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MULTIPLE SOLUTIONS TO FOURTH-ORDER ELLIPTIC PROBLEMS WITH STEEP POTENTIAL WELL

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Abstract. In this article, we are concerned with a class of fourth-order elliptic equations with sublinear perturbation and steep potential well. By using variational methods, we obtain that such equations admit at least two nontrivial solutions. We also explore the phenomenon of concentration of solutions.

1. Introduction

We consider the fourth-order elliptic problem (P_λ),

$$\begin{gathered}\Delta^2 u-\Delta u+\lambda V(x)u=f(x,u)+\alpha(x)|u|^{\nu-2}u, \quad \text{in } \mathbb{R}^N,\\ u\in H^2(\mathbb{R}^N),\end{gathered}\tag{1.1}$$

where $N\geq 5$, $\lambda>0$ a parameter, $\Delta^2=\Delta(\Delta)$ is the biharmonic operator, $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, $\alpha(x)$ is a weight function, $1<\nu<2$, and the potential V satisfies the following conditions:

- (V1) $V\in C(\mathbb{R}^N)$ and $V\geq 0$ on $\mathbb{R}^N$;
- (V2) there exists $c>0$ such that the set $\{V<c\}=\{x\in\mathbb{R}^N|V(x)<c\}$ is nonempty and has finite measure;
- (V3) $\Omega=\mathrm{int}V^{-1}(0)$ is nonempty and has smooth boundary with $\bar\Omega=V^{-1}(0)$.

In view of the concrete applications of fourth-order differential equations in mathematical physics, such as nonlinear oscillation in suspension bridge or static deflection of an elastic plate in a fluid; see [5, 9], in recent years, a lot of attention has been focused on the study of the existence of nontrivial solutions for fourth-order equations; see, for example, [1, 3, 4, 7, 10, 11, 12, 13, 14, 18, 17, 19].

For the case of problem on the bounded domains, Zhang and Wei [19] studied the existence of infinitely many solutions for the following problem when the nonlinearity involves a combination of superlinear and asymptotically linear terms:

$$\begin{gathered}\Delta^2 u-c\Delta u=f(x,u), \quad \text{in } \Sigma,\\ \Delta u=u=0, \quad \text{in } \partial\Sigma,\end{gathered}\tag{1.2}$$

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where Σ is a bounded domain of $\mathbb{R}^N$. Hu and Wang [7] studied the existence of solution for (1.2) under the conditions

$$\lim_{t\to 0}\frac{f(x,t)}{t}=p(x),\quad \lim_{t\to\infty}\frac{f(x,t)}{t}=l$$

uniformly a.e. $x\in\Sigma$, where $0<l\le+\infty$, $0\le p(x)\in L^\infty(\Sigma)$ and $|p|_\infty<\Lambda_1,\Lambda_1$ is the first eigenvalue of $(\Delta^2-c\Delta, H^2(\Sigma)\cap H_0^1(\Sigma))$.

The case of problem on the unbounded domain has begun to attract much attention; see, for example, [3, 10, 14, 17, 18]. The main difficulty is the lack of compactness for Sobolev embedding theorem in this case. To overcome this difficulty, the potential V was generally assumed to satisfy on the following two conditions:

- (V0) $\inf_{x\in\mathbb{R}^N}V(x)\ge a>0$ and for each $M>0$, $\operatorname{meas}\{x\in\mathbb{R}^N: V(x)\le M\}<+\infty$, where a is a constant and meas denote the Lebesgue measure in $\mathbb{R}^N$;
- (V0') $\inf_{x\in\mathbb{R}^N}V(x)\ge a>0$ and $V(x)\to+\infty$ as $|x|\to\infty$.

Under condition (V0), Yin and Wu [18] proved that (1.1) with $\lambda=1$ and $\alpha=0$ has infinitely many high energy solutions by using the symmetric mountain pass theorem. When $N=1$, under condition (V0'), Sun and Wu [14] studied multiple homoclinic solutions for a class of fourth-order differential equations with a sublinear perturbation. It is worth to emphasize that the hypothesis (V0) or (V0') is used to guarantee the compact embedding of Sobolev space. However, if (V0) or (V0') is replaced by (V1)-(V2), then the compactness of the embedding fails, which will become more delicate. More recently, Liu et al. [10] studied this case. Ye and Tang [17] improved the results of [10] under the conditions that the nonlinearity f is either superlinear or sublinear at infinity.

On the other hand, conditions (V1)–(V3) imply that λV represents a deep potential well whose depth is controlled by λ, which are first introduced by Bartsch and Wang [2] in the study of solutions for Schrödinger equations. From then on, these conditions have extensively been applied in the study of the existence of solutions for several types of nonlinear equations; see [8, 15, 20].

Motivated by the above facts, in this article we study the multiplicity of nontrivial solutions for problem (1.1) with steep potential well. We consider the case that the nonlinearity is a combination of superlinear or asymptotically linear terms and a sublinear perturbation. As far as we know, this case seems to be rarely concerned in the literature. Our aim is to generalize the result of [14] to fourth-order elliptic problem. In addition, the results in [10, 17] is also improved by considering the different nonlinearity.

Notation. Throughout this article, we denote by $|\cdot|_r$ the L^r-norm, $1\le r\le\infty$, and we use the symbols $p^\pm=\sup\{\pm p,0\}$ and $2^{**}=\frac{2N}{N-4}$. Also if we take a subsequence $\{u_n\}$, we shall denote it again by $\{u_n\}$. We use $o(1)$ to denote any quantity which tends to zero when $n\to\infty$.

We need the following minimization problem for each positive $k\in[1,2^{**}-1)$,

$$\lambda_1^{(k)}=\inf\Big\{\Big(\int_\Omega(|\Delta u|^2+|\nabla u|^2)dx\Big)^{\frac{k+1}{2}}: u\in H^2(\Omega)\cap H_0^1(\Omega), \int_\Omega q|u|^{k+1}dx=1\Big\}, \tag{1.3}$$

where q is a bounded function on $\bar\Omega$ with $q^+\ne 0$. Then $\lambda_1^{(k)}>0$, which is achieved by some $\phi_k\in H^2(\Omega)\cap H_0^1(\Omega)$ with $\int_\Omega q|u|^{k+1}dx=1$ and $\phi_k>0$ a.e. in Ω, by

Fatou's Lemma and the compactness of Sobolev embedding from $H^2(\Omega)\cap H_0^1(\Omega)$ into $L^{k+1}(\Omega)$.

