be the full measure set of all t such that $D_t u(t,\tilde{x})$ exists and

$$D_t u(t,\tilde{x}) \le g(t,\tilde{x},u(t,\tilde{x}),Du(t,\tilde{x}),D^2u(t,\tilde{x})).$$

By Preposition 2.6 there exists a sequence $t_m \to \tilde{t}^-$ such that $t_m \in A$ for every m, $D_t(u-\phi)(t_m,\tilde{x}) \ge 0$ and

$$D_t\phi(t_m,\tilde{x}) \le D_t u(t_m,\tilde{x}) \le g(t_m,\tilde{x},u(t_m,\tilde{x}),Du(t_m,\tilde{x}),D^2u(t_m,\tilde{x})).$$

Notice that by Definition 3.1 (i) and Remark 2.2 $u(\cdot,\tilde{x})$ is left-hand side continuous. Using this, Definition 3.1 (iii) and (4.2) we obtain by letting $t_m \to \tilde{t}$

$$D_t\phi(\tilde{t},\tilde{x}) \le g(\tilde{t},\tilde{x},u(\tilde{t},\tilde{x}),Du(\tilde{t},\tilde{x}),D^2u(\tilde{t},\tilde{x})).$$

Since $Du(\tilde{t},\tilde{x}) = D\phi(\tilde{t},\tilde{x})$, $D^2u(\tilde{t},\tilde{x}) \le D^2\phi(\tilde{t},\tilde{x})$ we have

$$D_t\phi(\tilde{t},\tilde{x}) \le g(\tilde{t},\tilde{x},u(\tilde{t},\tilde{x}),D\phi(\tilde{t},\tilde{x}),D^2\phi(\tilde{t},\tilde{x})).$$

Hence, u is a viscosity subsolution of (3.1). $\square$

Theorem 4.5. *Suppose that for $(\hat{t},\hat{x},\hat{u},\hat{p},\hat{X}) \in \Theta_T \times \mathbb{R} \times \mathbb{R}^n \times S[n]$*

$$\liminf_{(t,u,p,X)\to(\hat{t}^-,\hat{u},\hat{p},\hat{X})} g(t,\hat{x},u,p,X) \ge g(\hat{t},\hat{x},\hat{u},\hat{p},\hat{X}). \tag{4.3}$$

Then if $u \in LSC(E_T,\mathbb{R})$ is a supersolution of (3.1)*, then it is a viscosity supersolution of* (3.1).

If (4.2), (4.3) *hold (i.e. g is continuous (left-hand side in t)) and $u \in C(E_T,\mathbb{R})$ is a solution of* (3.1) *then it is a viscosity solution of* (3.1).

It is evident that for Theorem 4.4 and Theorem 4.5 a remark, similar to Remark 3.4 holds.

Remark 4.6. It is worth mentioning that by virtue of Theorem 4.4 and Theorem 4.5 we can use a subsolution and supersolution as an upper and lower function in Perron method which is a main tool in proving existence results for viscosity solutions.

In the following we discuss the optimality of assumptions in Theorem 4.4.

Example 4.7 (the necessity of (4.2) in t). Let $a \geq 0, b \in R$. Consider the equation

$$D_t u - aD^2 u + bDu = g(t) \quad \text{in } E_T = [0,2] \times \mathbb{R}, \tag{4.4}$$

where $g(t) = 0,\ t \in [0,1)$ and $g(t) = -1,\ t \in [1,2]$. Set $u(t,x) = 0,\ t \in (0,1]$ and $u(t,x) = 1 - t,\ t \in (1,2]$. It is not difficult to verify that u is a subsolution of (4.4). To show that u is not a viscosity subsolution w set $\tilde{t} = 1,\ \phi \equiv 0$ in Definition 4.1. (it will change if we redefine g by setting $g(1) = 0$).

Example 4.8 (the necessity of (4.2) in u). Consider equation (4.4) with $g(t)$ replaced by $g(u) = u,\ u \in (-\infty, e)$ and $g(u) = 0,\ u \in [e, \infty)$. Set $u(t,x) = e^t$, $t \in [0,1]$ and $u(t,x) = e$, $t \in (1,2]$. It is not difficult to verify that u is a subsolution. To show that u is not a viscosity subsolution we set $\tilde{t} = 1$, $\phi(t) = e^t$ in Definition 4.1 (it will change if we redefine g by setting $g(e) = e$).

Example 4.9 (the necessity of (4.2) in p). Consider the equation

$$D_t u = h(Du)x \quad \text{in } E_T = [0,2] \times \mathbb{R}, \tag{4.5}$$

where $h(p) = p,\ p \in (-\infty, e)$ and $g(p) = 0,\ p \in [e, \infty)$. Set $u(t,x) = e^t x,\ t \in [0,1]$ and $u(t,x) = ex,\ t \in (1,2]$. It is not difficult to verify that u is a subsolution (solution) of (4.5). To show that u is not a viscosity subsolution w set $(\tilde{t}, \tilde{x}) = (1, \tilde{x})$, $\tilde{x} > 0$, $\phi(t,x) = e^t x$ in Definition 4.1. (it will change if we redefine h by setting $h(e) = e$).

