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EXISTENCE AND UNIQUENESS OF POSITIVE ALMOST PERIODIC SOLUTIONS FOR SYSTEMS OF NONLINEAR DELAY INTEGRAL EQUATIONS

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Abstract. This article shows the existence and uniqueness of positive almost periodic solutions for some systems of nonlinear delay integral equations. After constructing a new fixed point theorem in the cone, which extend some existing results even in the case of scalar version, we apply it to a model of the evolution in time of two species in interaction.

1. Introduction

The theory of almost periodicity began with the pioneering papers of Bohr (1923) and developed by Bochner [3]. Almost periodicity as a structural property of functions is a generalization of pure periodicity, and certainly one of the important successes of this newer theory was the development of rather complete theory of Fourier series for almost periodic functions. This theory opens a way of studying a wide class of trigonometric series of the general type and of exponential series. The general property of almost periodicity can be illustrated by means of the particular example $f(t) = \sin 2\pi t + \sin 2\pi t\sqrt{2}$.

On the other hand, the existence of almost periodic solutions has became an interesting and important topic in the study of qualitative theory of differential and integral equations related to dynamical systems or flows. In the present work, we are concerned with the system

$$\begin{aligned} x(t) &= \int_{t-\tau_1(t)}^{t} \tilde{f}(s, x(s), y(s))ds \\ y(t) &= \int_{t-\tau_2(t)}^{t} \tilde{g}(s, x(s), y(s))ds \end{aligned} \tag{1.1}$$

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which is a model for the evolution in time of two species in interaction and is more general than the one studied in [7]

$$
\begin{aligned}
x(t) &= \int_{t-\tau_1}^{t} f(s, x(s), y(s))ds \\
y(t) &= \int_{t-\tau_2}^{t} g(s, x(s), y(s))ds
\end{aligned}
\tag{1.2}
$$

First of all we have interest to describe the meaning of the system (1.1) in the biologic context. $x(t), y(t)$ are, respectively, the numbers of individuals present in the populations x, y at time t and which live to the ages $\tau_1(t), \tau_2(t)$, and the functions f, g are, respectively, the numbers of new births per time unit in x, y. Also, we can describe (1.1) in the context of epidemics. $x(t), y(t)$ are the populations at time t of infectious individuals, $\tau_1(t), \tau_2(t)$ are the durations of infectivity and the functions f, g are the instantaneous rates of infection.

In our work, we show firstly an adequate fixed point theorem for vectorial version with two components (see theorem 2.7) which extend some existing results even in the case of scalar version and in the case of discrete systems (we refer the reader to [10, 11, 12, 22]) ,and then, we apply it for obtain the existence and uniqueness of positive almost periodic solutions for the system (1.1). Let us introduce a short history of the problem.

In 1976, Cooke and Kaplan [8] published an article where they formulate and study the existence of positive periodic solutions for the integral equation

$$
x(t) = \int_{t-\tau}^{t} f(s, x(s))ds. \tag{1.3}
$$

This model, explain the spread of some infectious diseases and have also been used as a growth equation for single species population when the birth rate varies seasonally. These authors proposed in the same article the system (1.2), which is a model of the evolution in time of two populations in interaction.

First results of the existence of positive periodic solutions in cases of cooperative and competitive type of system (1.2), has been obtained by Cañada and Zertiti [7] using the method of upper and lower solutions, which allows them to apply the Schauder's fixed point theorem. Since then, the existence of positive periodic solutions for other form of (1.2)

$$
\begin{aligned}
x(t) &= \int_{0}^{\tau_1(t)} f(t, s, x(t-s-l), y(t-s-l))ds \\
y(t) &= \int_{0}^{\tau_2(t)} g(t, s, x(t-s-l), y(t-s-l))ds
\end{aligned}
$$

has been discussed in [20, 24] via the method of upper and lower solutions and in [5, 21] by topological method. Also, the case of discrete systems was studied by Wen-Hai Pan and Wei Long [22].

However, as far as we know, many authors have studied the existence and uniqueness of periodic, almost periodic and almost automorphic solutions for various forms of (1.3) (see, e. g., [4, 10, 11, 12, 17, 23] and references therein), but we do not know any result concerning the existence of almost periodic solutions for the above systems.

In many works, the authors have investigated mixed monotone operators in Banach space, and obtained a lot of interesting and important results about the existence of almost periodic and almost automorphic solutions for the scalar case. Therefore, in this paper, we propose to extend this results by proving an adequate fixed point theorem for vectorial version with two components, and then, we apply it for obtain the existence and uniqueness of positive almost periodic solutions for system (1.1).

2. Preliminaries

We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers and by $C(E)$, where E is a metric set, the space of continuous functions defined on E with values in $\mathbb{R}$. For $f \in C(\mathbb{R})$(resp. $C(\mathbb{R} \times \mathbb{R}^+)$ or $C(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$), the translation of f is the function $\tau_s f(t) = f(t-s), t \in \mathbb{R}$, (resp. $\tau_s f(t,x) = f(t-s,x), (t,x) \in \mathbb{R} \times \mathbb{R}^+$ and $\tau_s f(t,x,y) = f(t-s,x,y), (t,x,y) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$).

