Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 121, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NONLINEAR DAMPED SCHRÖDINGER EQUATION IN TWO SPACE DIMENSIONS

TAREK SAANOUNI

Abstract. In this article, we study the initial value problem for a semi-linear damped Schrödinger equation with exponential growth nonlinearity in two space dimensions. We show global well-posedness and exponential decay.

1. Introduction

Consider the initial value problem for a damped semilinear Schrödinger equation

$$\begin{gathered} i\dot{u} + \Delta u - \alpha u + \omega\Delta\dot{u} - \mu\dot{u} = \epsilon f(u), \\ u|_{t=0} = u_0, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{1.1}$$

This equation arises for instance in plasma physics [15] or in optical fibers models [4]. Here and hereafter $(\alpha, \mu, \omega) \in \mathbb{R}^3_+$ and $\epsilon \in \{\pm 1\}$. The set $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and $u(t, x) : \mathbb{R}_+ \times \Omega \to \mathbb{C}$. The nonlinearity f satisfies the Hamiltonian form $f(z) = zF'(|z|^2)$, where $F \in C^1(\mathbb{R}_+)$ and vanishes on zero. Moreover, we assume that for all $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$|f(z_1) - f(z_2)|^2 \le C_\alpha |z_1 - z_2|^2 \big(e^{\alpha|z_1|^2} - 1 + e^{\alpha|z_2|^2} - 1\big), \quad \forall z_1, z_2 \in \mathbb{C}. \tag{1.2}$$

We define the energy of a solution u to (1.1) by

$$E(t) = E_\alpha(u(t)) := \int_\Omega \Big(|\nabla u(t)|^2 + \alpha|u(t)|^2 + \epsilon F(|u(t)|^2)\Big)\, dx.$$

The decay of the energy formally satisfies

$$\dot{E}(t) = -\omega\|\nabla\dot{u}\|^2_{L^2} - \mu\|\dot{u}\|^2_{L^2}.$$

If $\epsilon = -1$, the energy is positive and (1.1) is said to be defocusing, otherwise it is focusing.

In the monomial case $f(u) = u|u|^{p-1}$, local well-posedness in the energy space holds for any $1 < p < \infty$ [8, 6]. Moreover, the solution is global if $1 < p < 3$ or in the defocusing case [5]. So it is natural to consider problems with exponential nonlinearities, which have several applications, as for example the self trapped beams in plasma [10]. Moreover, the Moser-Trudinger estimate [1] provides another

2010 *Mathematics Subject Classification.* 35Q55.
Key words and phrases. Nonlinear damped Schrödinger equation; existence; uniqueness; Moser-Trudinger inequality, decay.

Submitted November 28, 2014. Published April 30, 2015.

motivation to consider exponential type nonlinearity in order to study semilinear Schrödinger equation in two space dimensions.

The two dimensional Schrödinger problem with exponential growth nonlinearity was studied in [14], where global well-posedness and scattering were proved. Later on, the critical type nonlinearity was considered in [7]. In fact, global well-posedness for small data in the subcritical and critical cases holds. Moreover, scattering in the subcritical case was established. The author [18] obtained a decay result in the critical case. Recently [17], global well-posedness and scattering in the energy space without any condition on the data, for some weaker exponential nonlinearity, were proved (the associated wave problem was treated in [11, 12]).

It is the aim of this article is to extend previous results about global well-posedness of the classical Schrödinger problem in two space dimensions with exponential type nonlinearity to the damped case.

he rest of the article is organized as follows. The second section states the main results and gives some tools needed in the sequel. The third section deals with local well-posedness of (1.1). In the last section we prove global well-posedness of (1.1) in the focusing case and an exponential decay of the energy.