Now, we give our main result.

Theorem 1.1. *Suppose that* (V1)-(V3) *hold. In addition, for each* $k\in[1,2^{**}-1)$, *we assume that the function* f *and* α *satisfy the following conditions:*

- (A1) $\alpha\in L^{\frac{\nu}{2-\nu}}(\mathbb{R}^N)$ *and* $\alpha>0$ *on* Ω*;*
- (F1) $f\in C(\mathbb{R}^N\times\mathbb{R})$, $f(x,s)\equiv 0$ *for all* $s<0$ *and* $x\in\mathbb{R}^N$*. Moreover, there exists* $p\in L^\infty(\mathbb{R}^N)$ *with*
$$|p^+|_\infty<\Theta:=\frac{(S^{**})^2}{|\{V<c\}|^{\frac{2^{**}-2}{2^{**}}}}$$
such that
$$\lim_{s\to 0^+}\frac{f(x,s)}{s^k}=p(x)$$
uniformly in $x\in\mathbb{R}^N$ *and* $\frac{f(x,s)}{s^k}\ge p(x)$ *for all* $s>0$ *and* $x\in\bar\Omega$*, where* S^{**} *is the best constant for the embedding of* $D^{2,2}(\mathbb{R}^N)$ *in* $L^{2^{**}}(\mathbb{R}^N)$*,* $D^{2,2}(\mathbb{R}^N)$ *will be defined in Section 2, and* $|\cdot|$ *is the Lebesgue measure;*
- (F2) *there exists* $q\in L^\infty(\mathbb{R}^N)$ *with* $q^+\ne 0$ *on* $\bar\Omega$ *such that*
$$\lim_{s\to\infty}\frac{f(x,s)}{s^k}=q(x)$$
uniformly in $x\in\mathbb{R}^N$*;*
- (F3) *there exist constants* $\theta>2$ *and* d_0 *satisfying* $0\le d_0<\frac{(\theta-2)}{2\theta}\Theta$ *such that*
$$F(x,s)-\frac{1}{\theta}f(x,s)s\le d_0s^2$$
for all $s>0$ *and* $x\in\mathbb{R}^N$*.*

Then we have the following results:

- (i) *If* $k=1$ *and* $\lambda_1^{(1)}<1$*, then there exist* $M>0$ *and* $\Lambda>0$ *such that for every* $|\alpha^+|_{\frac{2}{2-\nu}}\in(0,M)$*, problem* (1.1) *has at least two nontrivial solutions for all* $\lambda>\Lambda$*.*
- (ii) *If* $k\in(1,2^{**}-1)$*, then there exist* $M>0$ *and* $\Lambda>0$ *such that for every* $|\alpha^+|_{\frac{2}{2-\nu}}\in(0,M)$*, problem* (1.1) *has at least two nontrivial solutions for all* $\lambda>\Lambda$*.*

On the concentration of solutions we have the following results.

Theorem 1.2. *Let* $u_\lambda^{(1)}$*,* $u_\lambda^{(2)}$ *be two solutions obtained by Theorem 1.1. Then for every* $r\in[2,2^{**})$*,* $u_\lambda^{(1)}\to u_0^1$ *and* $u_\lambda^{(2)}\to u_0^2$ *strongly in* $L^r(\mathbb{R}^N)$ *as* $\lambda\to\infty$*, where* $u_0^1,u_0^2\in H^2(\Omega)\cap H_0^1(\Omega)$ *are two nontrivial solutions of the problem*
$$\begin{gathered}\Delta^2u-\Delta u=f(x,u)+\alpha(x)|u|^{\nu-2}u,\quad \text{in }\Omega,\\ u=0\in\partial\Omega.\end{gathered}\tag{1.4}$$

The article is organized as follows. In Section 2, we present some preliminaries. In Section 3 and 4, we give the proof of our main results.

2. Preliminaries

Let $D^{2,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$\|u\|_{D^{2,2}} = \Big(\int_{\mathbb{R}^N} |\Delta u|^2 dx\Big)^{1/2}.$$

From [3, (1.7)], the embedding $D^{2,2}(\mathbb{R}^N) \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$ is continuous, one has

$$\|u\|_{2^{**}} \le (S^{**})^{-1}\Big(\int_{\mathbb{R}^N} |\Delta u|^2 dx\Big)^{1/2}, \quad \forall u \in D^{2,2}(\mathbb{R}^N). \tag{2.1}$$

Let

$$X = \big\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2(x)dx < +\infty\big\}.$$

Then X is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\Delta u\Delta v + \nabla u\nabla v)dx + \int_{\mathbb{R}^N} V(x)u(x)v(x)dx$$

and the corresponding norm $\|u\|^2 = \langle u, u\rangle$. Note that $X \subset H^2(\mathbb{R}^N)$ and $X \subset L^r(\mathbb{R}^N)$ for all $r \in [2, 2^{**}]$ with the embedding being continuous. For any $p \in [2, 2^{**})$, the embeddings $X \hookrightarrow L^p_{\rm loc}(\mathbb{R}^N)$ are compact. For $\lambda > 0$, we also need the inner product

