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EXISTENCE OF INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENTIAL GROWTH

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Abstract. In this work we prove the existence of infinitely many nonradial solutions, that change sign, to the problem

$$\begin{gathered} -\Delta u = f(u) \quad \text{in } B \\ u = 0 \quad \text{on } \partial B, \end{gathered}$$

where B is the unit ball in $\mathbb{R}^2$ and f is a continuous and odd function with critical exponential growth.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function with $f(-t) = -f(t)$. Consider the problem

$$\begin{gathered} -\Delta u = f(u), \quad \text{in } \Omega, \\ \mathcal{B}u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

when $N \geq 4$, $\mathcal{B}u = u$ and $f(t) = |t|^{\frac{4}{N-2}} + \lambda t$, Brézis-Niremberg [5] proved that (1.1) admits a non-trivial positive solution, provided $0 < f'(0) < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Cerami-Solimini-Struwe [7] proved that if $N \geq 6$, problem (1.1) admits a solution with changes sign. Using this, they also proved that when $N \geq 7$ and Ω is a ball, (1.1) admits infinitely many radial solution which change sign.

Comte and Knaap [8] obtained infinitely many non-radial solutions that change sign for (1.1) on a ball with Neumann boundary condition $\mathcal{B}u = \frac{\partial u}{\partial \nu}$, for every $\lambda \in \mathbb{R}$. They obtained such solutions by cutting the unit ball into angular sectors. This approach was used by Cao-Han [6], where the authors dealt with the scalar problem (1.1) involving lower-order perturbation and by de Morais Filho et al. [11] to obtain multiplicity results for a class of critical elliptic systems related to the Brézis-Nirenberg problem with the Neumann boundary condition on a ball.

When $N = 2$, the notion of "critical growth" is not given by the Sobolev imbedding, but by the *Trudinger-Moser inequality* (see [14, 12]), which claims that for

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any $u \in H_0^1(\Omega)$,

$$\int_\Omega e^{\alpha u^2} dx < +\infty, \quad \text{for every } \alpha > 0. \tag{1.2}$$

Moreover, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$\sup_{||u||_{H_0^1(\Omega)} \le 1} \int_\Omega e^{\alpha u^2} dx \le C, \quad \text{for all } \alpha \le 4\pi. \tag{1.3}$$

Motivated by inequality in (1.3), we say that the nonlinearity f has critical exponential growth if f behaves like $e^{\alpha_0 s^2}$, as $|s| \to \infty$, for some $\alpha_0 > 0$. More precisely,

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0, \ \forall \alpha > \alpha_0 \quad \text{and} \quad \lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = +\infty, \ \forall \alpha < \alpha_0.$$

In this case, Adimurthi [1] proved that (1.1) admits a positive solution, provided that $\lim_{t\to\infty} tf(t)e^{\alpha t^2} = \infty$. Using a more weaker condition (see [1, Remark 4.2]), Adimurthi in [2] also proved the exitence of many solutions for the Dirichlet problem with critical exponential growth for the N-Laplacian. Adimurthi-Yadava [4], proved that (1.1) has a solution that changes sign and, when Ω is a ball in $\mathbb{R}^2$, (1.1) has infinitely many radial solutions that change sign. Inspired in [8], this paper is concerned with the existence of infinitely many non-radial sign changing solutions for (1.1) when f has critical exponential growth and Ω is a ball in $\mathbb{R}^2$. Our main result complements the studies made in [8, 11], because we consider the case where f has critical exponential growth in $\mathbb{R}^2$. It is important to notice that in both studies mentioned above was considered the Neumann boundary condition in order that the Pohozaev identity (see [13]) ensures that the problem (1.1) with the Dirichlet boundary condition, has no solutions for $\lambda < 0$ and $N \ge 3$. Since the Pohozaev identity is not available in dimension two, in our case we can use the Dirichlet boundary condition.

Here we use the following assumptions

(F1) There is $C > 0$ such that

$$|f(s)| \le Ce^{4\pi|s|^2}, \quad \text{for all } s \in \mathbb{R};$$

(F2) $\lim_{s\to 0} f(s)/s = 0$;

(H1) There are $s_0 > 0$ and $M > 0$ such that

$$0 < F(s) := \int_0^s f(t)dt \le M|f(s)| \quad \text{for all } |s| \ge s_0.$$

(H2) $0 < F(s) \le \frac{1}{2} f(s)s$ for all $s \in \mathbb{R} \setminus \{0\}$.

(H3) $\lim_{s\to\infty} sf(s)e^{-4\pi s^2} = +\infty$

Our main result reads as follows.

Theorem 1.1. *Let f be an odd and continuous function satisfying* (F1)–(F2) *and* (H1)–(H3). *Then* (1.1) *has infinitely many sign-changing solutions.*

2. Notation and auxiliary results

For each $m \in \mathbb{N}$, we define

$$A_m = \{x = (x_1, x_2) \in B : \cos\big(\frac{\pi}{2^m}\big)|x_1| < \sin\big(\frac{\pi}{2^m}\big)x_2\}.$$

So A_1 is a half-ball, A_2 an angular sector of angle $\pi/2$, and A_3 an angular sector of angle $\pi/4$ and so on; see Figure 1.

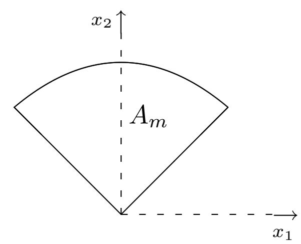

FIGURE 1. Angular sector A_m.

Using the above notation, we consider the auxiliary Dirichlet problem

$$\begin{gathered} -\Delta u = f(u), \quad \text{in } A_m, \\ u = 0, \quad \text{on } \partial A_m. \end{gathered} \tag{2.1}$$

We will use the Mountain Pass Theorem to obtain a positive solution of (2.1). Using this solution together with an anti-symmetric principle, we construct a sign-changing solution of problem (1.1).