Definition 2.1 ([9, 14])**.** A function $f \in C(\mathbb{R})$(resp. $C(\mathbb{R}\times\mathbb{R}^+)$ or $C(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$) is called almost periodic (resp. almost periodic in $t \in \mathbb{R}$, uniformly in $x \in \mathbb{R}^+$ or $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+$), if for each $\varepsilon > 0$(resp. $\varepsilon > 0$ and compact $K \subset \mathbb{R}^+$ or compact $K' \subset \mathbb{R}^+ \times \mathbb{R}^+$), there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains a number μ with the property that

$$\sup_{s\in\mathbb{R}} |\tau_\mu f(s) - f(s)| < \varepsilon$$

$$(\text{resp.} \sup_{(s,x)\in\mathbb{R}\times K} |\tau_\mu f(s,x) - f(s,x)| < \varepsilon$$

$$\text{or} \sup_{(s,x,y)\in\mathbb{R}\times K'} |\tau_\mu f(s,x,y) - f(s,x,y)| < \varepsilon).$$

Denote $AP(\mathbb{R})$ (resp. $AP(\mathbb{R}\times\mathbb{R}^+)$ or $AP(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$) the set of all such functions.

Definition 2.2. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called normal if for every sequence of real numbers $(S'_m)_m$ there exists a subsequence $(S_n)_n$ such that the sequence $[f(t+S_n)]_n$ converges uniformly to a function limit.

Theorem 2.3 (Bochner [3])**.** *A function f is almost periodic if and only if it is normal.*

Suppose that f belongs to $AP(\mathbb{R})$. Let $[\lambda_j]$ denote the set of all real numbers such that

$$\lim_{T\to+\infty} \int_0^T f(t)\exp(-i\lambda t)dt \neq 0.$$

It is well known that the set of numbers $[\lambda_j]$ in the above formula is countable. The set $[\sum_{j=1}^{N} n_j\lambda_j]$ for all integers N and integers n_j is called the module of $f(t)$, denoted by $\mathrm{mod}(f)$.

Lemma 2.4 ([14])**.** *Suppose that f and g are almost periodic. Then the following statements are equivalent:*

(i) $\mathrm{mod}(f) \supset \mathrm{mod}(g)$;
(ii) *For any sequence of real numbers $(S'_m)_m$, if $\lim_{m\to+\infty} f(t+S'_m) = f(t)$ for each $t \in \mathbb{R}$, then there exists a subsequence $(S_n)_n$ such that $\lim_{n\to+\infty} g(t+S_n) = g(t)$ for each $t \in \mathbb{R}$,*

Lemma 2.5 ([2, 9, 19]). *Assume that $f, g \in AP(\mathbb{R})$ and λ is any scalar. Then the following statements hold:*

- (i) $f+g$, $f.g$, λf, $f_\tau(t) = f(t+\tau)$, $\tilde{f}(t) = f(-t)$ *are almost periodic.*
- (ii) *The range $R_f = [f(t) : t \in \mathbb{R}]$ is precompact in $\mathbb{R}$, and so f is bounded.*
- (iii) *If f is almost periodic, then f is uniformly continuous.*
- (iv) *Let F be a uniformly continuous function and f be almost periodic. Then $F \circ f$ is almost periodic.*
- (v) *If $[f_n]_n$ is a sequence of almost periodic functions and $f_n \to f$ uniformly on $\mathbb{R}$, then f is almost periodic.*
- (vi) *$AP(\mathbb{R})$ equipped with the sup norm*
$$\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$$
turns out to be a Banach space.

Definition 2.6. Let E be a real Banach space. A closed convex set P in E is called a convex cone if the following conditions are satisfied

1. If $x \in P$, then $\lambda x \in P$ for any $\lambda \in \mathbb{R}^+$;
2. If $x \in P$ and $-x \in P$, then $x = 0$.

A cone P induces a partial ordering $\le$ in E defined by $x \le y$ if $y - x \in P$. A cone P is called normal if there exists a constant $N > 0$ such that $0 \le x \le y$ implies $\|x\| \le N\|y\|$, where $\|\cdot\|$ is the norm on E. We denote by $\mathring{P}$ the interior set of P. A cone P is called a solid cone if $\mathring{P} \neq \emptyset$.

Theorem 2.7. *Let P be a cone in a Banach space and $\Phi_1, \Phi_2 : \mathring{P} \times \mathring{P} \times \mathring{P} \to \mathring{P}$ are operators such that*