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^N} (\Delta u\Delta v + \nabla u\nabla v)dx + \int_{\mathbb{R}^N} \lambda V(x)u(x)v(x)dx$$

and the corresponding norm $\|u\|^2_\lambda = \langle u, u\rangle_\lambda$. It is clear that $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Set $X_\lambda = (X, \|u\|_\lambda)$. From (V1)–(V2), Hölder and Sobolev inequalities (2.1), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2)dx \\
&= \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2)dx + \int_{\{V<c\}} u^2(x)dx + \int_{\{V\ge c\}} u^2(x)dx \\
&\le \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2)dx + \Big(\int_{\{V<c\}} 1dx\Big)^{\frac{2^{**}-2}{2^{**}}} \Big(\int_{\{V<c\}} |u|^{2^{**}} dx\Big)^{\frac{2}{2^{**}}} \\
&\quad + \frac{1}{c}\int_{\{V\ge c\}} V(x)u^2(x)dx \\
&\le \Big(1 + |\{V < c\}|^{\frac{2^{**}-2}{2^{**}}} (S^{**})^{-2}\Big) \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2)dx \\
&\quad + \frac{1}{c}\int_{\{V\ge c\}} V(x)u^2(x)dx \\
&\le \max\big\{1 + |\{V < c\}|^{\frac{2^{**}-2}{2^{**}}}, \frac{1}{c}\big\} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx,
\end{aligned} \tag{2.2}$$

which implies that the imbedding $X \hookrightarrow H^2(\mathbb{R}^N)$ is continuous. Moreover, using the same conditions and techniques, for any $r \in [2, 2^{**}]$, we also have

$$\int_{\mathbb{R}^N} |u|^r dx \le \Big(\max\big\{|\{V < c\}|^{\frac{2^{**}-2}{2^{**}}}, \frac{(S^{**})^2}{\lambda c}\big\}\Big)^{\frac{2^{**}-r}{2^{**}-2}} (S^{**})^{-r}\|u\|^r_\lambda. \tag{2.3}$$

This implies that for any $\lambda \geq \frac{(S^{**})^2}{c}|\{V<c\}|^{\frac{2-2^{**}}{2^{**}}}$,

$$\int_{\mathbb{R}^N} |u|^r dx \leq |\{V<c\}|^{\frac{2^{**}-r}{2^{**}}}(S^{**})^{-r}\|u\|_\lambda^r. \tag{2.4}$$

In particular, for any $\lambda \geq \frac{(S^{**})^2}{c}|\{V<c\}|^{\frac{2-2^{**}}{2^{**}}}$,

$$\int_{\mathbb{R}^N} |u|^2 dx \leq |\{V<c\}|^{\frac{2^{**}-2}{2^{**}}}(S^{**})^{-2}\|u\|_\lambda^2 = \frac{1}{\Theta}\|u\|_\lambda^2, \tag{2.5}$$

where Θ is defined by condition (F1).

It is well-known that (1.1) is a variational problem and its solutions are the critical points of the functional defined in X by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + \lambda V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx - \frac{1}{\nu}\int_{\mathbb{R}^N} \alpha(x)|u|^\nu dx. \tag{2.6}$$

Furthermore, the functional J_λ is of class C^1 in X, and that

$$\begin{aligned} J'_\lambda(u), v\rangle = &\int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v)dx + \int_{\mathbb{R}^N} \lambda V(x)uv\,dx \\ &- \int_{\mathbb{R}^N} f(x,u)v dx - \int_{\mathbb{R}^N} \alpha(x)|u|^{\nu-2}uv\,dx. \end{aligned} \tag{2.7}$$

Hence, if $u \in X$ is a critical point of J_λ, then u is a solution of problem (1.1). Moreover, we have the following results.

Lemma 2.1. *Suppose that* (V1)–(V3) *hold. In addition, for each $k \in [1, 2^{**}-1)$, we assume that f satisfies* (F3). *Then for each nontrivial solution u_λ of* (1.1)*, we have*

$$J_\lambda(u_\lambda) \geq K := -\big(1-\frac{\nu}{2}\big)\frac{(\theta-\nu)|\alpha^+|_{\frac{2}{2-\nu}}}{\nu\theta\Theta^{\frac{\nu}{2}}}\Big[\frac{(\theta-\nu)|\alpha^+|_{\frac{2}{2-\nu}}}{\Theta^{\frac{\nu}{2}-1}(\theta\Theta-2\Theta-2\theta d_0)}\Big]^{\frac{\nu}{2-\nu}}.$$

Proof. If u_λ is a nontrivial solution of (1.1), then

$$\int_{\mathbb{R}^N} (|\Delta u_\lambda|^2 + |\nabla u_\lambda|^2 + \lambda V(x)u_\lambda^2)dx = \int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda dx + \int_{\mathbb{R}^N} \alpha(x)|u_\lambda|^\nu dx.$$

Moreover, by (F3), we have

$$\int_{\mathbb{R}^N} [F(x,u_\lambda) - \frac{1}{\theta}f(x,u_\lambda)u_\lambda]dx \leq \int_{\mathbb{R}^N} d_0 u_\lambda^2 dx.$$

By (2.5), one has

$$\begin{aligned} J_\lambda(u_\lambda) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u_\lambda|^2 + |\nabla u_\lambda|^2 + \lambda V u_\lambda^2)dx \\ &\quad - \int_{\mathbb{R}^N} F(x,u_\lambda)dx - \frac{1}{\nu}\int_{\mathbb{R}^N} \alpha(x)|u_\lambda|^\nu dx \\ &\geq \frac{1}{2}\|u_\lambda\|_\lambda^2 - d_0\int_{\mathbb{R}^N} u_\lambda^2 dx - \frac{1}{\theta}\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda dx - \frac{1}{\nu}\int_{\mathbb{R}^N} \alpha(x)|u_\lambda|^\nu dx \\ &\geq \big(\frac{1}{2}-\frac{1}{\theta}\big)\|u_\lambda\|_\lambda^2 - d_0\int_{\mathbb{R}^N} u_\lambda^2 dx - \big(\frac{1}{\nu}-\frac{1}{\theta}\big)\int_{\mathbb{R}^N} \alpha(x)|u_\lambda|^\nu dx \\ &\geq \big(\frac{\theta-2}{2\theta}-\frac{d_0}{\Theta}\big)\|u_\lambda\|_\lambda^2 - \frac{(\theta-\nu)|\alpha^+|_{\frac{2}{2-\nu}}}{\nu\theta\Theta^{\frac{\nu}{2}}}\|u_\lambda\|_\lambda^\nu \geq K\,. \end{aligned} \tag{2.8}$$

□

Next, we give a useful theorem. It is the variant version of the mountain pass theorem, which allows us to find a so-called Cerami type (PS) sequence.