According to Figueiredo, Miyagaki and Ruf [9], to obtain a positive solution of (2.1) it is sufficient to assume that the limit in (H3) satisfies

(H3') $\lim_{s\to+\infty} sf(s)e^{-4\pi s^2} \geq \beta > \frac{1}{2\pi d_m^2}$, where d_m is the radius of the largest open ball contained in A_m.

Hypothesis (H3) was initially considered in Adimurthi [1]. This hypothesis will be fundamental to ensure not only the existence but also the multiplicity of sign-changing solutions. As we will see bellow, assuming (H3) in place of (H3'), we have the existence of positive solution of (2.1), for every $m \in \mathbb{N}$. This is the content of the next result.

Theorem 2.1. *Under the assumptions* (F1)–(F2) *and* (H1)–(H3)*, problem* (2.1) *has a positive solution, for every* $m \in \mathbb{N}$.

3. Proof of Theorem 2.1

In what follows, for an open set $\Theta \subset \mathbb{R}^2$ we denote $L^q(\Theta)$ and $H_0^1(\Theta)$ norms by

$$|u|_{q,\Theta} = \Big(\int_\Theta |u|^q\Big)^{1/q}, \quad \|u\|_\Theta = \Big(\int_\Theta |\nabla u|^2\Big)^{1/2},$$

respectively. Since we are interested in positive solutions to (2.1), we assume that

$$f(s) = 0, \quad \text{for all } s \leq 0.$$

Associated with problem (2.1), we have the functional $I : H_0^1(A_m) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{A_m} |\nabla u|^2 - \int_{A_m} F(u).$$

In our case, ∂A_m is not of class C^1. However, the functional I is well defined. In fact, for each $u \in H_0^1(A_m)$, let us consider $u^* \in H_0^1(B)$ the zero extension of u in

B defined by

$$u^*(x) = \begin{cases} u(x), & \text{if } x \in A_m, \\ 0, & \text{if } x \in B \setminus A_m. \end{cases}$$

Clearly,

$$\|u\|_{A_m} = \|u^*\|_B.$$

Then, from (F1) and the Trudinger-Moser inequality (1.2)

$$\Big|\int_{A_m} F(u)\Big| = \Big|\int_B F(u^*)\Big| \le \int_B |F(u^*)| \le C\int_B e^{4\pi|u^*|^2} < \infty.$$

Moreover, using a standard argument we can prove that the functional I is of class C^1 with

$$I'(u)v = \int_{A_m} \nabla u \nabla v - \int_{A_m} f(u)v, \quad \text{for all } u, v \in H_0^1(A_m).$$

Therefore, critical points of I are precisely the weak solutions of (2.1). The next lemma ensures that the functional I has the mountain pass geometry.

Lemma 3.1. (a) *There are $r, \rho > 0$ such that $I(u) \ge \rho > 0$ for all $\|u\|_{A_m} = r$.*
(b) *There is $e \in H_0^1(A_m)$ such that $\|e\|_{A_m} > r$ and $I(e) < 0$.*

Proof. Using the definition of I and the growth of f, we obtain

$$I(u) \ge \frac{1}{2}\int_{A_m} |\nabla u|^2 - \frac{\epsilon}{2}\int_{A_m} |u|^2 - C\int_{A_m} |u|^q e^{\beta|u|^2},$$

or equivalently,

$$I(u) \ge \frac{1}{2}\int_B |\nabla u^*|^2 - \frac{\epsilon}{2}\int_B |u^*|^2 - C\int_B |u^*|^q e^{\beta|u^*|^2}.$$

By the Poincaré's inequality,

$$I(u) \ge \frac{1}{2}\int_B |\nabla u^*|^2 - \frac{\epsilon}{2\lambda_1}\int_B |\nabla u^*|^2 - C\int_B |u^*|^q e^{\beta|u^*|^2},$$

where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(B))$. Fixing $\epsilon > 0$ sufficiently small, we have $C_1 := \frac{1}{2} - \frac{\epsilon}{2\lambda_1} > 0$, from where it follows that

$$I(u) \ge C_1\int_B |\nabla u^*|^2 - C\int_B |u^*|^q e^{\beta|u^*|^2}.$$

Notice that, from Trudinger-Moser inequality (1.3), $e^{\beta|u^*|^2} \in L^2(B)$ and by continuous embedding $|u^*|^q \in L^2(B)$. Since $H_0^1(B) \hookrightarrow L^{2q}(B)$ for all $q \ge 1$, by Hölder's inequality

$$\begin{aligned} \int_B |u^*|^q e^{\beta|u^*|^2} &\le \Big(\int_B |u^*|^{2q}\Big)^{1/2}\Big(e^{2\beta|u^*|^2}\Big)^{1/2} \\ &\le |u^*|_{2q,B}^q \Big(\int_B e^{2\beta|u^*|^2}\Big)^{1/2} \\ &\le C\|u^*\|_B^q \Big(\int_B e^{2\beta|u^*|^2}\Big)^{1/2}. \end{aligned}$$

We claim that for $r > 0$ small enough, we have

$$\sup_{\|u^*\|_B = r} \int_B e^{2\beta|u^*|^2} < \infty.$$

In fact, note that

$$\int_B e^{2\beta|u^*|^2} = \int_B e^{2\beta\|u^*\|_B^2(\frac{|u^*|}{\|u^*\|_B})^2}.$$

Choosing $r > 0$ small enough such that $\alpha := 2\beta r^2 < 4\pi$ and using the Trudinger-Moser inequality (1.3),

$$\sup_{\|u^*\|_B=r} \int_B e^{2\beta|u^*|^2} \le \sup_{\|v\|_B\le 1} \int_B e^{\alpha|v|^2} < \infty.$$

Thus,

$$I(u) \ge C_1\|u^*\|_B^2 - C_2\|u^*\|_B^q.$$

Fixing $q > 2$, we derive

$$I(u) \ge C_1 r^2 - C_2 r^q := \rho > 0,$$

for $r = \|u\|_{A_m} = \|u^*\|_B$ small enough, which shows that the item (a) holds.