We mention that C will be used to denote a constant which may vary from line to line. We use $A \lesssim B$ to denote an estimate of the form $A \le CB$ for some absolute constant C. We denote Lebesgue space $L^p := L^p(\Omega)$ and Sobolev space $H_0^1 := H_0^1(\Omega)$ endowed with the complete norm $\|\cdot\|_{H_0^1} := \|\nabla \cdot\|_{L^2}$. Finally, if $T > 0$ and X is an abstract space, we denote $C_T(X) := C([0,T], X)$ and $L_T^p(X) := L^p([0,T], X)$.

2. Results and background

In this section, we give the main results of this paper and some technical tools needed in the sequel. For $u \in H_0^1$, we define the quantities

$$
\begin{gathered}
I_\alpha(u) := \int_\Omega \Big(|\nabla u|^2 + \alpha |u|^2 - \bar{u} f(u)\Big)\,dx; \\
m := \inf_{0 \ne u \in H_0^1} \{E(u),\, I(u) = 0\}, \quad N := \{0 \ne u \in H_0^1 : I(u) = 0\}; \\
N^+ := \{u \in H_0^1 : I(u) > 0\} \cup \{0\}; \\
(u,v)_* := \omega(\nabla u, \nabla v)_{L^2} + \mu(u,v)_{L^2}, \quad \|\cdot\|_*^2 := (u,u)_*.
\end{gathered}
$$

$E_T := C_T(H_0^1)$ endowed with the norm $\|\cdot\|_T := \|\cdot\|_{L_T^\infty(H_0^1)}$. If $u = u(t)$, we denote for simplicity $I(t) = I_\alpha(u(t))$. The first result is about the existence of a unique local solution to (1.1).

Theorem 2.1. *Assume that $\mu > 0$, the nonlinearity satisfies* (1.2), *and $u_0 \in H_0^1$. Then there exists $T > 0$ and a unique local solution to the Cauchy problem* (1.1), *in the energy space*

$$C([0,T], H_0^1).$$

Moreover,

1. *the solution satisfies decay of the energy;*
2. *the solution is global in the defocusing case.*

In the next result, we assume that the nonlinearity satisfies the supplementary condition: There exist $r_0, a > 0$ such that

$$F(r_0) > 0 \quad \text{and} \quad rf(r) \geq (1+a)F(r) \quad \text{for all} \quad r \in \mathbb{R}_+. \tag{2.1}$$

In the focusing case, we give a result of global existence and exponential decay.

Theorem 2.2. *Assume that $\epsilon = -1$, $\omega > 0$ and the nonlinearity satisfies* (1.2) *with* (2.1)*. Let $u_0 \in N^+$ such that $E(0) < m$. Then the solution u given by the previous result is global and satisfies*

1. *$u(t) \in N^+$ for any time;*
2. *for $0 < \alpha$ large enough, there exists $\gamma > 0$ such that*

$$0 < \|u(t)\|_{H_0^1} \lesssim e^{-\gamma t}, \quad \forall t \in \mathbb{R}_+.$$

Remark 2.3. The following function satisfies conditions of Theorem 2.2,

$$f(u) := \frac{1}{2} u(1+|u|^2)^{\frac{-1}{2}} \Big(\mathrm{e}^{(1+|u|^2)^{\frac{1}{2}}} - e(1+|u|^2)^{\frac{1}{2}} \Big).$$

Proof. We have $F(r) = e^{(1+r)^{\frac{1}{2}}} - \frac{e}{2}(r+2) = e^t - \frac{e}{2}(t^2+1)$, where $t := \sqrt{1+r}$. From direct computations, we have

$$rF'(r) = \frac{1}{2}(-1+t^2)(\frac{e^t}{t} - e);$$

$$\phi_a(t) := 2(rF'(r) - (1+a)F(r)) = (t - \frac{1}{t} - 2(1+a))e^t + ea(1+t^2) + 2e;$$

$$\phi_a'(t) = (t - \frac{1}{t} - 1 - 2a + \frac{1}{t^2})e^t + 2eat, \quad \phi_a(1) = 0 = \phi_a'(1);$$

$$\phi_a''(t) = (t - \frac{1}{t} + \frac{2}{t^2} - \frac{2}{t^3} - 2a)e^t + 2ea, \quad \phi_a''(1) = 0;$$

$$\phi_a'''(t) = (t - \frac{1}{t} + \frac{3}{t^2} - \frac{6}{t^3} + \frac{6}{t^4} + 1 - 2a)e^t, \quad \phi_a'''(1) = 2(2-a)e.$$