In a similar way, considering the equation $D_t u = \frac{1}{2} h(Du) x^2$ we can show the necessity of (4.2) in X. The last example shows that upper absolutcontinuity plays crucial role in Theorem 4.4.

Example 4.10 (the necessity of (i) in Definition 3.1). Let $z : [0,1] \to [0,1]$ be the Cantor function. Of course, $z \notin UAC([c,1], \mathbb{R})$ for $c \in (0,1)$ and $z' = 0$ a.e in $[0,1]$. We will show that z is not a viscosity subsolution of $z' \leq 0$. Set $\phi(t) = t$. It follows from the construction of the Cantor function that there exists $\tilde{t} \in (0,1)$ such that $z(\tilde{t}) - \tilde{t} = \sup\{z(t) - t : t \in [0,1]\} > 0$. Since $\phi'(\tilde{t}) = 1$ and $\phi \in C^1(0,1)$ we have a contradiction with Definition 4.1 ($g \equiv 0$).

References

[1] P. Brandi, C. Marcelli; *Haar inequality in hereditary setting and applications*, Rend. Sem. Mat. Univ. Padova 96 (1996), 177-194.

[2] M. Cinquini Cibrario; *Nuove ricerche sui sistemi di equazioni non lineari a derivate parziali in piú variabili indipendenti*, Rend, Sem, Mat, Fis. Univ. Milano, 52 (1982), 531-550.

[3] M. Cinquini Cibrario; *A class of systems of nonlinear partial differential equations in several independent variables*, Ann. Mat. Pura Appl. 140 (4) (1985), 223 - 253.

[4] M. G. Crandall, H. Ishii, P. L. Lions; *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27 (1992), 1-67.

[5] M. G. Crandall, P. L. Lions; *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. 277 (1983), 1-42.

[6] L. C. Evans; *Partial Differential Equations*, 1998 American Mathematical Society.

[7] A. Haar; *Atti del Congresso Internationale dei Mathematici (1928)*, Bologna 3, p. 5-10.

[8] E. R. Hassan, W. Rzymowski; *Extremal solutions of a discontinuous scalar differential equation*, Nonlinear Analysis, Theory, Meth. and Appl. 37 (1999), 997-1017.

[9] S. N. Kruzkov; *Generalized solutions of first order nonlinear equations in several independent variables I.* Mat. Sb. 70 (1966), 394-415; II, Mat. Sb. (N.S) 72 (1967), 93-116. (in Russian)

[10] C. M. Lee; *An analogue of the theorem of Hake-Alexandroff-Looman*, Fund. Math. 100 (1978), 69-74.

[11] M. Nagumo; *Uber die Ungleichung* $\frac{\partial u}{\partial x} > f(x, y, u, \frac{\partial u}{\partial y})$, Jap. J. Math. 15. (1938), 51-56.

[12] M. Nagumo; *Notes on the theory of partial differential equations III*, Kansu-Hoteisiki oyobi OyO-Kaiseki, 15 (December 1939), 15 -26. (in Japanese)

[13] M. Nagumo et S. Simoda; *Notes sur l'inegalite differentielle concernant les equations du type parabolique*, Proc. Jap. Ac. 27 (1951), 536-539.

[14] E. Liz, R. L. Pouso; *Upper and Lower Solutions with "Jumps"*. J. Math. Anal. Appl. 222 (1998), 484-493.

[15] V. D. Ponomarev; *Absolute upper semicontinuity*, Mat. Zametki, 22 (3) (1977), 395-399 (in Russian).

[16] R. L. Pouso; *Upper and lower solutions for first-order discontinuous ordinary differential equations*,Journal of Mathematical Analysis and Applications, 244 (2) (2000), 466-482,

[17] J. Ridder; *Ueber die gegenseitigen Beziehungen verschieder approximativ stetiger Denjoy-Perron-Integrall*, Fund. Math. 22 (1934), 136-162.

[18] J. Szarski; *Differential Inequalities*, PWN, Warszawa, 1967.

[19] W. Walter; *Differential and Integral Inequalities*, Springer, Berlin, 1970.

[20] H. Westphal; *Zur Abschatzung der Losungen nichtlinearer parabolischer Differentialgleichungen*, Math. Z., 51 (1949), 690-695.

Krzysztof A. Topolski

Institute of Mathematics, University of Gdańsk, Wit Stwosz 57, 80-952 Gdańsk, Poland

E-mail address: matkt@mat.ug.edu.pl