- (A1) $\Phi_1(\cdot, u, y)$ *is nondecreasing and* $\Phi_1(x, \cdot, y)$, $\Phi_1(x, u, \cdot)$ *are nonincreasing;* $\Phi_2(\cdot, u, y)$, $\Phi_2(x, \cdot, y)$ *are nonincreasing and* $\Phi_2(x, u, \cdot)$ *is nondecreasing.*
- (A2) *There exist a constant $\alpha_0 \in [0, 1)$ and functions $\phi_i : (0, 1) \times \mathring{P} \times \mathring{P} \times \mathring{P} \to (0, +\infty)$, $i = 1, 2$ such that $\phi_i(\alpha, x, u, y) > \alpha$ and*
$$\begin{gathered}\Phi_1(\alpha x, \frac{1}{\alpha} u, \frac{1}{\alpha} y) \ge \phi_1(\alpha, x, u, y)\Phi_1(x, u, y),\\ \Phi_2(\frac{1}{\alpha} x, \frac{1}{\alpha} u, \alpha y) \ge \phi_2(\alpha, x, u, y)\Phi_2(x, u, y),\end{gathered}$$
for each $x, u, y \in \mathring{P}$, for each $\alpha \in (\alpha_0, 1)$.
- (A3) *There exist $x_0, x^0, y_0, y^0 \in \mathring{P}$ with $x_0 \le x^0, y_0 \le y^0$ such that*
$$\begin{gathered}x_0 \le \Phi_1(x_0, x^0, y^0), \quad \Phi_1(x^0, x_0, y_0) \le x^0,\\ y_0 \le \Phi_2(x^0, y^0, y_0), \quad \Phi_2(x_0, y_0, y^0) \le y^0\end{gathered} \tag{2.1}$$
and for each $\alpha \in (\alpha_0, 1)$,
$$\begin{gathered}\underline{\phi_1}(\alpha) = \inf_{y\in[y_0,y^0],x,u\in[x_0,x^0]} \phi_1(\alpha, x, u, y) > \alpha,\\ \underline{\phi_2}(\alpha) = \inf_{x\in[x_0,x^0],v,y\in[y_0,y^0]} \phi_2(\alpha, x, v, y) > \alpha\end{gathered}$$

Then $\Phi : \mathring{P} \times \mathring{P} \times \mathring{P} \times \mathring{P} \to \mathring{P} \times \mathring{P}$ defined by $\Phi(x, u, v, y) = (\Phi_1(x, u, y), \Phi_2(x, v, y))$ has a unique fixed point $(x^*, y^*) \in [x_0, x^0] \times [y_0, y^0]$; that is,
$$\Phi(x^*, x^*, y^*, y^*) = (x^*, y^*).$$

Moreover, constructing successively the iterative sequences

$$u_{n+1} = \Phi_1(u_n, u_n, v_n), \quad v_{n+1} = \Phi_2(u_n, v_n, v_n)$$

for any initial $(u_0, v_0) \in [x_0, x^0] \times [y_0, y^0]$, *we have*

$$\|u_n - x^*\| \to 0, \quad \|v_n - y^*\| \to 0, \quad \text{as } n \to +\infty.$$

Proof. Construct the sequences

$$\begin{aligned} x_{n+1} &= \Phi_1(x_n, x^n, y^n), \quad x^{n+1} = \Phi_1(x^n, x_n, y_n), \\ y_{n+1} &= \Phi_2(x^n, y^n, y_n), \quad y^{n+1} = \Phi_2(x_n, y_n, y^n). \end{aligned}$$

From (A3), it is easy to show by induction that

$$\begin{aligned} &x_0 \le x_1 \le \dots \le x_n \le \dots \le x^n \le \dots \le x^1 \le x^0, \\ &y_0 \le y_1 \le \dots \le y_n \le \dots \le y^n \le \dots \le y^1 \le y^0. \end{aligned} \tag{2.2}$$

Let

$$r_n = \sup[r > 0 : x_n \ge r x^n \text{ and } y_n \ge r y^n].$$

It follows that $x_n \ge r_n x^n$, $y_n \ge r_n y^n$, $n = 1, 2, \dots$, and then

$$x_{n+1} \ge x_n \ge r_n x^n \ge r_n x^{n+1}, \quad y_{n+1} \ge y_n \ge r_n y^n \ge r_n y^{n+1}, \quad n = 1, 2, \dots$$

Therefore $r_{n+1} \ge r_n$, which implies that $(r_n)_n$ is increasing with $r_n \le 1$.

Set $r^* = \lim_{n \to +\infty} r_n$. We claim that $r^* = 1$. In fact, if we suppose to the contrary that $r_n \le r^* < 1$, we distinguish two cases.

Case 1: there exists k such that $r_k = r^*$. In this case we have that for all $n \ge k$,

$$\begin{aligned} x_{n+1} &= \Phi_1(x_n, x^n, y^n) \ge \Phi_1\Big(r^* x^n, \frac{1}{r^*} x_n, \frac{1}{r^*} y_n\Big) \ge \phi_1(r^*, x^n, x_n, y_n) x^{n+1}, \\ y_{n+1} &= \Phi_2(x^n, y^n, y_n) \ge \Phi_2\Big(\frac{1}{r^*} x_n, \frac{1}{r^*} y_n, r^* y^n\Big) \ge \phi_2(r^*, x_n, y_n, y^n) y^{n+1}. \end{aligned}$$

Thus,

$$x_{n+1} \ge \min\{\underline{\phi_1}(r^*), \underline{\phi_2}(r^*)\} x^{n+1}, \quad y_{n+1} \ge \min\{\underline{\phi_1}(r^*), \underline{\phi_2}(r^*)\} y^{n+1}.$$

This implies that $r_{n+1} \ge \min\{\underline{\phi_1}(r^*), \underline{\phi_2}(r^*)\} > r^*$. This is a contradiction. PAGE 5.