Now, taking $\phi_a'''(t) = (\psi(t) + 1 - 2a)e^t$, where $t^4\psi(t) = t^5 - t^3 + 3t^2 - 6t + 6 \geq 0$ for $t \geq 1$. Which implies that (2.1) is satisfied for any $a \in (0, 1/2)$. $\square$

In the two-dimensional space, we have the Sobolev injections [2],

$$H_0^1 \hookrightarrow L^p, \quad \text{for any} \quad 2 \leq p < \infty,$$

and it is false for $p = \infty$. The critical Sobolev embedding is described with the so called Orlicz space [3], which is given by the following Moser-Trudinger inequality [1, 13, 19].

Proposition 2.4. *Let $\alpha \in (0, 4\pi)$. Then there exists a constant $\mathcal{C}_\alpha$ such that for all $u \in H_0^1$ satisfying $\|\nabla u\|_{L^2} \leq 1$, one has*

$$\int_\Omega \Big(e^{\alpha |u(x)|^2} - 1 \Big)\, dx \leq \mathcal{C}_\alpha \|u\|_{L^2}^2.$$

Moreover,

1. *the above inequality is false when $\alpha \geq 4\pi$;*
2. *$\alpha = 4\pi$ becomes admissible if we consider $\|u\|_{H_0^1} \leq 1$ rather than $\|\nabla u\|_{L^2} \leq 1$. In this case, one has*

$$\sup_{\|u\|_{H_0^1} \leq 1} \int_\Omega e^{4\pi |u(x)|^2}\, dx < \infty$$

and this is false for $\alpha > 4\pi$ [1].

3. Proof of Theorem 2.1

We prove well-posedness of the Cauchy problem (1.1) in the energy space. We take in this section $\epsilon = 1$, in fact the sign of the nonlinearity has no local effect.

3.1. Local well-posedness.

Lemma 3.1. *Let $T > 0$, $u_0 \in H_0^1$ and $u \in C_T(H_0^1)$. Then there exists a unique $v \in E_T$ such that*

$$\begin{gathered} i\dot{v} + \Delta v - \alpha v + \omega\Delta\dot{v} - \mu\dot{v} = f(u) \quad \text{on } [0,T]\times\Omega, \\ v|_{t=0} = u_0, \\ v|_{\partial\Omega} = 0. \end{gathered} \tag{3.1}$$

Proof. Let $W_h := \langle w_1, \dots, w_h\rangle$, where $\{w_j\}$ is a complete system of eigenvectors of $-\Delta$ in H_0^1 such that $\|w_j\|_{L^2} = 1$. Then, $\{w_j\}$ is orthogonal and complete on L^2 and H_0^1. Denote the associated eigenvalues $\{\lambda_j\}$. Let

$$u_0^h := \sum_1^h \Re\Big(\int_\Omega \overline{\nabla u_0}\nabla w_j\Big) w_j.$$

Then, $u_0^h \in W_h$ and $u_0^h \to u_0$ in H_0^1. For $h \ge 1$, we seek for h functions $\gamma_1^h, \dots, \gamma_h^h$ in $C^2[0,T]$ such that $v_h(t) := \sum_{j=1}^h \gamma_j^h(t) w_j$ solves, for any $\eta \in W_h$, the problem

$$\begin{gathered} \int_\Omega \Big[i\dot{v}_h(t) + \Delta v_h(t) - \alpha v_h + \omega\Delta\dot{v}_h(t) - \mu\dot{v}_h(t) - f(u)\Big]\eta = 0, \\ v_h(0) = u_0^h. \end{gathered} \tag{3.2}$$