To prove (b), first notice that

Claim 1. For each $\epsilon > 0$, there exists $\overline{s}_\epsilon > 0$ such that

$$F(s) \le \epsilon f(s)s, \quad \text{for all } x \in A_m,\ |s| \ge \overline{s}_\epsilon.$$

In fact, from hypothesis (H1)

$$\Big|\frac{F(s)}{sf(s)}\Big| \le \frac{M}{|s|}, \quad \text{for all } |s| \ge s_0.$$

For $p > 2$, claim 1 with $\epsilon = 1/p > 0$, guarantees the existence of $\overline{s}_\epsilon > 0$ such that

$$pF(s) \le f(s)s, \quad \text{for all } s \ge \overline{s}_\epsilon,$$

which implies the existence of constants $C_1, C_2 > 0$ satisfying

$$F(s) \ge C_1|s|^p - C_2, \quad \text{for all } s \ge 0.$$

Thus, fixing $\varphi \in C_0^\infty(A_m)$ with $\varphi \ge 0$ and $\varphi \ne 0$. For $t \ge 0$, we have

$$\int_{A_m} F(t\varphi) \ge \int_{A_m} (C_1|t\varphi|^p - C_2) \ge C_1|t|^p \int_{A_m} |\varphi| - C_2|A_m|,$$

from where it follows that

$$\int_{A_m} F(t\varphi) \ge C_3|t|^p - C_4. \tag{3.1}$$

From (3.1), if $t \ge 0$,

$$I(t\varphi) \le \frac{t^2}{2}\|\varphi\|_{A_m}^2 - C_3|t|^p + C_4.$$

Since $p > 2$, $I(t\varphi) \to -\infty$ as $t \to +\infty$. Fixing t_0 large enough and let $e = t_0\varphi$, we obtain

$$\|e\|_{A_m} \ge r \quad \text{and} \quad I(e) < 0.$$

□

The next lemma is crucial for proving that the energy functional I satisfies the Palais-Smale condition and its proof can be found in [9].

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and (u_n) be a sequence of functions in $L^1(\Omega)$ such that u_n converging to $u \in L^1(\Omega)$ in $L^1(\Omega)$. Assume that $f(u_n(x))$ and $f(u(x))$ are also $L^1(\Omega)$ functions. If*

$$\int_\Omega |f(u_n)u_n| \leq C, \quad \text{for all } n \in \mathbb{N},$$

then $f(u_n)$ converges in $L^1(\Omega)$ to $f(u)$.

Lemma 3.3. *The functional I satisfies the $(PS)_d$ condition, for all $d \in (0, 1/2)$.*

Proof. Let $d < 1/2$ and (u_n) be a $(PS)_d$ sequence for the functional I; i.e.,

$$I(u_n) \to d \quad \text{and} \quad I'(u_n) \to 0, \quad \text{as } n \to +\infty.$$

For each $n \in \mathbb{N}$, let us define $\epsilon_n = \sup_{\|v\|_{A_m} \leq 1}\{|I'(u_n)v|\}$, then

$$|I'(u_n)v| \leq \epsilon_n \|v\|_{A_m},$$

for all $v \in H_0^1(A_m)$, where $\epsilon_n = o_n(1)$. Thus

$$\frac{1}{2}\int_{A_m} |\nabla u_n|^2 - \int_{A_m} F(u_n) = d + o_n(1), \quad \forall n \in \mathbb{N}, \tag{3.2}$$

$$\Big|\int_{A_m} \nabla u_n \nabla v - \int_{A_m} f(u_n)v\Big| \leq \epsilon_n \|v\|_{A_m}, \quad \text{for all } n \in \mathbb{N},\ v \in H_0^1(A_m). \tag{3.3}$$

From (3.2) and Claim 1, for any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2}\|u_n\|_{A_m}^2 = \frac{1}{2}\int_{A_m} |\nabla u_n|^2 \leq \epsilon + d + \int_{A_m} F(u_n) \leq C_\epsilon + \epsilon \int_{A_m} f(u_n)u_n,$$

for all $n \geq n_0$. Using (3.3) with $v = u_n$, we obtain

$$\big(\frac{1}{2} - \epsilon\big)\|u_n\|_{A_m}^2 \leq C_\epsilon + \epsilon\|u_n\|_{A_m}, \quad \text{for all } n \geq n_0.$$

Thus, the sequence (u_n) is bounded. Since $H_0^1(A_m)$ is a reflexive Banach space, there exits $u \in H_0^1(A_m)$ such that, for some subsequence,

$$u_n \rightharpoonup u \quad \text{in } H_0^1(A_m).$$

Furthermore, from compact embedding,

$$u_n \to u \quad \text{in } L^q(A_m),\ q \geq 1,$$
$$u_n(x) \to u(x) \quad \text{a.e. in } A_m.$$

On the other hand, using (3.3) with $v = u_n$, we obtain

$$-\epsilon_n \|u_n\|_{A_m} \leq \int_{A_m} |\nabla u_n|^2 - \int_{A_m} f(u_n)u_n,$$

which implies

$$\int_{A_m} f(u_n)u_n \leq \|u_n\|_{A_m}^2 - \epsilon_n\|u_n\|_{A_m} \leq C, \quad \text{for all } n \in \mathbb{N}.$$