Case 2: $r_n < r^* < 1$, $n = 1, 2, \dots$. Setting $\eta_1(\alpha, x, u, y) = \frac{\phi_1(\alpha, x, u, y)}{\alpha} - 1$ and $\eta_2(\alpha, x, v, y) = \frac{\phi_2(\alpha, x, v, y)}{\alpha} - 1$, for all $\alpha \in (0, 1)$, for all $x, u, v, y \in \mathring{P}$, we have that

$$\begin{aligned} x_{n+1} &= \Phi_1(x_n, x^n, y^n) \\ &\ge \Phi_1\Big(r_n x^n, \frac{1}{r_n} x_n, \frac{1}{r_n} y_n\Big) \\ &= \Phi_1\Big(\frac{r_n}{r^*} r^* x^n, \frac{r^*}{r_n} \frac{1}{r^*} x_n, \frac{r^*}{r_n} \frac{1}{r^*} y_n\Big) \\ &\ge \phi_1\Big(\frac{r_n}{r^*}, r^* x^n, \frac{1}{r^*} x_n, \frac{1}{r^*} y_n\Big) \Phi_1\Big(r^* x^n, \frac{1}{r^*} x_n, \frac{1}{r^*} y_n\Big) \\ &\ge \frac{r_n}{r^*} \phi_1\Big(r^*, x^n, x_n, y_n) \Phi_1(x^n, x_n, y_n\Big) \\ &\ge \frac{r_n}{r^*} r^* [1 + \eta_1(r^*, x^n, x_n, y_n)] \Phi_1(x^n, x_n, y_n). \end{aligned}$$

This implies $x_{n+1} \ge r_n \left[1 + \eta_1(r^*, x^n, x_n, y_n)\right] x^{n+1}$.

Also we obtain $y_{n+1} \geq r_n[1+\eta_2(r^*, x_n, y_n, y^n)]y^{n+1}$. Thus
$$x_{n+1} \geq r_n \min\{\frac{\underline{\phi_1}(r^*)}{r^*}, \frac{\underline{\phi_2}(r^*)}{r^*}\}x^{n+1}, \quad y_{n+1} \geq r_n \min\{\frac{\underline{\phi_1}(r^*)}{r^*}, \frac{\underline{\phi_2}(r^*)}{r^*}\}y^{n+1}.$$
It follows that
$$r_{n+1} \geq r_n \min\{\frac{\underline{\phi_1}(r^*)}{r^*}, \frac{\underline{\phi_2}(r^*)}{r^*}\}.$$
Therefore,
$$r^* \geq r^* \min\{\frac{\underline{\phi_1}(r^*)}{r^*}, \frac{\underline{\phi_2}(r^*)}{r^*}\} > r^*.$$
Which is a contradiction. Hence $\lim_{n\to+\infty} r_n = r^* = 1$. Now, for any natural number p we have
$$\begin{gathered}0 \leq x_{n+p} - x_n \leq x^n - x_n \leq x^n - r_n x^n \leq (1-r_n)x^0, \\ 0 \leq x^n - x^{n+p} \leq x^n - x_n \leq x^n - r_n x^n \leq (1-r_n)x^0\end{gathered}$$
and
$$\begin{gathered}0 \leq y_{n+p} - y_n \leq y^n - y_n \leq y^n - r_n y^n \leq (1-r_n)y^0, \\ 0 \leq y^n - y^{n+p} \leq y^n - y_n \leq y^n - r_n y^n \leq (1-r_n)y^0.\end{gathered}$$
Since P is normal cone, we have
$$\begin{gathered}\|x_{n+p} - x_n\| \leq N(1-r_n)\|x^0\|, \quad \|x^n - x^{n+p}\| \leq N(1-r_n)\|x^0\|, \\ \|y_{n+p} - y_n\| \leq N(1-r_n)\|y^0\|, \quad \|y^n - y^{n+p}\| \leq N(1-r_n)\|y^0\|.\end{gathered}$$
Here N is the normality constant. So $[x_n], [x^n], [y_n], [y^n]$ are cauchy sequences. Thus, there exist $u_*, u^* \in [x_0, x^0]$ and $v_*, v^* \in [y_0, y^0]$ such that $x_n \to u_*$, $x^n \to u^*$, $y_n \to v_*$ and $y^n \to v^*$ when $n \to +\infty$. By (2.2), we have
$$\begin{gathered}0 \leq u^* - u_* \leq x^n - x_n \leq (1-r_n)x^0, \\ 0 \leq v^* - v_* \leq y^n - y_n \leq (1-r_n)y^0.\end{gathered}$$
Thus, $u^* = u_*$ and $v^* = v_*$. Let $x^* = u^* = u_*$ and $y^* = v^* = v_*$, we obtain $\Phi(x^*, x^*, y^*, y^*) = (x^*, y^*)$.