Taking $\eta = \bar{w}_j$ in (3.2), we obtain

$$\begin{gathered} (-i + \omega\lambda_j + \mu)\dot{\gamma}_j^h(t) + (\alpha + \lambda_j)\gamma_j^h(t) = -\int_\Omega f(u(t))\bar{w}_j\,dx, \\ \gamma_j^h(0) = \lambda_j \Re\Big(\int_\Omega \bar{u}_0 w_j\,dx\Big). \end{gathered}$$

Since $\int_\Omega f(u(t))w_j\,dx \in C[0,T]$, we have a unique solution γ_j^h to the previous problem. This yields to a solution v_h defined as above and satisfying (3.2). In particular, $v_h \in C^2([0,T], H_0^1)$. Taking $\eta = \overline{\dot{v}_h}$ in (3.2), yields

$$\begin{aligned} &\|\nabla v_h(t)\|_{L^2}^2 + \alpha\|v_h(t)\|_{L^2}^2 + 2\int_0^t \|\dot{v}_h(s)\|_*^2\,ds \\ &= \|\nabla u_0^h\|_{L^2}^2 + \alpha\|u_0^h\|_{L^2}^2 - 2\int_0^t \Re\Big(\int_\Omega \overline{f(u(s))}\dot{v}_h(s)\,dx\Big)\,ds. \end{aligned}$$

Now, by Moser-Trudinger inequality, via the identity $2|ab| \le \delta|a|^2 + \frac{1}{\delta}|b|^2$, for $\delta > 0$ near to zero, we have

$$\begin{aligned} 2\int_0^t \Re\Big(\int_\Omega \overline{f(u(s))}\dot{v}_h(s)\,dx\Big)\,ds &\le \frac{1}{\delta}\int_0^t\int_\Omega |f(u(s))|^2\,dx\,ds + \delta\int_0^t\int_\Omega |\dot{v}_h(s)|^2\,dx\,ds \\ &\le \frac{1}{\delta}C_T + \delta\int_0^t\int_\Omega |\dot{v}_h(s)|^2\,dx\,ds \end{aligned}$$

$$\leq C_T + \int_0^t \|\dot v_h(s)\|_*^2\,ds.$$

In fact, with Moser-Trudinger inequality, for any $0 < \alpha < \frac{4\pi}{\|u\|_T^2}$,

$$\begin{aligned}\int_\Omega |f(u(s))|^2\,dx &\leq C_\alpha \int_\Omega \Big(e^{\alpha\|u\|_T^2(\frac{|u(s)|}{\|u\|_T})^2} - 1\Big)\,dx\\ &\leq C_\alpha \int_\Omega |u(s)|^2\,dx \leq C_\alpha \|u\|_T^2 = C_T.\end{aligned}$$

Thus, $\|v_h\|_T^2 + \int_0^T \|\dot v_h(t)\|_*^2 \leq C_T$. So, $\{v_h\}$ is bounded in $H_0^1((0,T)\times\Omega)$. Then, taking the weak limit $v_h \rightharpoonup v$ in (3.2), we obtain a weak solution v to (3.1). Since $v \in H_0^1((0,T)\times\Omega)$, we obtain $v \in C([0,T], H_0^1(\Omega))$. The existence part of the Lemma is proved.

Now, for two solutions v_1, v_2 of (3.1) and $w := v_1 - v_2$, subtracting the equations and testing with $\overline{\dot w}$, we obtain

$$\|\nabla w(t)\|_{L^2}^2 + \alpha\|w(t)\|_{L^2}^2 + 2\int_0^t \|w(s)\|_*^2\,ds = 0.$$