Theorem 2.2 ([6]). *Let E be a real Banach space and its dual space E^*. Suppose that $I \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho} I(u)$$

for some $\mu < \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $\hat{c} \ge \eta$ be characterized by

$$\hat{c} = \inf_{\gamma\in\Gamma} \max_{0\le\tau\le1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e. Then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to \hat{c} \ge \eta \quad \textit{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0, \textit{as} \quad n \to \infty.$$

In what follows, we give a lemma which ensures that the functional J_λ has mountain pass geometry.

Lemma 2.3. *Suppose that (V1)–(V2) hold. In addition, for each $k \in [1, 2^{**} - 1)$, we assume that the function f satisfies (F1)–(F2). Then there exist $M > 0$, $\rho > 0$ and $\eta > 0$ such that*

$$\inf\{J_\lambda(u) : u \in X_\lambda, \|u\|_\lambda = \rho\} > \eta$$

for all $\lambda \ge \frac{(S^{**})^2}{c}|\{V < c\}|^{\frac{2-2^{**}}{2^{**}}}$ and $|\alpha^+|_{\frac{2}{2-\nu}} < M$.

Proof. For any $\epsilon > 0$, from (F1)–(F2) there exists $C_\epsilon > 0$ such that

$$F(x,s) \le \frac{|p^+|_\infty + \epsilon}{2} s^2 + \frac{C_\epsilon}{r}|s|^r, \quad \forall s \in \mathbb{R}, \tag{2.9}$$

where $\max\{2, k+1\} < r < 2^{**}$. Then by (2.5) and Sobolev inequality (2.1), for every $u \in X_\lambda$ and $\lambda \ge \frac{(S^{**})^2}{c}|\{V < c\}|^{\frac{2-2^{**}}{2^{**}}}$, we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + \lambda V u^2)dx \\
&\quad - \int_{\mathbb{R}^N} F(x,u)dx - \frac{1}{\nu}\int_{\mathbb{R}^N} \alpha(x)|u|^\nu dx \\
&\ge \frac{1}{2}\|u\|_\lambda^2 - \frac{|p^+|_\infty + \epsilon}{2}\int_{\mathbb{R}^N} u^2 dx \\
&\quad - \frac{C_\epsilon}{r}\int_{\mathbb{R}^N} u^r dx - \frac{1}{\nu}\int_{\mathbb{R}^N} \alpha(x)|u|^\nu dx \\
&\ge \frac{1}{2}\Big(1 - \frac{(|p^+|_\infty + \epsilon)|\{V < c\}|^{\frac{2^{**}-2}{2^{**}}}}{(S^{**})^2}\Big)\|u\|_\lambda^2 \\
&\quad - \frac{C_\epsilon\{V < c\}|^{\frac{2^{**}-r}{2^{**}}}}{r(S^{**})^r}\|u\|_\lambda^r - \frac{|\alpha^+|_{\frac{2}{2-\nu}}}{\Theta^{\frac{\nu}{2}}}\|u\|_\lambda^\nu \\
&:= \frac{1}{2}\Big(1 - \frac{(|p^+|_\infty + \epsilon)|\{V < c\}|^{\frac{2^{*}-2}{2^{*}}}}{(S^{**})^2}\Big)(\|u\|_\lambda^2 - A\|u\|_\lambda^\nu - B\|u\|_\lambda^r).
\end{aligned} \tag{2.10}$$

Therefore, by (F1) and [16, Lemma 3.1], fixing $\epsilon \in (0, \Theta - |p^+|_\infty)$, we have that there exist $t_B > 0, M > 0$ such that, for $\|u\|_\lambda = t_B > 0$,

$$J_{\lambda,a}(u) \geq \frac{1}{2}\Big(1 - \frac{(|p^+|_\infty + \epsilon)|\{V < c\}|^{\frac{2^{**}-2}{2^{**}}}}{(S^{**})^2}\Big)\Psi_{A,B}(t_B) > 0$$

provided that

$$|\alpha^+|_{\frac{2}{2-\nu}} < M,$$

where $\Psi_{A,B}(t) = t^2 - At^\nu - Bt^r, A, B > 0$. It is easy to see that there is $\eta > 0$ such that this lemma holds. $\square$

Lemma 2.4. *Suppose that* (V1)–(V3) *hold. In addition, for each $k \in [1, 2^{**} - 1)$, we assume that the function f satisfies* (F1)–(F2). *Let $\rho > 0$ be as in Lemma 2.3, then we have the following results:*

(i) *If $k = 1$ and $\lambda_1^{(1)} < 1$, then there exists $e \in X$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,a}(e) < 0$ for every $\lambda > 0$.*
(ii) *If $k \in (1, 2^{**}-1)$, then there exists $e \in X$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,a}(e) < 0$ for every $\lambda > 0$.*

Proof. (i) Since $\lambda_1^{(1)} < 1$ and $\nu < 2$, from (V3), (F1)–(F2) and Fatou's Lemma it follows that

$$\begin{aligned}
\lim_{t\to+\infty} \frac{J_\lambda(t\phi_1)}{t^2} &= \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta\phi_1|^2 + |\nabla\phi_1|^2 + \lambda V\phi_1^2)dx - \lim_{t\to+\infty}\int_{\mathbb{R}^N} \frac{F(x, t\phi_1)}{t^2\phi_1}\phi_1 dx \\
&\leq \frac{1}{2}\int_\Omega (|\Delta\phi_1|^2 + |\nabla\phi_1|^2)dx - \frac{1}{2}\int_\Omega q|\phi_1|^2 dx \\
&\leq \frac{1}{2}\Big(1 - \frac{1}{\lambda_1^{(1)}}\Big)\int_\Omega (|\Delta\phi_1|^2 + |\nabla\phi_1|^2)dx < 0,
\end{aligned}$$

where ϕ_1 is defined in the minimum problem (1.3). Thus, $J_\lambda(t\phi_1) \to -\infty$ as $t \to +\infty$. Hence, there exists $e \in X$ with $\|e\|_\lambda > \rho$ such that $J_\lambda(e) < 0$.