Suppose that $(x_*, y_*) \in [x_0, x^0] \times [y_0, y^0]$ is a fixed point of Φ. Then, from the definition of x_n, x^n, y_n, y^n we have $x_n \leq x_* \leq x^n$, $y_n \leq y_* \leq y^n$, and by the normality of P, we get $x^* = x_*$ and $y^* = y_*$. Also, we have for any initial $(u_0, v_0) \in [x_0, x^0] \times [y_0, y^0]$, $x_n \leq u_n \leq x^n$ and $y_n \leq v_n \leq y^n$, where $u_n = \Phi_1(u_{n-1}, u_{n-1}, v_{n-1})$ and $v_n = \Phi_1(u_{n-1}, v_{n-1}, v_{n-1})$. Therefore, $\|u_n - x^*\| \to 0$, $\|v_n - y^*\| \to 0$ as $n \to +\infty$. □

3. Existence and uniqueness of almost periodic solution

In this section, we show the existence and uniqueness of positive almost periodic solution for system (1.1). Throughout the rest of this article, we assume that the functions $\tilde{f}$ and $\tilde{g}$ admit a decomposition
$$\tilde{f}(t,x,y) = h(t,x)f(t,x,y) \quad \text{and} \quad \tilde{g}(t,x,y) = k(t,y)g(t,x,y).$$
Firstly, we introduce some notations and lemmas. Set, uniformly in $t \in \mathbb{R}$,
$$\liminf_{y\to0^+} \liminf_{y\neq0, x\to+\infty} \frac{f(t,x,y)}{x} = f_{(+\infty,0^+)}(t), \quad \liminf_{u\to0^+} h(t,u) = h_{0^+}(t),$$

$$
\begin{gathered}
\liminf_{y\to+\infty}\liminf_{x\to 0^+}\frac{f(t,x,y)}{x}=f_{(0^+,+\infty)}(t),\quad \liminf_{u\to+\infty}h(t,u)=h_{+\infty}(t),\\
\limsup_{y\to 0^+}\limsup_{y\neq 0,x\to+\infty}\frac{f(t,x,y)}{x}=f^{(+\infty,0^+)}(t),\quad \limsup_{u\to 0^+}h(t,u)=h^{0^+}(t),\\
\liminf_{x\to 0^+}\liminf_{x\neq 0,y\to+\infty}\frac{g(t,x,y)}{x}=g_{(+\infty,0^+)}(t), quad \liminf_{u\to 0^+}k(t,u)=k_{0^+}(t),\\
\liminf_{x\to+\infty}\liminf_{y\to 0^+}\frac{g(t,x,y)}{x}=g_{(0^+,+\infty)}(t),\quad \liminf_{u\to+\infty}k(t,u)=k_{+\infty}(t),\\
\limsup_{x\to 0^+}\limsup_{x\neq 0,x\to+\infty}\frac{g(t,x,y)}{x}=g^{(+\infty,0^+)}(t),\quad \limsup_{u\to 0^+}k(t,u)=k^{0^+}(t).
\end{gathered}
$$

Denote by P the following set in the Banach space $AP(\mathbb{R})$

$$P=[x\in AP(\mathbb{R}):x(t)\geq 0,\ \forall t\in\mathbb{R}].$$

It is not difficult to verify that P is a normal and solid cone in $AP(\mathbb{R})$ and

$$\mathring{P}=\{x\in P:\exists\varepsilon>0 \text{ such that } x(t)>\varepsilon,\ \forall t\in\mathbb{R}\}.$$

We will need the following three lemmas in the proof of our result.

Lemma 3.1. *Suppose that $f\in AP(\mathbb{R}\times\mathbb{R}^+)$ and $x\in P$ (resp. $f\in AP(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$ and $(x,y)\in P\times P$). Then $f(\cdot,x(.))\in AP(\mathbb{R})$ (resp. $f(\cdot,x(.),y(.))\in AP(\mathbb{R})$).*

Lemma 3.2. *Let $f\in AP(\mathbb{R})$ and $\tau\in AP(\mathbb{R})$. Then*

$$F(t)=\int_{t-\tau(t)}^{t}f(s)ds\in AP(\mathbb{R}).$$

Proofs of the two lemmas above and more details can be found in [9, 13].

For $c\in AP(\mathbb{R})$ and $\tau\in AP(\mathbb{R})$, we denote by $r(L_{(\tau,c)})=\lim_{n\to+\infty}\|(L_{(\tau,c)})^n\|^{1/n}$ the spectral radius of the linear operator $L_{(\tau,c)}:AP(\mathbb{R})\to AP(\mathbb{R})$ defined by

$$L_{(\tau,c)}(x)(t)=\int_{t-\tau(t)}^{t}c(s)x(s)ds,\quad \forall x\in AP(\mathbb{R}),\ \forall t\in\mathbb{R}\,.$$

Lemma 3.3. *Let $\tau\in AP(\mathbb{R})$ is positive function, $\rho\in AP(\mathbb{R})$ is nonnegative function such that $\mathring{D}\neq\emptyset$, where $D=[s\in\mathbb{R}:\rho(s)=0]$. Then, for each $x\in P\setminus[0]$ the function z defined by*

$$z(t)=\int_{t-\tau(t)}^{t}\rho(s)x(s)ds$$

is in $\mathring{P}$.