The proof of Lemma 3.1 is complete. □

We are ready to prove local well-posedness of (1.1). We denote $R_0 := \|\nabla u_0\|_{L^2}$, and for $R > 0$ define the closed subset of the complete metric space E_T,

$$X_T := \{u \in E_T : \|u\|_T \leq R,\ u(0) = u_0\}.$$

Take the function $\phi(u) := v$, the solution to (3.1). We shall prove that, for some $T, R > 0$, ϕ is a contraction on X_T. Recall the identity

$$\begin{aligned}&\|\nabla v(t)\|_{L^2}^2 + \alpha\|v(t)\|_{L^2}^2 + 2\int_0^t \|\dot v(s)\|_*^2\,ds\\ &= \|\nabla u_0\|_{L^2}^2 + \alpha\|u_0\|_{L^2}^2 - 2\int_0^t \Re\Big(\int_\Omega \overline{f(u(s))}\dot v(s)\,dx\Big)\,ds.\end{aligned}$$

Moreover, for any $0 < \delta < \min\{\mu, 4\pi/R^2\}$, by Moser-Trudinger inequality

$$\begin{aligned}&2\int_0^t \int_\Omega |f(u(s))||\dot v(s)|\,ds\\ &\leq \frac{1}{\delta}\int_0^t \int_\Omega |f(u(s))|^2\,ds + \delta\int_0^t \int_\Omega |\dot v(s)|^2\,ds\\ &\leq \frac{C_\delta}{\delta}\int_0^t \int_\Omega \Big(e^{\delta|u|^2} - 1\Big)\,ds + \delta\int_0^t \int_\Omega |\dot v(s)|^2\,ds\\ &\leq \frac{C_\delta}{\delta}\int_0^t \int_\Omega |u(s)|^2\,ds + \delta\int_0^t \int_\Omega |\dot v(s)|^2\,ds\\ &\leq \frac{C_\delta}{\delta}TR^2 + \frac{\delta}{\mu}\int_0^t \|\dot v(s)\|_*^2\,ds \leq \frac{C_\delta}{\delta}TR^2 + \int_0^t \|\dot v_h(s)\|_*^2\,ds.\end{aligned}$$

This implies

$$\|\nabla v(t)\|_{L^2}^2 + \alpha\|v(t)\|_{L^2}^2 \leq C\alpha R_0^2 + \frac{C_\delta}{\delta}TR^2.$$

Taking $R^2 > 2C\alpha R_0^2$, yields

$$\|v\|_T^2 \leq \Big(\frac{1}{2} + \frac{C_\delta}{\delta}T\Big)R^2.$$

So $\phi(X_T) \subset X_T$ for small $T > 0$. Let prove that ϕ is contractive. Take $u_1, u_2 \in X_T$, $v_i := \phi(u_i)$, $v = v_1 - v_2$ and $u = u_1 - u_2$. Then, for any $\eta \in H_0^1$ and almost every $t \in [0, T]$,

$$\int_\Omega \Big(i\dot{v}\eta - \alpha v\eta - \nabla v\nabla\eta - \omega\nabla\dot{v}\nabla\eta - \mu\dot{v}\eta \Big)\, dx = \int_\Omega \Big(f(u_1) - f(u_2) \Big)\eta\, dx.$$

Taking the real part in the previous identity for $\eta = \overline{\dot{v}}$, via (1.2) yields, for any $\varepsilon > 0$,

$$\begin{aligned}
\|\nabla v(t)\|_{L^2}^2 &\le \|\nabla v(t)\|_{L^2}^2 + \alpha\|v(t)\|_{L^2}^2 + 2\int_0^t \|\dot{v}(s)\|_*^2\, ds \\
&= -2\int_0^t \Re\Big(\int_\Omega (f(u_1) - f(u_2))\overline{\dot{v}}\, dx \Big)\, ds \\
&\le \int_0^t \int_\Omega \Big[\frac{1}{\varepsilon}\Big| f(u_1) - f(u_2) \Big|^2 + \varepsilon|\dot{v}|^2 \Big]\, dx\, ds \\
&\le \varepsilon\int_0^t \|\dot{v}\|_{L^2}^2\, ds + \frac{1}{\varepsilon}\int_0^t \int_\Omega \Big| f(u_1) - f(u_2) \Big|^2\, dx\, ds \\
&\le \varepsilon\int_0^t \|\dot{v}\|_{L^2}^2\, ds + \frac{1}{\varepsilon}\int_0^t \Big[\int_\Omega |u|^2\Big(e^{\varepsilon|u_1|^2} - 1 + e^{\varepsilon|u_2|^2} - 1 \Big)\, dx \Big]\, ds.
\end{aligned}$$