(ii) By (F2), $k > 1$, $\nu < 2$ and Fatou's Lemma, we have

$$\begin{aligned}
\lim_{t\to+\infty} \frac{J_\lambda(t\phi_k)}{t^{k+1}} &= -\lim_{t\to+\infty}\int_{\mathbb{R}^N} \frac{F(x, t\phi_k)}{t^{k+1}\phi_k}\phi_k dx \\
&\leq -\frac{1}{k+1}\int_\Omega q|\phi_k|^{k+1}dx \\
&= -\frac{1}{k+1} < 0,
\end{aligned}$$

where ϕ_k is defined in minimizing problem (1.3). Thus, $J_\lambda(t\phi_k) \to -\infty$ as $t \to +\infty$. Hence, there exists $e \in X$ with $\|e\|_\lambda > \rho$ such that $J_\lambda(e) < 0$. $\square$

3. Proof of Theorem 1.1

First we define

$$\alpha_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{0\leq t\leq 1} J_\lambda(\gamma(t)),$$
$$\alpha_0(\Omega) = \inf_{\gamma\in\bar{\Gamma}_\lambda(\Omega)} \max_{0\leq t\leq 1} J_\lambda|_{H^2(\Omega)\cap H_0^1(\Omega)}(\gamma(t)),$$

where $J_\lambda|_{H^2(\Omega)\cap H_0^1(\Omega)}$ is a restriction of J_λ on $H^2(\Omega) \cap H_0^1(\Omega)$,

$$\Gamma_\lambda = \{\gamma \in C([0, 1], X_\lambda) : \gamma(0) = 0, \gamma(1) = e\},$$

$$\bar{\Gamma}_\lambda(\Omega) = \{\gamma \in C([0,1], H^2(\Omega) \cap H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Note that

$$J_\lambda|_{H^2(\Omega)\cap H_0^1(\Omega)}(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 + |\nabla u|^2)dx - \int_\Omega F(x,u)dx - \frac{1}{\nu}\int_\Omega \alpha(x)|u|^\nu dx,$$

and $\alpha_0(\Omega)$ is independent of λ. Moreover, if (F1)–(F3) hold, similar to the proofs of Lemmas 2.3 and 2.4, we can conclude that $J_\lambda|_{H^2(\Omega)\cap H_0^1(\Omega)}$ also satisfies the mountain pass hypothesis in Theorem 2.2. Note that $H^2(\Omega) \cap H_0^1(\Omega) \subset X_\lambda$ for all $\lambda > 0$, then $0 < \eta \le \alpha_\lambda \le \alpha_0$ for all $\lambda \ge \frac{(S^{**})^2}{c}|\{V < c\}|^{\frac{2-2^{**}}{2^{**}}}$. Then for each $k \in [1, 2^{**} - 1)$, we can take a positive number D such that $0 < \eta \le \alpha_\lambda \le \alpha_0 < D$ for all $\lambda \ge \frac{(S^{**})^2}{c}|\{V < c\}|^{\frac{2-2^{**}}{2^{**}}}$. Thus, by Lemmas 2.3 and 2.4 and Theorem 2.2, we obtain that for each $\lambda \ge \frac{(S^{**})^2}{c}|\{V < c\}|^{\frac{2-2^{**}}{2^{**}}}$, there exists $\{u_n\} \subset X_\lambda$ such that

$$J_\lambda(u_n) \to \alpha_\lambda > 0 \quad \text{and} \quad (1 + \|u_n\|)\|J'_\lambda(u_n)\|_{X_\lambda^{-1}} \to 0, \quad \text{as } n \to \infty, \tag{3.1}$$

where $0 < \eta \le \alpha_\lambda \le \alpha_0 < D$.

Lemma 3.1. *Suppose that* (V1)–(V3) *hold. In addition, for each* $k \in [1, 2^{**} - 1)$, *we assume that* f *satisfies* (F1)-(F3). *Then for each* $\lambda \ge \frac{(S^{**})^2}{c}|\{V < c\}|^{\frac{2-2^{**}}{2^{**}}}$ *and* $\{u_n\}$ *defined by* (3.1), *we have that* $\{u_n\}$ *is bounded in* X_λ.

Proof. For n large enough, by (F3) and (2.2), we have

$$\begin{aligned}
\alpha_\lambda + 1 &\ge J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\
&= (\frac{1}{2} - \frac{1}{\theta})\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N}[\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)]dx \\
&\quad + (\frac{1}{\theta} - \frac{1}{\nu})\int_{\mathbb{R}^N}\alpha(x)|u_n|^\nu dx \\
&\ge (\frac{1}{2} - \frac{1}{\theta})\|u_n\|_\lambda^2 - d_0\int_{\mathbb{R}^N}u_n^2 dx + (\frac{1}{\theta} - \frac{1}{\nu})\int_{\mathbb{R}^N}\alpha(x)|u_n|^\nu dx \\
&\ge (\frac{1}{2} - \frac{1}{\theta} - \frac{d_0}{\Theta})\|u_n\|_\lambda^2 - (\frac{1}{\nu} - \frac{1}{\theta})\frac{|\alpha^+|_{\frac{2}{2-\nu}}}{\Theta^{\frac{\nu}{2}}}\|u_n\|_\lambda^\nu,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in X_λ. □

Next, we shall investigate the compactness conditions for the functional J_λ. Recall that a C^1 functional I satisfies Cerami condition at level c ($(C)_c$-condition for short) if any sequence $\{u_n\} \in E$ and $I(u_n) \to c$ and $(1 + u_n)\|I'(u_n)\|_{E^*} \to 0$ has a convergent subsequence, and such sequence is called a $(C)_c$-sequence.

Proposition 3.2. *Suppose that* (V1)–(V3) *hold. In addition, for each* $k \in [1, 2^{**} - 1)$, *we assume that the function* f *satisfies* (F1)-(F3). *Then for each* $D \ge 0$, *there exists* $\bar{\Lambda}_0 = \Lambda(D) \ge \frac{2\theta d_0}{c(\theta-2)} > 0$ *such that* J_λ *satisfies the* $(C)_\alpha-$*condition in* X_λ *for all* $\alpha < D$ *and* $\lambda > \bar{\Lambda}_0$.