Proof. Recall that if $x\in AP(\mathbb{R})$. Then the closure, in the uniform topology, of the set $\mathrm{Hull}(x)=[x(t+\beta)]_{\beta\in\mathbb{R}}$ is compact in the uniform topology. Let $C_x=[t\in\mathbb{R}:x(t)=0]$. One can show, by using the compactness of $Hull(x)$, that there exists $M>0$ such that if $v\in Hull(x)$ and $[a,b]\subset C_v$ then $b-a\leq M$. Now, if there is $t_0\in\mathbb{R}$ such that $z(t_0)=0$, Choose $n\in\mathbb{N}$ satisfying $n\tau(t_0)>M$. Then

$$z(t_0)=\int_{t_0-\tau(t_0)}^{t_0}\rho(s)x(s)\,ds=0.$$

It follows $x(s)=0$ in the interval $[t_0-\tau(t_0),t_0]$. Repeating the process with the points $t_0-\tau(t_0)$ and t_0 we obtain $x(s)=0$ in the interval $[t_0-2\tau(t_0),t_0]$. If we repeat the process n times we obtain $x(s) = 0$ in the interval $[t_0 - n\tau(t_0), t_0]$ which is contradiction. Thus, $z(t) > 0$ for all $t \in \mathbb{R}$. Suppose that $\inf_{t\in\mathbb{R}} z(t) = 0$. Then there exists a sequence $(\alpha_n)_n \subseteq \mathbb{R}$ such that $z(\alpha_n) \to 0$ as $n \to +\infty$. Since $[x(t+\alpha_n)]$ and $[z(t+\alpha_n)]$ are precompact, we may consider $x(t+\alpha_n) \to v(t)$ uniformly on $\mathbb{R}$ and $z(t+\alpha_n) \to w(t)$ uniformly on $\mathbb{R}$, where $v \in Hull(x), w \in Hull(z)$. Then we have $w(0) = 0$, and it is easily checked that $v(s) = 0$ on an interval of length greater than M, which contradicts our previous assertion. Thus $z \in \mathring{P}$. $\square$

We list the following assumptions that we will use them throughout the rest of this article:

- (H1) $f, g \in AP(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$, $h, k \in AP(\mathbb{R} \times \mathbb{R}^+)$ are nonnegative functions and $\tau_1, \tau_2 \in AP(\mathbb{R})$ are positive functions.
- (H2) for all $s \in \mathbb{R}$, the functions $f(s, ., y)$ and $g(s, x, .)$ are nondecreasing and $f(s, x, .)$, $g(s, ., y)$, $h(s, .)$, $k(s, .)$ are nonincreasing.
- (H3) There exist positive functions φ_1, φ_2 defined on $(0, 1) \times (0, +\infty)$, ψ_1, ψ_2 defined on $(0, 1) \times (0, +\infty) \times (0, +\infty)$ such that
$$h(s, \frac{1}{\alpha}x) \geq \varphi_1(\alpha, x)h(s, x), \quad f(s, \alpha x, \frac{1}{\alpha}y) \geq \psi_1(\alpha, x, y)f(s, x, y),$$
$$k(s, \frac{1}{\alpha}x) \geq \varphi_2(\alpha, x)k(s, x), \quad g(s, \frac{1}{\alpha}x, \alpha y) \geq \psi_2(\alpha, x, y)f(s, x, y)$$
and
$$\varphi_i(\alpha, x) > \alpha, \ \psi_i(\alpha, x, y) > \alpha, \ i = 1, 2,$$
for all $x, y > 0$, all $\alpha \in (0, 1)$, all $s \in \mathbb{R}$. Moreover, for any $0 < a \leq b < +\infty$ and $0 < c \leq d < +\infty$,
$$\inf_{y\in[c,d],\ x,u\in[a,b]} \varphi_1(\alpha, u)\psi_1(\alpha, x, y) > \alpha,$$
$$\inf_{x\in[a,b],\ u,y\in[c,d]} \varphi_2(\alpha, u)\psi_2(\alpha, x, y) > \alpha,$$
for all $\alpha \in (0, 1)$.

Now, we are in a position to present the existence and uniqueness theorem.

Theorem 3.4. *Assume that* (H1)–(H3) *hold and*

(i)
$$\min_{t\in\mathbb{R}} \int_{t-\tau_1(t)}^{t} h_{0^+}(s)f_{(+\infty,0^+)}(s)ds > 0, \quad \min_{t\in\mathbb{R}} \int_{t-\tau_2(t)}^{t} k_{0^+}(s)g_{(+\infty,0^+)}(s)ds > 0.$$

(ii)
$$r\Big(L_{(\tau_1,h_{+\infty}f_{(0^+,+\infty)})}\Big) > 1, \quad r\Big(L_{(\tau_2,k_{+\infty}g_{(0^+,+\infty)})}\Big) > 1.$$

Moreover $\mathring{D}_1 \neq \emptyset$, $\mathring{D}_2 \neq \emptyset$, *where* $D_1 = \{s \in \mathbb{R} : h_{+\infty}(s)f_{(0^+,+\infty)}(s) = 0\}$ *and* $D_2 = \{s \in \mathbb{R} : k_{+\infty}(s)g_{(0^+,+\infty)}(s) = 0\}$.