Now, for $0 < \delta < \frac{\pi}{R^2}$, with Moser-Trudinger inequality via Sobolev embedding, we have

$$\begin{aligned}
\int_\Omega |u|^2\Big(e^{\delta|u_1|^2} - 1 + e^{\delta|u_2|^2} - 1 \Big)\, dx &\le \|u\|_{L^4}^2\Big(\|e^{\delta|u_1|^2} - 1\|_{L^2} + \|e^{\delta|u_2|^2} - 1\|_{L^2} \Big) \\
&\le \|u\|_T^2\Big(\|e^{2\delta|u_1|^2} - 1\|_{L^1}^{\frac{1}{2}} + \|e^{2\delta|u_2|^2} - 1\|_{L^1}^{\frac{1}{2}} \Big) \\
&\le C_\delta\|u\|_T^2\Big(\|u_1\|_{L^2} + \|u_2\|_{L^2} \Big) \lesssim R\|u\|_T^2.
\end{aligned}$$

Finally, taking $0 < \varepsilon < \min\{2, \frac{\pi}{R^2}\}$, yields

$$\|\phi(u_1) - \phi(u_2)\|_T \lesssim \sqrt{RT}\|u_1 - u_2\|_T.$$

Thus ϕ is a contraction of X_T for $T > 0$ small enough. With Picard Theorem, there exists a unique fixed point u which is a solution to (1.1). Uniqueness follows arguing as previously and applying the precedent inequality for two solutions to (1.1), which belong to X_T with a continuity argument for some $T > 0$ small enough.

3.2. **Global existence in the defocusing case.** We recall two important facts. First, the time of local existence depends only on the quantity $\|\nabla u_0\|_{L^2}$. Second the energy dominates the H_0^1 norm. Let u be the maximal solution of (1.1) in the space E_T for any $0 < T < T^*$ with initial data u_0, where $0 < T^* \le +\infty$ is the lifespan of u. We shall prove that u is global. By contradiction, suppose that $T^* < +\infty$, we consider for $0 < s < T^*$, the following problem

$$\begin{gathered}
i\dot{v} + \Delta v - \alpha v + \omega\Delta\dot{v} - \mu\dot{v} = f(v), \\
v(s, .) = u(s, .), \\
v|_{\partial\Omega} = 0.
\end{gathered} \tag{3.3}$$

By the same arguments used in the local existence and taking

$$0 < \delta \le \min\big\{ \mu, \frac{\pi}{E(0)} \big\},$$

we can find a real $\tau > 0$ and a solution v to (3.3) on $[s, s+\tau]$. According to the section of local existence and using decay of the energy, τ does not depend on s. Thus, if we let s be close to T^* such that $s+\tau > T^*$, we can extend v for times higher than T^*. This fact contradicts the maximality of T^*. We obtain the result claimed in Theorem 2.1.

4. Proof of Theorem 2.2

We are interested on the focusing case associated to the problem (1.1), so here and hereafter, we fix $\epsilon = -1$. By (2.1) we have [9], $m = J(\varphi) > 0$ where φ is the ground state solution of

$$-\Delta\varphi + \alpha\varphi = f(\varphi).$$

If there exists $t_0 > 0$ such that $u(t_0) \notin N^+$, then $I(t_0) \le 0$. With a continuity argument, there exists a time $t_1 \in (0, t_0)$ such that $I(t_1) = 0$ and $E(t_1) < m$ which contradicts the definition of m. Let us prove that u is global. For any real number $0 < \varepsilon < 1$,