Proof. Let $\{u_n\}$ be a sequence with $\alpha < D$. Then, by Lemma 3.1, $\{u_n\}$ is bounded in X_λ. Therefore, there exist a subsequence $\{u_n\}$ and u_0 in X_λ such that

$$\begin{gathered}
u_n \rightharpoonup u_0 \quad \text{weakly in } X_\lambda; \\
u_n \to u_0 \quad \text{strongly in } L^r_{\rm loc}(\mathbb{R}^N), \quad \text{for } 2 \le r < 2^{**}.
\end{gathered} \tag{3.2}$$

Moreover, $J'_\lambda(u_0)=0$. Now we show that $u_n\to u_0$ strongly in X_λ. Let $v_n=u_n-u_0$. By $\alpha\in L^{\frac{2}{2-\nu}}(\mathbb{R}^N)$ and (3.2), we have

$$\int_{\mathbb{R}^N}\alpha(x)|u|^\nu dx\to 0. \tag{3.3}$$

From (V2) it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} v_n^2dx &= \int_{\{V\ge c\}} v_n^2dx+\int_{\{V<c\}} v_n^2dx\\
&\le \frac{1}{\lambda c}\int_{\{V\ge c\}}\lambda V v_n^2dx+\int_{\{V<c\}} v_n^2dx\\
&\le \frac{1}{\lambda c}\int_{\mathbb{R}^N}\lambda V v_n^2dx+o(1)\\
&= \frac{1}{\lambda c}\|v_n\|_\lambda^2+o(1)
\end{aligned}\tag{3.4}$$

Then, by Hölder and Sobolev inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^N}|v_n|^rdx &\le \Big(\int_{\mathbb{R}^N}v_n^2dx\Big)^{\frac{2^{**}-r}{2^{**}-2}}\Big(\int_{\mathbb{R}^N}|v_n|^{2^{**}}dx\Big)^{\frac{r-2}{2^{**}-2}}\\
&\le \Big(\int_{\mathbb{R}^N}v_n^2dx\Big)^{\frac{2^{**}-r}{2^{**}-2}}\Big[(S^{**})^{-2^{**}}\Big(\int_{\mathbb{R}^N}|\Delta v_n|^2dx\Big)^{2^{**}/2}\Big]^{\frac{r-2}{2^{**}-2}}\\
&\le \Big(\frac{1}{\lambda c}\Big)^{\frac{2^{**}-r}{2^{**}-2}}(S^{**})^{-\frac{2^{**}(r-2)}{2^{**}-2}}\|v_n\|_\lambda^r+o(1).
\end{aligned}\tag{3.5}$$

Moreover, by (F1)-(F2) and Brezis-Lieb Lemma, we have

$$J_\lambda(v_n)=J_\lambda(u_n)-J_\lambda(u_0)+o(1)\quad\text{and}\quad J'_\lambda(v_n)=o(1).$$

Consequently, from this with (F3), (3.2) and Lemma 2.1, we obtain

$$\begin{aligned}
D-K &\ge \alpha-J_\lambda(u_0)\\
&\ge J_\lambda(v_n)-\frac{1}{\theta}\langle J'_\lambda(v_n),v_n\rangle+o(1)\\
&= \frac{(\theta-2)}{2\theta}\int_{\mathbb{R}^N}(|\Delta v_n|^2+|\nabla v_n|^2+\lambda V v_n^2)dx\\
&\quad +\int_{\mathbb{R}^N}\Big(\frac{1}{\theta}f(x,v_n)v_n-F(x,v_n)\Big)dx+o(1)\\
&\ge \frac{(\theta-2)}{2\theta}\|v_n\|_\lambda^2-d_0\int_{\mathbb{R}^N}v_n^2dx+o(1)\\
&\ge \Big(\frac{\theta-2}{2\theta}-\frac{d_0}{\lambda c}\Big)\|v_n\|_\lambda^2+o(1),
\end{aligned}$$

which implies that for every $\lambda>\frac{2\theta d_0}{c(\theta-2)}$, one has

$$\|v_n\|_\lambda^2\le\frac{2\theta\lambda c(D-K)}{(\theta-2)c\lambda-2\theta d_0}+o(1). \tag{3.6}$$

By (2.4), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^r dx &\le \frac{|\{V<c\}|^{\frac{2^{**}-r}{2^{**}}}}{(S^{**})^r}\|u\|_\lambda^r \\
&\le \frac{|\{V<c\}|^{\frac{2^{**}-r}{2^{**}}}}{(S^{**})^r}\Big(\frac{2\theta\lambda c(D-K)}{(\theta-2)c\lambda-2\theta d_0}\Big)^{r/2}+o(1).
\end{aligned} \tag{3.7}
$$

Since $\langle J'_{\lambda,a}(v_n), v_n\rangle = o(1)$ and

$$
\int_{\mathbb{R}^N} f(x,v_n)v_n dx \le (|p^+|_\infty+\epsilon)\int_{\mathbb{R}^N} v_n^2 dx + C_\epsilon\int_{\mathbb{R}^N} v_n^r dx. \tag{3.8}
$$