From Lemma 3.2, $f(u_n) \to f(u)$ in $L^1(A_m)$. Then, there is $h \in L^1(A_m)$ such that

$$|f(u_n(x))| \leq h(x), \quad \text{a.e. in } A_m,$$

and from (H1), $|F(u_n)| \leq Mh(x)$, a.e. in A_m. Furthermore,

$$F(u_n(x)) \to F(u(x)) \quad \text{a.e. in } A_m.$$

Consequently, by the Lebesgue's dominated convergence,

$$\int_{A_m} F(u_n) - \int_{A_m} F(u) = o_n(1).$$

Thus, from (3.2),

$$\frac{1}{2}\|u_n\|_{A_m}^2 - \int_{A_m} F(u) - d = o_n(1),$$

which implies

$$\lim_{n\to\infty} \|u_n\|_{A_m}^2 = 2\Big(d + \int_{A_m} F(u)\Big). \tag{3.4}$$

Using again (3.3) with $v = u_n$, we obtain

$$\Big|\|u_n\|_{A_m}^2 - \int_{A_m} f(u_n)u_n\Big| \le o_n(1),$$

from where we derive

$$\begin{aligned}\Big|\int_{A_m} f(u_n)u_n - 2\Big(d + \int_{A_m} F(u)\Big)\Big| &\le \Big|\|u_n\|_{A_m}^2 - \int_{A_m} f(u_n)u_n\Big| \\ &\quad + \Big|\|u_n\|_{A_m}^2 - 2\Big(d + \int_{A_m} F(u)\Big)\Big|.\end{aligned}$$

Then

$$\lim_{n\to\infty} \int_{A_m} f(u_n)u_n = 2\Big(d + \int_{A_m} F(u)\Big).$$

Furthermore, from (H2),

$$2\int_{A_m} F(u) \le 2\lim_{n\to\infty} \int_{A_m} F(u_n) \le \lim_{n\to\infty} \int_{A_m} f(u_n)u_n = 2d + 2\int_{A_m} F(u),$$

which implies that $d \ge 0$.

Claim 2. For any $v \in H_0^1(A_m)$,

$$\int_{A_m} \nabla u \nabla v = \int_{A_m} f(u)v.$$

In fact, let us fix $v \in H_0^1(A_m)$ and notice that

$$\begin{aligned}&\Big|\int_{A_m} \nabla u\nabla v - \int_{A_m} f(u)v\Big| \\ &\le \Big|\int_{A_m} \nabla u_n\nabla v - \int_{A_m} \nabla u\nabla v\Big| + \Big|\int_{A_m} f(u_n)v - \int_{A_m} f(u)v\Big| \\ &\quad + \Big|\int_{A_m} \nabla u_n\nabla v - \int_{A_m} f(u_n)v\Big|.\end{aligned}$$

Using Lemma 3.2, the weak convergence $u_n \rightharpoonup u$ in $H_0^1(A_m)$ and the estimate in (3.3), we derive

$$\Big|\int_{A_m} \nabla u\nabla v - \int_{A_m} f(u)v\Big| \le o_n(1) + \|v\|_{A_m} o_n(1),$$

and the proof of Claim 2 is complete.

Note that from (H2) and Claim 2,

$$J(u) \ge \frac{1}{2}\int_{A_m} |\nabla u|^2 - \frac{1}{2}\int_{A_m} f(u)u = 0.$$

Now, We split the proof into three cases:

Case 1. The level $d = 0$. By the lower semicontinuity of the norm,

$$\|u\|_{A_m} \le \liminf_{n\to\infty} \|u_n\|_{A_m},$$

then

$$\frac{1}{2}\|u\|^2_{A_m} \le \frac{1}{2}\|u_n\|^2_{A_m}.$$

Using (3.4),

$$0 \le I(u) \le \frac{1}{2}\liminf \|u_n\|^2_{A_m} - \int_{A_m} F(u),$$

which implies

$$0 \le I(u) \le \int_{A_m} F(u) - \int_{A_m} F(u) = 0,$$

from where $I(u) = 0$, or equivalently,

$$\|u\|^2_{A_m} = 2\int_{A_m} F(u).$$

Using again (3.4), we derive

$$\|u_n\|^2_{A_m} - \|u\|^2_{A_m} = o_n(1),$$

since $H^1_0(A_m)$ be a Hilbert space, $u_n \to u$ in $H^1_0(A_m)$. Therefore, I satisfies the Palais-Smale at the level $d = 0$.

Case 2. The level $d \neq 0$ and the weak limit $u \equiv 0$. We will show that this can not occur for a Palais-Smale sequence.

Claim 3. There are $q > 1$ and a constant $C > 0$ such that

$$\int_{A_m} |f(u_n)|^q < C, \quad \text{for all } n \in \mathbb{N}.$$

In fact, from (3.4), for each $\epsilon > 0$,

$$\|u_n\|^2_{A_m} \le 2d + \epsilon, \quad \text{for all } n \ge n_0,$$

for some $n_0 \in \mathbb{N}$. Furthermore, from (F1),

$$\int_{A_m} |f(u_n)|^q \le C\int_{A_m} e^{4\pi q u_n^2} = C\int_B e^{4\pi\|u_n^*\|^2_B(\frac{u_n}{\|u_n^*\|_B})^2}.$$

By the Trudinger-Moser inequality (1.3), the last integral in the equality above is bounded if $4\pi q\|u_n^*\|^2_B < 4\pi$ and this occur if we take $q > 1$ sufficiently close to 1 and ϵ small enough, because $d < 1/2$, which proves the claim.