$$\begin{aligned} E(t) &= \alpha\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 - \int_\Omega F(|u|^2)\,dx \\ &= \alpha\|u\|_{L^2}^2 + \varepsilon\|\nabla u\|_{L^2}^2 + (1-\varepsilon)\Big(I(t) - \alpha\|u\|_{L^2}^2 + \int_\Omega \bar{u}f(u)\,dx\Big) \\ &\quad - \int_\Omega F(|u|^2)\,dx \\ &\ge \alpha\varepsilon\|u\|_{L^2}^2 + \varepsilon\|\nabla u\|_{L^2}^2 + (1-\varepsilon)I(t) + ((1-\varepsilon)(1+a) - 1)\int_\Omega F(|u|^2)\,dx. \end{aligned}$$

Thus, using the fact that f satisfies (2.1), we have for any $0 < \varepsilon < \frac{a}{2+a}$,

$$E(0) \ge E(t) \ge \alpha\varepsilon\|u\|_{L^2}^2 + \varepsilon\|\nabla u\|_{L^2}^2 + \varepsilon\int_\Omega F(|u|^2)\,dx. \tag{4.1}$$

Thus $\|\nabla u(t)\|_{L^2}$ is bounded and u is global.

Now, we prove an exponential decay of the solution to (1.1). Note that since $u(t) \in N^+$ and f satisfies (2.1), we have $E \ge I > 0$. We denote, for some $0 < \varepsilon < \min\{\alpha, \frac{a}{2+a}\}$ (so satisfying (4.1)), the real function

$$L(t) := E(t) + \frac{\varepsilon\omega}{2}\int_\Omega |\nabla u(t)|^2\,dx.$$

By (4.1), we have $E \lesssim L \lesssim E$. Taking account of (1.1), we compute, for $0 < \varepsilon < \frac{\mu}{1+\mu}$,

$$\begin{aligned} \dot{L} &= \dot{E} - \varepsilon\Big(\alpha\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 - \int_\Omega \bar{u}f(u)\,dx + \int_\Omega [\mu\Re(\bar{u}\dot{u}) - \Im(\bar{u}\dot{u})]\,dx\Big) \\ &\le -\mu\|\dot{u}\|_{L^2}^2 - \varepsilon\Big(E + \int_\Omega F(|u|^2)\,dx - \int_\Omega \bar{u}f(u)\,dx + \int_\Omega [\mu\Re(\bar{u}\dot{u}) - \Im(\bar{u}\dot{u})]\,dx\Big) \\ &\le -\mu\|\dot{u}\|_{L^2}^2 - \varepsilon E + \varepsilon\int_\Omega |uf(u)|\,dx + \frac{\varepsilon}{2}(1+\mu)(\|\dot{u}\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\le -\varepsilon E + \varepsilon\int_\Omega |uf(u)|\,dx + \frac{\varepsilon}{2}(1+\mu)\|u\|_{L^2}^2. \end{aligned}$$

With (4.1), we have $E \geq \varepsilon\|u\|_{H_0^1}^2$, thus, using Moser-Trudinger inequality, for any $0 < \delta < \frac{4\pi\varepsilon}{E(0)}$,

$$\begin{aligned}\int_\Omega |uf(u)|\,dx &\leq C_\varepsilon \int_\Omega \Big(e^{\delta|u|^2} - 1\Big)\,dx \\ &\leq C_\varepsilon \int_\Omega \Big(e^{\delta\|u\|_{H_0^1}^2(\frac{|u|}{\|u\|_{H_0^1}})^2} - 1\Big)\,dx \\ &\leq C_\varepsilon\|u\|_{L^2}^2.\end{aligned}$$

So,

$$\dot{L}(t) \leq -\varepsilon\Big(E - (C_\varepsilon + \frac{1+\mu}{2})\|u\|_{L^2}^2\Big).$$

Now, also with (4.1), for $\alpha > \frac{2}{\varepsilon}[\frac{1+\mu}{2} + C_\varepsilon]$, we have

$$E - \big(\frac{1+\mu}{2} + C_\varepsilon\big)\|u\|_{L^2}^2 \geq \frac{\varepsilon}{2}E.$$

Finally, we conclude with a Gronwall argument via the inequalities

$$\dot{L}(t) \lesssim -E(t) \lesssim -L(t).$$

References

[1] S. Adachi and K. Tanaka; *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponent*, Proc. Amer. Math. Society, vol. 128 no. 7, (1999), 2051–2057.