(iii)
$$r\Big(L_{(\tau_1,h^{0^+}f^{(+\infty,0^+)})}\Big) < 1, \quad r\Big(L_{(\tau_2,k^{0^+}g^{(+\infty,0^+)})}\Big) < 1.$$

Then, system (1.1) *has exactly one almost periodic solution* $(x^*, y^*) \in \mathring{P} \times \mathring{P}$. *Moreover, for any initial* $(u_0, v_0) \in \mathring{P} \times \mathring{P}$ *and*
$$u_{n+1}(t) = \int_{t-\tau_1(t)}^{t} h(s, u_n(s))f(s, u_n(s), v_n(s))ds,$$

$$v_{n+1}(t) = \int_{t-\tau_2(t)}^{t} k(s, v_n(s))g(s, u_n(s), v_n(s))ds$$

we have $\|u_n - x^*\| \to 0$, $\|v_n - y^*\| \to 0$, *as* $n \to +\infty$.

Proof. We prove that all hypotheses of theorem 2.7 are satisfied for adequate operators Φ_1 and Φ_2. Consider the nonlinear operator Φ defined by $\Phi(x, u, v, y) = (\Phi_1(x, u, y), \Phi_2(x, v, y))$, where

$$\Phi_1(x, u, y)(t) = \int_{t-\tau_1(t)}^{t} h(s, u(s)f(s, x(s), y(s))ds,$$
$$\Phi_2(x, v, y)(t) = \int_{t-\tau_2(t)}^{t} k(s, v(s)g(s, x(s), y(s))ds$$

for all $x, u, v, y \in \mathring{P}$ and all $t \in \mathbb{R}$. From (H1), (H3), lemmas 2.5, 3.1 and 3.2, we obtain $\Phi_1(x, u, y) \in AP(\mathbb{R})$ and $\Phi_2(x, v, y) \in AP(\mathbb{R})$ for all $x, u, v, y \in \mathring{P}$. Since

$$\min_{t\in\mathbb{R}} \int_{t-\tau_1(t)}^{t} h_{0^+}(s)f_{(+\infty,0^+)}(s)ds > 0,$$

there exists a positive number $\varepsilon > 0$ such that

$$\min_{t\in\mathbb{R}} \int_{t-\tau_1(t)}^{t} (h_{0^+}(s) - \varepsilon)(f_{(+\infty,0^+)}(s) - \varepsilon)ds > 0.$$

it follows that there exist numbers δ, M with $0 < \delta < M$ such that

$$h(s, u) \geq (h_{0^+}(s) - \varepsilon), \quad \forall u \leq \delta, \ \forall s \in \mathbb{R}, \quad \text{and}$$
$$f(s, x, y) \geq (f_{(+\infty,0^+)}(s) - \varepsilon)x, \quad \forall x \geq M, \ \forall y \leq \delta, \ \forall s \in \mathbb{R}.$$

Let $x, u, y \in \mathring{P}$. We consider $\alpha \in (0, 1)$ satisfying the inequalities $\frac{1}{\alpha}(\min_{t\in\mathbb{R}} x(t)) \geq M$, $\alpha(\max_{t\in\mathbb{R}} y(t)) \leq \delta$ and $\alpha(\max_{t\in\mathbb{R}} u(t)) \leq \delta$. Then for all $t \in \mathbb{R}$,

$$\begin{aligned}
&\Phi_1(x, u, y)(t) \\
&= \int_{t-\tau_1(t)}^{t} h(s, u(s))f(s, x(s), y(s))ds \\
&= \int_{t-\tau_1(t)}^{t} h(s, \frac{1}{\alpha}\alpha u(s))f(s, \alpha\frac{1}{\alpha}x(s), \frac{1}{\alpha}\alpha y(s))ds \\
&\geq \int_{t-\tau_1(t)}^{t} \varphi_1(\alpha, \alpha u(s))h(s, \alpha u(s))\psi_1(\alpha, \frac{1}{\alpha}x(s), \alpha y(s))f(s, \frac{1}{\alpha}x(s), \alpha y(s))ds \\
&\geq \alpha^2 \int_{t-\tau_1(t)}^{t} h(s, \alpha u(s))f(s, \frac{1}{\alpha}x(s), \alpha y(s))ds \\
&\geq \alpha \int_{t-\tau_1(t)}^{t} (h_{0^+}(s) - \varepsilon)(f_{(+\infty,0^+)}(s) - \varepsilon)x(s)ds \\
&\geq \alpha \min_{s\in\mathbb{R}} x(s) \int_{t-\tau_1(t)}^{t} (h_{0^+}(s) - \varepsilon)(f_{(+\infty,0^+)}(s) - \varepsilon)ds > 0.
\end{aligned}$$

This implies that $\Phi_1 : \mathring{P} \times \mathring{P} \times \mathring{P} \to \mathring{P}$. Analogously, $\Phi_2 : \mathring{P} \times \mathring{P} \times \mathring{P} \to \mathring{P}$.