Then, using (3.3) with $v = u_n$, we obtain

$$\Big|\int_{A_m} |\nabla u_n|^2 - \int_{A_m} f(u_n)u_n\Big| \le \epsilon_n\|u_n\|_{A_m} \le \epsilon_n C, \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$\|u_n\|^2_{A_m} \le o_n(1) + \int_{A_m} f(u_n)u_n, \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Furthermore, from Hölder inequality, we can estimate the integral above as follows

$$\int_{A_m} f(u_n)u_n \le \Big(\int_{A_m} |f(u_n)|^q\Big)^{1/q}\Big(\int_{A_m} |u_n|^{q'}\Big)^{1/q'}, \quad \text{for all } n \in \mathbb{N},$$

and since $u_n \to 0$ in $L^{q'}(A_m)$, $\int_{A_m} f(u_n)u_n = o_n(1)$. Then, from (3.5),

$$\|u_n\|_{A_m}^2 \to 0, \quad \text{as } n \to \infty, \tag{3.6}$$

which contradicts (3.4), because

$$\|u_n\|_{A_m}^2 \to 2d \neq 0, \quad \text{as } n \to \infty,$$

proving that $d \neq 0$ and $u = 0$ does not occur.

Case 3. The level $d \neq 0$ and the weak limit $u \neq 0$. Since

$$I(u) = \frac{1}{2}\|u\|_{A_m}^2 - \int_{A_m} F(u) \leq \liminf_n \Big(\frac{1}{2}\|u_n\|_{A_m}^2 - \int_{A_m} F(u_n)\Big) = d,$$

we have $I(u) \leq d$.

Claim 4. $I(u) = d$. Suppose by contradiction that $I(u) < d$, from definition of I,

$$\|u\|v^2 < 2\Big(d + \int_{A_m} F(u)\Big). \tag{3.7}$$

On the other hand, if we consider the functions

$$v_n = \frac{u_n^*}{\|u_n^*\|_B},\ n \in \mathbb{N},$$
$$v = u^*\Big[2\Big(d + \int_B F(u^*)\Big)\Big]^{-1/2},$$

we have $\|v_n\|_B = 1$ e $\|v\|_B < 1$. Furthermore, since

$$\begin{aligned}\int_B \nabla v_n \nabla\varphi &= \|u_n\|_B^{-1}\int_{A_m} \nabla u_n \nabla\varphi \\ &\to \Big[2\Big(d + \int_B F(u^*)\Big)\Big]^{-1/2}\int_B \nabla u \nabla\varphi = \int_B \nabla v \nabla\varphi,\end{aligned}$$

for every $\varphi \in C_0^\infty(B)$, i.e.,

$$\int_B \nabla v_n \nabla\varphi - \int_B \nabla v \nabla\varphi = o_n(1),$$

we have $v_n \rightharpoonup v$ in $H_0^1(B)$.

Claim 3.4. *There are $q > 1$ and $n_0 \in \mathbb{N}$ such that*

$$\int_{A_m} |f(u_n)|^q < C, \quad \textit{for all } n \geq n_0.$$

To prove this claim, we need the following result due to Lions [10].

Proposition 3.5. *Let (u_n) be a sequence in $H_0^1(\Omega)$ such that $|\nabla u_n|_{2,\Omega} = 1$ for all $n \in \mathbb{N}$. Furthermore, suppose that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ with $|\nabla u|_{2,\Omega} < 1$. If $u \neq 0$, then for each $1 < p < \frac{1}{1-|\nabla u|_{2,\Omega}^2}$, we have*

$$\sup_{n\in\mathbb{N}} \int_\Omega e^{4\pi p u_n^2} < \infty.$$

From hypothesis (F1),

$$\int_{A_m} |f(u_n)|^q \leq C\int_{A_m} e^{4\pi q u_n^2} = C\int_B e^{4\pi q\|u_n^*\|_B^2 v_n^2}. \tag{3.8}$$

The last integral in the above expression is bounded. In fact, by Proposition 3.5, it is suffices to prove that there are $q, p > 1$ and $n_0 \in \mathbb{N}$ such that

$$q\|u_n^*\|_B^2 \le p < \frac{1}{1-\|v\|_B^2}, \quad \text{for all } n \ge n_0. \tag{3.9}$$

To prove that (3.9) occur, notice that $I(u) \ge 0$ and $d < 1/2$, which implies that

$$2 < \frac{1}{d - I(u)},$$

from where it follows that

$$2\Big(d + \int_B F(u^*)\Big) < \frac{d + \int_B F(u^*)}{d - I(u)} = \frac{1}{1-\|v\|_B^2}.$$

Thus, for $q > 1$ sufficiently close to 1,

$$2q\Big(d + \int_B F(u)\Big) < \frac{1}{1-\|v\|_B^2}.$$

From (3.4), there are $p > 1$ and $n_0 \in \mathbb{N}$ such that

$$q\|u_n^*\|_B^2 \le p < \frac{1}{1-\|v\|_B^2},$$

for all $n \ge n_0$ which implies that (3.9) occur. Therefore, Claim 3.4 holds.

Now, we will show that $u_n \to u$ in $H_0^1(A_m)$. First, notice that from Hölder inequality and (3.4),

$$\int_{A_m} f(u_n)(u_n - u) \le \int_{A_m} (|f(u_n)|^q)^{1/q} \Big(\int_{A_m} |u_n - u|^{q'}\Big)^{1/q'} \le C|u_n - u|_{q',A_m},$$

where $1/q + 1/q' = 1$. Since $u_n \to u$ in $L^{q'}(A_m)$,

$$\int_{A_m} f(u_n)(u_n - u) = o_n(1). \tag{3.10}$$

Using (3.3) with $v = u_n - u$ and (3.10), we obtain $\langle u_n - u, u_n\rangle = o_n(1)$, and since $u_n \rightharpoonup u$ in $H_0^1(A_m)$,

$$\|u_n - u\|_{A_m}^2 = \langle u_n - u, u_n\rangle - \langle u_n - u, u\rangle = o_n(1).$$