[2] D. R. Adams; *Sobolev Spaces*. Academic Press, (1975).

[3] H. Bahouri, M. Majdoub, N. Masmoudi; *On the lack of compactness in the 2D critical Sobolev embedding*, Journal of Functional Analysis, vol 260, iss 1, (2011), 208–252.

[4] K. J. Blow, N. J. Doran; *Global and local chaos in the pumped nonlinear Schrödinger equation*, Phys. Rev. Lett., vol. 52, (1984), 526–529.

[5] T. Cazenave; *An introduction to nonlinear Schrödinger equations*, Textos de Metodos Matematicos 26, Instituto de Matematica UFRJ, (1996).

[6] T. Cazenave, F. B. Weissler; *Critical nonlinear Schrödinger equation*, Non. Anal. TMA, 14, (1990), 807–836.

[7] J. Colliander, S. Ibrahim, M. Majdoub, N. Masmoudi; *Energy critical NLS in two space dimensions*, J. Hyperbolic Differ. Equ., vol. 6 (2009), no. 3, 549–575.

[8] J. Ginibre, G. Velo; *Scattering theory in the energy space for a class of nonlinear schrödinger equations*, J. Math. Pures Appl. (9), vol. 64, no. 4, (1985), 363–401.

[9] L. Jeanjean, S. Lecoz; *Instability for standing waves of nonlinear Klein-Gordon equations via mountain-pass arguments*, Trans. Amer. Math. Soc., vol. 361, no 10, (2009), 5401–5416.

[10] J. F. Lam, B. Lippman, F. Trappert; *Self trapped laser beams in plasma*, Phys. Fluid, vol. 20, (1997), 1176–1179.

[11] O. Mahouachi, T. Saanouni; *Global well posedness and linearization of a semilinear wave equation with exponential growth*, Georgian Math. J, vol. 17, no. 3, (2010), 543–562.

[12] O. Mahouachi and T. Saanouni; *Well and ill-posedness issues for a class of 2D wave equation with exponential growth*, J. P. D. E, vol. 24, no. 4, (2011), 361–384.

[13] J. Moser; *A sharp form of an inequality of N. Trudinger*, Ind. Univ. Math. J., vol. 20, (1971), 1077–1092.

[14] M. Nakamura, T. Ozawa; *Nonlinear Schrödinger equations in the Sobolev Space of Critical Order*, J. Funct. Anal, vol. 155, (1998), 364–380.

[15] K. Nozaki, N. Bekki; *Low-dimensional chaos in a driven damped, nonlinear Schrdinger equation*, Physica D, vol. 21, (1986), 381–393.

[16] B. Ruf; *A sharp Moser-Trudinger type inequality for unbounded domains in $\mathbb{R}^2$*, J. Funct. Anal, vol. 219, (2004), 340–367.

[17] T. Saanouni; *Global well-posedness and scattering of a 2D schrödinger equation with exponential growth*, Bull. Belg. Math. Soc. Simon Stevin, vol. 17, (2010), 441–462.

[18] T. Saanouni; *Decay of solutions to a 2D Schrödinger equation with exponential growth*, J. Partial Differ. Equ., vol. 24 (2011), no. 1, 37–54.

[19] N. S. Trudinger; *On imbedding into Orlicz spaces and some applications*, J. Math. Mech., vol. 17, (1967), 473–484.

Tarek Saanouni

University Tunis El Manar, Faculty of Sciences of Tunis, LR03ES04 partial differential quations and applications, 2092 Tunis, Tunisia

E-mail address: `Tarek.saanouni@ipeiem.rnu.tn`