It follows from (3.3)-(3.7) that

$$
\begin{aligned}
o(1) &= \Big(\int_{\mathbb{R}^N} |\Delta v_n|^2+|\nabla v_n|^2 dx + \int_{\mathbb{R}^N}\lambda V v_n^2 dx\Big) \\
&\quad - (|p^+|_\infty+\epsilon)\int_{\mathbb{R}^N} v_n^2 dx - C_\epsilon\int_{\mathbb{R}^N} v_n^r dx \\
&\ge \|v_n\|_\lambda^2 - \frac{|p^+|_\infty+\epsilon}{\lambda c}\|v_n\|_\lambda^2 - C_\epsilon\Big(\int_{\mathbb{R}^N} v_n^r dx\Big)^{(r-2)/r}\Big(\int_{\mathbb{R}^N} v_n^r dx\Big)^{2/r} \\
&\ge \Big(1-\frac{|p^+|_\infty+\epsilon}{\lambda c}\Big)\|v_n\|_\lambda^2 - C_\epsilon\Big[\frac{|\{V<c\}|^{\frac{2^{**}-r}{2^{**}}}}{(S^{**})^r}\Big(\frac{2\theta\lambda c(D-K)}{(\theta-2)c\lambda-2\theta d_0}\Big)^{r/2}\Big]^{(r-2)/r} \\
&\quad\times\Big[\big(\frac{1}{\lambda c}\big)^{\frac{2^{**}-r}{2^{**}-2}}(S^{**})^{-\frac{2^{**}(r-2)}{2^{**}-2}}\Big]^{2/r}\|v_n\|_\lambda^2 \\
&\ge \Big(1-\frac{|p^+|_\infty+\epsilon}{\lambda c} - C_\epsilon\Big[\frac{|\{V<c\}|^{\frac{2^{**}-r}{2^{**}}}}{(S^{**})^r}\Big(\frac{2\theta\lambda c(D-K)}{(\theta-2)c\lambda-2\theta d_0}\Big)^{r/2}\Big]^{(r-2)/r} \\
&\quad\times\Big[\big(\frac{1}{\lambda c}\big)^{\frac{2^{**}-r}{2^{**}-2}}(S^{**})^{-\frac{2^{**}(r-2)}{2^{**}-2}}\Big]^{2/r}\Big)\|v_n\|_\lambda^2,
\end{aligned}
$$

Therefore, there exists $\bar\Lambda_0 = \Lambda(D) \ge \frac{2\theta d_0}{c(\theta-2)} > 0$ such that $u_n \to u_0$ strongly in X_λ for $\lambda > \bar\Lambda_0$. □

Proof of Theorem 1.1. By Lemmas 2.3 and 2.4 and Theorem 2.2, we obtain that for each

$$
\lambda > \Lambda := \max\Big\{\frac{(S^{**})^2}{c}|\{V<c\}|^{\frac{2-2^{**}}{2^{**}}}, \frac{2\theta d_0}{c(\theta-2)}\Big\},
$$

there exists C_{α_λ}-sequence $\{u_n\}$ for J_λ on X_λ. Then, by Proposition 3.2 and $0 < \alpha_\lambda \le \alpha_0(\Omega) < D$, we can obtain that there exist a subsequence $\{u_n\}$ and $u_\lambda^{(1)} \in X_\lambda$ such that $u_n \to u_\lambda^{(1)}$ strongly in X_λ as $n\to\infty$ and for λ large enough. Moreover, $J_\lambda(u_\lambda^{(1)}) = \alpha_\lambda \ge \eta > 0$ and $u_\lambda^{(1)}$ is a nontrivial solution for (1.1).

The second solution for (1.1) will be constructed by the local minimization. We will first show that there exists $\varphi \in X_\lambda$ such that $J_\lambda(l\varphi) < 0$ for all $l > 0$ small enough. Indeed, we can take $\varphi \in H^2(\Omega)\cap H_0^1(\Omega)$ with $\int_\Omega \alpha(x)|u|^\nu dx > 0$. Using

(F1), we have, for all $l > 0$ small enough,

$$
\begin{aligned}
J_\lambda(l\varphi) &= \frac{l^2}{2}\int_\Omega |\Delta\varphi|^2 + |\nabla\varphi|^2 dx - \int_\Omega F(x, l\varphi)dx - \frac{1}{\nu}\int_\Omega \alpha(x)|l\varphi|^\nu dx \\
&\le \frac{l^2}{2}\int_\Omega |\Delta\varphi|^2 + |\nabla\varphi|^2 dx - l^k\int_\Omega p(x)|\varphi(x)|^k dx - \frac{l^\nu}{\nu}\int_\Omega \alpha(x)|\varphi|^\nu dx \\
&< 0.
\end{aligned} \tag{3.9}
$$

It follows from that the minimum of the functional J_λ on any closed ball in X_λ with center 0 and radius $R < \rho$ satisfying $J_\lambda(u) \ge 0$ for all $u \in X_\lambda$ with $\|u\|_\lambda = R$ is achieved in the corresponding open ball and thus yields a nontrivial solution $u_\lambda^{(2)}$ of problem (1.1) satisfying $J_\lambda(u_\lambda^{(2)}) < 0$ and $\|u_\lambda^{(2)}\| < R$. Moreover, (3.9) implies that there exist $l_0 > 0$ and $\kappa < 0$ being independent of λ such that $J_\lambda(l_0\varphi) = \kappa$ and $\|l_0\varphi\| < R$. Therefore, we can conclude that

$$
J_{\lambda_n}(u_n^{(2)}) \le \kappa < 0 \le \eta \le \alpha_{\lambda_n} = J_{\lambda_n}(u_n^{(1)})
$$

for all $\lambda \ge \Lambda$. This completes the proof. □

4. Proof of Theorem 1.2

In this section, we investigate the concentration of solutions and give a proof.

Proof of Theorem 1.2. For any sequence $\lambda_n \to \infty$, let $u_n^{(i)} := u_{\lambda_n}^{(i)}, i = 1, 2$ be the critical points of J_{λ_n} obtained in Theorem 1.1. Since

$$
J_{\lambda_n}(u_n^{(2)}) \le \kappa < 0 \le \eta \le \alpha_{\lambda_n} = J_{\lambda_n}(u_n^{(1)}) < D, \tag{4.1}
$$

$$
D \ge \alpha_{\lambda_n}(u_n^{(i)}) \ge \big(\frac{\theta-2}{2\theta} - \frac{d_0}{\Theta}\big)\|u_n^{(i)}\|_{\lambda_n}^2 - \frac{(\theta-\nu)|\alpha^+|_{\frac{2}{2-\nu}}}{\nu\theta\Theta^{\frac{\nu}{2}}}\|u_n^{(i)}\|_{\lambda_n}^\nu, \tag{4.2}
$$