Then $\|u_n\|_{A_m}^2 \to \|u\|_{A_m}^2$ and this together with (3.4) contradicts (3.7). Which proves that $I(u) = d$, i.e.,

$$\|u\|_{A_m}^2 = 2\Big(d + \int_{A_m} F(u)\Big).$$

Furthermore, from (3.4), $\|u_n\|_{A_m} \to \|u\|_{A_m}$ as $n \to \infty$. Therefore

$$u_n \to u \text{ em } H_0^1(A_m).$$

□

From Lemma 3.1 and the Mountain pass Theorem without compactness conditions (see [15]), there is a $(PS)_{c_m}$ sequence $(u_n) \subset H_0^1(A_m)$ such that

$$I(u_n) \to c_m \quad textand \quad I'(u_n) \to 0,$$

where

$$c_m = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], H_0^1(A_m)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}.$$

To conclude the proof of existence of positive solution for (2.1), it remains to show that $c_m \in (-\infty, 1/2)$. For this, we introduce the following Moser's functions (see [12]):

$$\overline{w}_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\ln(n))^{1/2}, & 0 \le |x| \le 1/n \\ \frac{\ln(1/|x|)}{(\ln(n))^{1/2}}, & 1/n \le |x| \le 1 \\ 0, & |x| \ge 1 \end{cases}$$

Let $d_m > 0$ and $x_m \in A_m$ such that $B_{d_m}(x_m) \subset A_m$ and define

$$w_n(x) = \overline{w}_n\Big(\frac{x - x_m}{d_m}\Big),$$

we have $w_n \in H_0^1(A_m)$, $\|w_n\|_{A_m} = 1$ and $\operatorname{supp} w_n \subset B_{d_m}(x_m)$.

We claim that the exists $n \in \mathbb{N}$ such that $\max_{t\ge 0} I(tw_n) < \frac{1}{2}$. In fact, suppose by contradiction that this is not the case. Then, there exist $t_n > 0$ such that

$$\max_{t\ge 0} I(tw_n) = I(t_n w_n) \ge \frac{1}{2}. \tag{3.11}$$

It follows from (3.11) and (H1) that

$$t_n^2 \ge 1. \tag{3.12}$$

Furthermore, $\frac{d}{dt} I(tw_n)\,|_{t=t_n} = 0$, i.e.,

$$t_n^2 = \int_{A_m} f(t_n w_n) t_n w_n, \tag{3.13}$$

which implies that

$$t_n^2 \ge \int_{B_{d_m/n}(x_m)} f(t_n w_n) t_n w_n. \tag{3.14}$$

In what follows, we fix a positive constant β_m satisfying

$$\beta_m > \frac{1}{2\pi d_m^2}. \tag{3.15}$$

From (H3), there exists $s_m = s_m(\beta_m) > 0$ such that

$$f(s)s \ge \beta_m e^{4\pi s^2}, \text{ for all } s \ge s_m. \tag{3.16}$$

Using (3.16) in (3.14) and the definition of w_n in $B_{d_m/n}(0)$, we obtain

$$t_n^2 \ge \beta_m \pi \frac{d_m^2}{n^2} e^{2t_n^2 \ln(n)} \tag{3.17}$$

for n large enough, or equivalently,

$$t_n^2 \ge \beta_m \pi d_m^2 e^{2\ln(n)(t_n^2 - 1)}, \tag{3.18}$$

it implies that the sequence (t_n) is bounded. Moreover, from (3.18) and (3.12), $t_n^2 \to 1$ as $n \to \infty$. Now, let us define

$$C_n = \{x \in B_{d_m}(x_m) : t_n w_n(x) \ge s_m\}, \quad D_n = B_{d_m}(x_m) \setminus C_n.$$

With the above notations and using (3.13),

$$t_n^2 \ge \int_{B_{d_m/n}(x_m)} f(t_n w_n) t_n w_n = \int_{C_n} f(t_n w_n) t_n w_n + \int_{D_n} f(t_n w_n) t_n w_n$$

and by (3.16),

$$t_n^2 \geq \int_{D_n} f(t_n w_n) t_n w_n + \beta_m \int_{C_n} e^{4\pi t_n^2 w_n^2}$$

or equivalently,

$$t_n^2 \geq \int_{D_n} f(t_n w_n) t_n w_n + \beta_m \int_{B_{d_m}(x_m)} e^{4\pi t_n^2 w_n^2} - \beta_m \int_{D_n} e^{4\pi t_n^2 w_n^2}. \tag{3.19}$$

Notice that

$$\begin{gathered} w_n(x) \to 0 \quad \text{a.e. in } B_{d_m}(x_m), \\ \chi_{D_n}(x) \to 1 \text{ a.e. in } B_{d_m}(x_m), \\ e^{4\pi t_n^2 w_n^2} \chi_{D_n} \leq e^{4\pi t_n^2 s_m^2} \in L^1(B_{d_m}(x_m)). \end{gathered}$$

Then, by Lebesgue's dominated convergence

$$\lim_n \int_{D_n} e^{4\pi t_n^2 w_n^2} = \lim_n \int_{B_{d_m}(x_m)} e^{4\pi t_n^2 w_n^2} \chi_{D_n} = \int_{B_{d_m}(x_m)} 1 = \pi d_m^2. \tag{3.20}$$

Furthermore,

$$\begin{gathered} f(t_n w_n) t_n w_n \chi_{D_n} \leq C t_n w_n e^{4\pi t_n^2 w_n^2} \leq C s_m e^{4\pi s_m^2} \in L^1(B_{d_m}(x_m)), \\ f(t_n w_n(x)) t_n w_n(x) \chi_{D_n}(x) \to 0 \text{ a.e. in } B_{d_m}(x_m). \end{gathered}$$

Thus, using again Lebesgue's dominated convergence,

$$\lim_n \int_{D_n} f(t_n w_n) t_n w_n = 0 \tag{3.21}$$

Passing to the limit $n \to \infty$ in (3.19) and using (3.20) and (3.21),

$$1 \geq \beta_m \lim_n \int_{B_{d_m}(x_m)} e^{4\pi t_n^2 w_n^2} - \beta_m \pi d_m^2.$$