On the other hand, from (H2) it easy to show that Φ_1 and Φ_2 satisfy assumption (A1) of theorem 2.7. We prove that assumption (A2) holds. Let $x, u, y \in \mathring{P}$ and

$\alpha \in (0,1)$. By setting

$$a(x,u,y) = \min\{\inf_{s\in\mathbb{R}} x(s), \inf_{s\in\mathbb{R}} u(s), \inf_{s\in\mathbb{R}} y(s)\},$$
$$b(x,u,y) = \max\{\sup_{s\in\mathbb{R}} x(s), \sup_{s\in\mathbb{R}} u(s), \sup_{s\in\mathbb{R}} y(s)\},$$

we have $0 < a(x,u,y) \le b(x,u,y) < +\infty$ and $x(s), u(s), y(s) \in \{a(x,u,y), b(x,u,y)\}$, for all $s \in \mathbb{R}$. We define

$$\phi_i(\alpha, x, u, y) = \inf_{\beta,\gamma,\eta \in \{a(x,u,y), b(x,u,y)\}} \varphi_i(\alpha,\gamma)\psi_i(\alpha,\beta,\eta), \quad i = 1, 2.$$

By (H3), it easy to see that $\phi_i(\alpha, x, u, y) > \alpha$ for all $x, u, y \in \mathring{P}$ and for all $\alpha \in (0,1)$. Also, we have

$$\begin{aligned}
\Phi_1(\alpha x, \frac{1}{\alpha}u, \frac{1}{\alpha}y)(t) &= \int_{t-\tau_1(t)}^{t} h(s, \frac{1}{\alpha}u(s)) f(s, \alpha x(s), \frac{1}{\alpha}y(s)) ds \\
&\ge \int_{t-\tau_1(t)}^{t} \varphi_1(\alpha, u(s))\psi_1(\alpha, x(s), y(s)) h(s, u(s)) f(s, x(s), y(s)) ds \\
&\ge \phi_1(\alpha, x, u, y) \int_{t-\tau_1(t)}^{t} h(s, u(s)) f(s, x(s), y(s)) ds.
\end{aligned}$$

Which means that

$$\Phi_1(\alpha x, \frac{1}{\alpha}u, \frac{1}{\alpha}y) \ge \phi_1(\alpha, x, u, y)\Phi_1(x, u, y)$$

for each $x, u, y \in \mathring{P}$ and $\alpha \in (0,1)$. Analogously we obtain

$$\Phi_2(\frac{1}{\alpha}x, \frac{1}{\alpha}u, \alpha y) \ge \phi_2(\alpha, x, u, y)\Phi_2(x, u, y)$$

for each $x, u, y \in \mathring{P}$ and $\alpha \in (0,1)$. Thus, assumption (A2) in theorem 2.7 is satisfied.

Finally, by combining lemma 3.3 and the same reasoning as in the proof of [20, corollary 3.2], we obtain the existence of $(x_0, y_0), (x^0, y^0) \in \mathring{P} \times \mathring{P}$ satisfying (2.1) in assumption (A3) of theorem 2.7. $\square$

Remark 3.5. Recall that in [7], Cañada and Zertiti studied system (1.2) which is a special case of system (1.1) where $\tau_1(t) \equiv \tau_1$, $\tau_2(t) \equiv \tau_2$, $h(t,x) \equiv 1$ and $k(t,x) \equiv 1$. In deed, they defined three particular cases of system (1.2):

(a) Of a competition type if

$$f(t,x,y) \nearrow x,\ f(t,x,y) \searrow y;\ g(t,x,y) \searrow x,\ g(t,x,y) \nearrow y.$$

(b) Of a cooperative type if

$$f(t,x,y) \nearrow x,\ f(t,x,y) \nearrow y;\ g(t,x,y) \nearrow x,\ g(t,x,y) \nearrow y.$$

(c) Of a prey-predator type if

$$f(t,x,y) \nearrow x,\ f(t,x,y) \searrow y;\ g(t,x,y) \nearrow x,\ g(t,x,y) \nearrow y.$$

Then, the authors gave some results about the existence of positive periodic solutions of just the competition and the cooperative types. In our case, it is clear that it contains the case (a), but one can give similar theorems to the theorem 2.7 for obtain the existence and uniqueness of positive almost periodic solutions of (1.1) in two other cases that generalize system (1.2) in the cases (b) and (c).

Example 3.6. Let us consider system (1.1) by setting

$$
\begin{gathered}
f(t,x,y)=\{1+\sin^2 2\pi\big(\sin\pi t+\sin(\sqrt{2}\pi t)\big)\}x\frac{x^2y+3}{x^2y+1},\quad h(t,x)=1\\
g(s,x,y)=\{1+\frac{1}{2}|\cos 2\pi\big(\sin\pi t+\sin\sqrt{(2\pi t)}\big)|\}y\frac{x^2y^3+4}{x^2y^3+2},\quad k(t,x)=1\\
\psi_1(\alpha,x,y)=\alpha\frac{\alpha x^2y+3}{\alpha x^2y+1}\frac{x^2y+1}{x^2y+3},\quad \psi_2(\alpha,x,y)=\alpha\frac{\alpha x^2y^3+4}{\alpha x^2y^3+2}\frac{x^2y^3+2}{x^2y^3+4},\\
\varphi_1(\alpha,x)=\varphi_2(\alpha,x)=1\quad\text{and}\quad \tau_1(t)=\frac{3+\frac{1}{2}\sin^2 t}{8},\quad \tau_2(t)=\frac{4+\frac{1}{3}\cos^2 t}{7}.
\end{gathered}
$$

for all $(t,x,y)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$ and for all $\alpha\in(0,1)$. Then, all hypotheses of theorem 3.4 are verified. Therefore, system (1.1) with the above functions f,g,h,k,τ_1 and τ_2 has a unique positive almost periodic solution.

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