one has

$$
\|u_n^{(i)}\|_{\lambda_n} \le C, \tag{4.3}
$$

where C is a constant independent of λ_n. Therefore, we may assume that $u_n^{(i)} \rightharpoonup u_0^{(i)}$ weakly in X and $u_n^{(i)} \to u_0^{(i)}$ strongly in $L_{\rm loc}^r(\mathbb{R}^N)$ for $2 \le r < 2^{**}$. By Fatou's Lemma, we have

$$
\int_{\mathbb{R}^N} V(x)(u_0^{(i)})^2 dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x)(u_n^{(i)})^2 dx \le \liminf_{n\to\infty}\frac{\|u_n^{(i)}\|_{\lambda_n}^2}{\lambda_n} = 0,
$$

this implies that $u_0^{(i)} = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$, and $u_0^{(i)} \in H^2(\Omega) \cap H_0^1(\Omega)$. Now, for any $\varphi \in C_0^\infty$, since $\langle J'_{\lambda_n}(u_n^{(i)}), \varphi\rangle = 0$, it is easy to check that

$$
\int_\Omega (\Delta u_0^{(i)}\Delta\varphi + \nabla u_0^{(i)}\nabla\varphi) = \int_{\mathbb{R}^N}[f(x, u_0^{(i)}) + \alpha(x)|u_0^{(i)}|^{\nu-2}u_0^{(i)}]\varphi dx.
$$

That is, $u_0^{(i)}$ is a weak solution in $H^2(\Omega) \cap H_0^1(\Omega)$.

Now we show that $u_n^{(i)} \to u_0^{(i)}$ strongly in $L^r(\mathbb{R}^N)$ for $2 \le r < 2^{**}$. Otherwise, there exist $\delta > 0, R_0 > 0$ and $x_n \in \mathbb{R}^N$ such that

$$
\int_{B^N(x_n, R_0)} (u_n^{(i)} - u_0^{(i)})^2 dx \ge \delta.
$$

Since $|B^N(x_n, R_0)| \cap \{V < c\} \to 0$ as $x_n \to \infty$, by Hölder inequality, we have

$$\int_{B^N(x_n,R_0)\cap\{V<c\}} (u_n^{(i)} - u_0^{(i)})^2 dx \to 0.$$

Consequently,

$$\begin{aligned}
0 &= \|u_n^{(i)}\|_{\lambda_n}^2 \\
&\geq \lambda_n c \int_{B(x_n,R_0)\cap\{V\geq c\}} (u_n^{(i)})^2 dx \\
&= \lambda_n c \int_{B(x_n,R_0)\cap\{V\geq c\}} (u_n^{(i)} - u_0^{(i)})^2 dx \\
&= \lambda_n c \Big[\int_{B(x_n,R_0)} (u_n^{(i)} - u_0^{(i)})^2 dx - \int_{B(x_n,R_0)\cap\{V<c\}} (u_n^{(i)} - u_0^{(i)})^2 dx \Big] \\
&\to \infty,
\end{aligned} \tag{4.4}$$

which contradicts (4.3). Therefore, $u_n^{(i)} \to u_0^{(i)}$ in $L^r(\mathbb{R}^N)$ for $2 \leq r < 2^{**}$. Moreover, using (A1), Hölder inequality and $u_n^{(i)} \to u_0^{(i)}$ in $L^2(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \alpha(x)|u_n^{(i)}|^\nu dx \to \int_{\mathbb{R}^N} \alpha(x)|u_n^{(i)}|^{\nu-2} u_n^{(i)} u_0^{(i)} dx.$$

By (F1)–(F2), we have

$$\int_{\mathbb{R}^N} f(x, u_n^{(i)}) u_n^{(i)} dx \to \int_{\mathbb{R}^N} f(x, u_n^{(i)}) u_0^{(i)} dx.$$

Since $\langle J'_{\lambda_n}(u_n^{(i)}), u_n^{(i)} \rangle = \langle J'_{\lambda_n}(u_n^{(i)}), u_0^{(i)} \rangle = 0$, we have

$$\begin{gathered}
\|u_n^{(i)}\|_{\lambda_n}^2 = \int_{\mathbb{R}^N} f(x, u_n^{(i)}) u_n^{(i)} dx + \int_{\mathbb{R}^N} \alpha(x)|u_n^{(i)}|^\nu dx, \\
\langle u_n^{(i)}, u_0^{(i)} \rangle = \int_{\mathbb{R}^N} f(x, u_n^{(i)}) u_0^{(i)} dx + \int_{\mathbb{R}^N} \alpha(x)|u_n^{(i)}|^{\nu-2} u_n^{(i)} u_0^{(i)} dx.
\end{gathered}$$

Then by (V3) and $u_0^{(i)} \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\lim_{n\to\infty} \|u_n^{(i)}\|_{\lambda_n}^2 = \lim_{n\to\infty} \langle u_n^{(i)}, u_0^{(i)} \rangle_{\lambda_n} = \|u_0^{(i)}\|^2.$$

On the other hand, by the weakly lower semi-continuity of norm, one has

$$\|u_0^{(i)}\|^2 \leq \liminf_{n\to\infty} \|u_n^{(i)}\|^2 \leq \liminf_{n\to\infty} \|u_n^{(i)}\|_{\lambda_n}^2.$$

Hence, $u_n^{(i)} \to u_0^{(i)}$ in X. Using (4.1) and the constants κ, η are independent of λ, we have

$$\begin{gathered}
\frac{1}{2} \int_\Omega |\Delta u_0^{(1)}|^2 + |\nabla u_0^{(1)}|^2 - \int_\Omega F(x, u_0^{(1)}) dx - \int_\Omega \alpha(x)|u_0^{(1)}|^\nu dx \geq \eta > 0, \\
\frac{1}{2} \int_\Omega |\Delta u_0^{(2)}|^2 + |\nabla u_0^{(2)}|^2 - \int_\Omega F(x, u_0^{(2)}) dx - \int_\Omega \alpha(x)|u_0^{(2)}|^\nu dx \leq \kappa < 0,
\end{gathered}$$

which imply that $u_0^{(i)} \neq 0, i = 1, 2$ and $u_0^{(1)} \neq u_0^{(2)}$. This completes the proof. $\square$

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