Since $t_n^2 \geq 1$, we obtain

$$1 \geq \beta_m \lim_n \Big[\int_{B_{d_m}(x_m)} e^{4\pi w_n^2} \Big] - \beta_m \pi d_m^2. \tag{3.22}$$

On the other hand, since

$$\int_{B_{d_m}(x_m)} e^{4\pi w_n^2} = d_m^2 \int_{B_1(0)} e^{4\pi \overline{w}_n^2} = d_m^2 \big\{ \frac{\pi}{n^2} e^{4\pi \frac{1}{2\pi} \ln(n)} + 2\pi \int_{1/n}^1 e^{4\pi \frac{1}{2\pi} \frac{[\ln(1/r)]^2}{\ln(n)}} r dr \big\},$$

making a changing of variables $s = \ln(1/r)/\ln(n)$,

$$\int_{B_{d_m}(x_m)} e^{4\pi w_n^2} = \pi d_m^2 + 2\pi d_m^2 \ln(n) \int_0^1 e^{2s^2 \ln(n) - 2s \ln(n)},$$

and since

$$\lim_{n\to\infty} \Big[2\ln(n) \int_0^1 e^{2\ln(n)(s^2 - s)} ds \Big] = 2,$$

we have

$$\lim_{n\to\infty} \int_{B_{d_m}(x_m)} e^{4\pi w_n^2} = \pi d_m^2 + 2\pi d_m^2 = 3\pi d_m^2.$$

Using the last limit in (3.22), we obtain

$$1 \geq 3\beta_m \pi d_m^2 - \beta \pi d_m^2 = 2\beta \pi d_m^2,$$

from where we derive

$$\beta_m \leq \frac{1}{2\pi d_m^2},$$

which contradicts the choice of β_m in (3.15). Then,

$$\max_{t\geq 0} I(tw_n) < \frac{1}{2},$$

proving that $c_m < 1/2$, for any $m \in \mathbb{N}$ fixed arbitrarily.

4. Proof of Theorem 1.1

We shall use the following proposition.

Proposition 4.1. *Let A be an angular sector contained on the positive half plane of $\mathbb{R}^2$ such that one of its boundary lies in x_1 axis, and denote such boundary of A by $B_0 = \{x = (x_1, x_2) \in A : \ x_2 = 0\}$. Consider A' the reflection A with respect to x_1 axis. Suppose that u is a solution of the problem*

$$\begin{gathered} -\Delta u = f(u), \quad \text{in } A, \\ u = 0, \quad \text{on } B_0, \end{gathered} \tag{4.1}$$

where f is a real, continuous and odd function. Then, the function $\tilde{u}$ such that $\tilde{u} = u$ in A and $\tilde{u}$ is antisymmetric with respect to x_1 axis,

$$\tilde{u}(x_1, x_2) = \begin{cases} u(x_1, x_2), & \text{in } A \\ -u(x_1, -x_2), & \text{in } A' \\ 0, & \text{on } B_0 \end{cases}$$

satisfies

$$-\Delta \tilde{u} = f(\tilde{u}) \quad \text{in } A \cup A'.$$

Proof. Since u is a solution of (4.1), we have

$$\int_A \nabla u \nabla \varphi = \int_A f(u)\varphi, \quad \text{for all } \varphi \in C_c^\infty(A).$$

We want to prove that

$$\int_{A\cup A'} \nabla \tilde{u} \nabla \phi = \int_{A\cup A'} f(\tilde{u})\phi, \quad \text{for all } \phi \in C_0^\infty(A \cup A').$$

For any $\phi \in C_0^\infty(A \cup A')$,

$$\int_{A\cup A'} f(\tilde{u})\phi = \int_A f(u(x_1, x_2))\phi(x_1, x_2) + \int_{A'} f(-u(x_1, -x_2))\phi(x_1, x_2).$$

Since f is an odd function,

$$\begin{aligned} \int_{A\cup A'} f(\tilde{u})\phi &= \int_A f(u(x_1, x_2))\phi(x_1, x_2) + \int_{A'} f(-u(x_1, -x_2))\phi(x_1, x_2) \\ &= \int_A f(u(x_1, x_2))\phi(x_1, x_2) - \int_{A'} f(u(x_1, -x_2))\phi(x_1, x_2) \\ &= \int_A f(u(x_1, x_2))\phi(x_1, x_2) - \int_A f(u(x_1, x_2))\phi(x_1, -x_2). \end{aligned}$$

Thus

$$\int_{A\cup A'} f(\tilde{u})\phi = \int_A f(u)\psi, \tag{4.2}$$

where $\psi(x_1,x_2)=\phi(x_1,x_2)-\phi(x_1,-x_2)$. On the other hand,

$$\begin{aligned}\int_{A\cup A'}\nabla\tilde u\nabla\phi &= \int_A \nabla u(x_1,x_2)\nabla\phi(x_1,x_2)-\int_{A'}\nabla u(x_1,-x_2)\nabla\phi(x_1,x_2)\\ &= \int_A \nabla u(x_1,x_2)\nabla\phi(x_1,x_2)-\int_A \nabla u(x_1,x_2)\nabla(\phi(x_1,-x_2))\\ &= \int_A \nabla u(x_1,x_2)\nabla(\phi(x_1,x_2)-\phi(x_1,-x_2)).\end{aligned}$$

Then

$$\int_{A\cup A'}\nabla\tilde u\nabla\phi=\int_A\nabla u\nabla\psi. \tag{4.3}$$

The function ψ does not in general belong to $C_0^\infty(A)$. Therefore, ψ can not be used as a function test (in the definition of weak solution on $H^1(A)$). On the other hand, if we consider the sequence of functions (η_k) in $C^\infty(\mathbb{R})$, defined by

$$\eta_k(t)=\eta(kt),\quad t\in\mathbb{R},\ k\in\mathbb{N},$$

where $\eta\in C^\infty(\mathbb{R})$ is a function such that

$$\eta(t)=\begin{cases}0, & \text{if } t<1/2,\\ 1, & \text{if } t>1.\end{cases}$$

Then

$$\varphi_k(x_1,x_2):=\eta_k(x_2)\psi(x_1,x_2)\in C_0^\infty(A),$$

which implies that

$$\int_A\nabla u\nabla\varphi_k=\int_A f(u)\varphi_k,\quad k\in\mathbb{N}. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we can conclude the proof, in view of the following limits

$$\int_A\nabla u\nabla\varphi_k\to\int_A\nabla u\nabla\psi, \tag{4.5}$$

$$\int_A f(u)\varphi_k\to\int_A f(u)\psi, \tag{4.6}$$

as $k\to\infty$. To see that (4.5) occur, notice that

$$\int_A\nabla u\nabla\varphi_k=\int_A\eta_k\nabla u\nabla\psi+\int_A\frac{\partial u}{\partial x_2}k\eta'(kx_2)\psi.$$

Clearly,

$$\int_A\eta_k\nabla u\nabla\psi\to\int_A\nabla u\nabla\psi,\quad\text{as } k\to\infty.$$

Therefore, it remains to show that

$$\int_A\frac{\partial u}{\partial x_2}k\eta'(kx_2)\psi\to 0\quad\text{as } k\to\infty. \tag{4.7}$$

In fact this occurs,

$$\Big|\int_A\frac{\partial u}{\partial x_2}k\eta'(kx_2)\psi\Big|\le kMC\int_{0<x_2<1/k}\Big|\frac{\partial u}{\partial x_2}\Big|x_2\le MC\int_{0<x_2<1/k}\Big|\frac{\partial u}{\partial x_2}\Big|,$$

where $C=\sup_{t\in[0,1]}|\eta'(t)|$ and $M>0$ is such that

$$|\psi(x_1,x_2)|\le M|x_2|,\quad\text{for all } (x_1,x_2)\in A\cup A',$$

and since

$$\int_{0<x_2<1/k} |\frac{\partial u}{\partial x_2}| \to 0, \quad \text{as } k \to \infty,$$

the limit in (4.7) occur. The item (4.6) is an immediately consequence of the Lebesgue's dominated convergence.

Now, for each $m \in \mathbb{N}$, we apply the Proposition 4.1 to the solution u of problem (2.1). Let A'_m be the reflection of A_m in one of its sides. On $A_m \cup A'_m$, wecan the define the function $\tilde{u}$ such that $\tilde{u} = u$ on A_m, and $\tilde{u}$ is antisymmetric with respect to the side of reflection. Now, let A''_m be the reflection of $A_m \cup A'_m$ in one of its sides and $\tilde{\tilde{u}}$ the function defined on $A_m \cup A'_m \cup A''_m$ such that $\tilde{\tilde{u}} = \tilde{u}$ on $A_m \cup A'_m$ and $\tilde{\tilde{u}}$ is antisymmetric with respect to the side of reflection. Repeating this procedure, after finite steps, we finally obtain a function defined on the whole unit ball B, denoted by u_m. Clearly, u_m satisfies the Dirichlet condition on the boundary ∂B. That is, u_m is a sign-changing solution of problem (1.1). Since for every $m \in \mathbb{N}$, problem (2.1) admits a positive solution, we conclude that there exist infinitely many sign-changing solutions, and the proof of Theorem 1.1 is complete. □

In Figure 2, we represent the signal of three solutions, corresponding to the cases $m = 1$, $m = 2$, and $m = 3$, respectively. The blue color represents the regions where the solutions are negative and the red color, the regions where the solutions are positive.

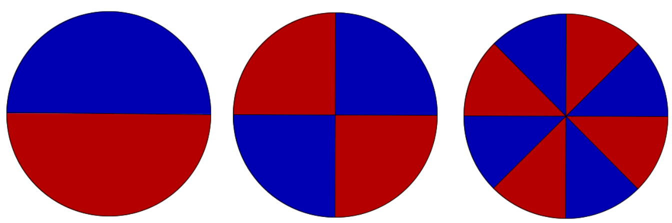

FIGURE 2. Signal of solutions

We show in Figure 3 the profile of solution for the case $m = 2$.

Remark 4.2. It is possible to make a version of Theorem 1.1 with Neumann boundary condition using the same arguments that we used here, but we have to work with another version of Trudinger-Moser inequality in $H^1(\Omega)$ due to Adimurthi-Yadava [3], which says that if Ω is a bounded domain with smooth boundary, then for any $u \in H^1(\Omega)$,

$$\int_\Omega e^{\alpha u^2} < +\infty, \quad \text{for all } \alpha > 0. \tag{4.8}$$

Furthermore, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$\sup_{||u||_{H^1(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2} \leq C, \quad \text{for all } \alpha \leq 2\pi. \tag{4.9}$$

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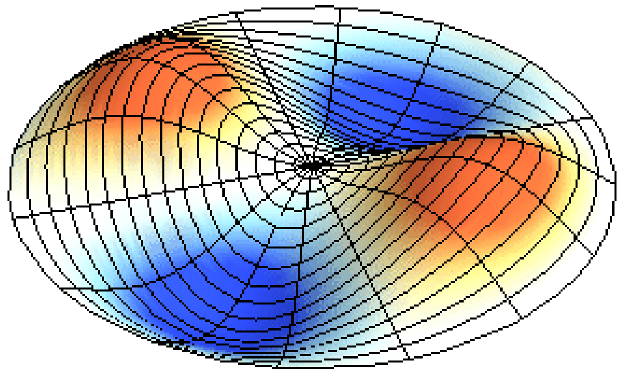

FIGURE 3. Case $m = 2$

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