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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SUBLINEAR ORDINARY DIFFERENTIAL EQUATIONS AT RESONANCE

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Abstract. Using a Z_2 type index theorem, we show the existence and multiplicity of solutions for the sublinear ordinary differential equation

$$\mathcal{L}u(t) = \mu u(t) + W_u(t, u(t)), \quad 0 \le t \le L$$

with suitable periodic or boundary conditions. Here $\mathcal{L}$ is a linear positive selfadjoint operator, μ is a parameter between two egienvalues of this operator, and W_u is the gradient of a potential function.

1. Introduction

In the study of physical, chemical and biological systems, many ordinary differential equation models can be set in the form

$$\mathcal{L}u(t) = \mu u(t) + W_u(t, u(t)), \quad 0 \le t \le L, \tag{1.1}$$

(cf. [3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14] and their references) where $\mathcal{L}$ is a linear positive selfadjoint operator on $L^2([0, L], \mathbb{R}^n)$, μ is a real parameter, the potential $W(t, u) : \mathbb{R} \times \mathbb{R}^n \to R$ is a C^1-function, and $W_u(t, u) = \partial W/\partial u$ denotes the gradient of $W(t, u)$ with respect to the variable u. We say that (1.1) is sublinear if W satisfies $\lim_{|u|\to\infty} W(t, u)/|u|^2 = 0$.

Throughout this article, $\|\cdot\|_{L^q}$ denotes the norm of the usual space $L^q := L^q([0, L], \mathbb{R}^n)$ with $1 \le q \le \infty$, and we always assume that, for an appropriate Hilbert space $(X, \|\cdot\|) \subset L^2$ with the corresponding inner product $\langle \cdot, \cdot \rangle$, solutions of (1.1) are exactly the critical points of the corresponding functional

$$\Phi(u) = I(u) - J(u), \quad u \in X \tag{1.2}$$

where

$$I(u) = \frac{1}{2}(\|u\|^2 - \mu\|u\|^2_{L^2}), \quad J(u) = \int_0^L W(t, u(t))dt, \tag{1.3}$$

the problem

$$\mathcal{L}u(t) = \lambda u(t), \tag{1.4}$$

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has eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$, the corresponding eigenspaces $\mathcal{N}_j = \{v_j\}$ $(j \geq 1)$ have finite dimensions. For simplicity, we first consider the case of $\dim \mathcal{N}_j = 1$ for all $j \geq 1$, and more general case shall be discussed later. Further, we assume that there exists some $k \in N$ such that $\mu \in [\lambda_k, \lambda_{k+1})$. One says (1.1) is at resonance if $\mu = \lambda_k$, and (1.1) is sublinear.

Now we can state our main result as follows.

Theorem 1.1. *Suppose that $(X, \|\cdot\|) \subset L^2$ is a Hilbert space, continuously embedded into L^q for all $q \in [1, \infty]$, and $\{v_j(t)\}$ is an orthogonal basis in X and L^2 such that*

$$\|v_j(t)\|^2 = 1 = \lambda_j \|v_j(t)\|_{L^2}^2, \quad \forall j \geq 1. \tag{1.5}$$

Furthermore, assume that the functional $J(u) \in C^1(X, R)$ satisfies $J(0) = 0$, $J'(u)$ is a compact operator, and

- (J1) *$J(u) = J(-u)$ for all $u \in X$,*
- (J2) *there exists $K > 0$ such that $|J'(u)w| \leq K\|w\|_{L^1}$ for all $u, w \in X$,*
- (J3) *there exist $p \in N, M > 0, \rho > 0$ such that $M > \lambda_{k+p} - \lambda_k$, and*

$$J(u) \geq \frac{1}{2} M \|u\|_{L^2}^2 \quad \text{for } \|u\|_{L^\infty} \leq \rho,$$

- (J4) *$J(u) \to \pm\infty$ if $u \in \mathcal{N}_j$ for all $j \geq 1$, and $\|u\| \to \infty$.*

Then, there exist at least p distinct pairs $(u, -u)$ of critical points of $\Phi(u)$. If $\mu \in (\lambda_k, \lambda_{k+1})$, then (J4) can be ommitted.

The above theorem will be proved using the following Z_2 type index theorem.

Theorem 1.2 ([1])**.** . *Let Y be a Banach space, and $f \in C^1(Y, R)$ be even satisfying the Palais-Smale condition. Suppose that: (i) there exist a subspace V of Y with $\dim V = r$ and $\delta > 0$ such that $\sup_{w \in V, \|w\| = \delta} f(w) < f(0)$; (ii) there exists a closed subspace W of Y with $\operatorname{Codim} W = s < r$ such that $\inf_{w \in W} f(w) > -\infty$. Then f possesses at least $r - s$ distinct pairs $(u, -u)$ of critical points.*

For the convenience of the reader, let us recall that the functional f is said to satisfy the Palais-Smale condition: if any sequence $\{u_j\}$ in Y be such that $f(u_j)$ is bounded and $f'(u_j) \to 0$, possesses a convergent subsequence.

This article is organized as follows. In Section 2, we prove some lemmas for the functional $\Phi(u)$ defined by (1.2). In section 3, the proof of Theorem 1.1 and its some extensions shall be given. Section 4 is devoted to apply Theorem 1.1 to sublinear Hamiltonian systems as well as Extended Fisher-Kolmogorov type equations, and the existence and multiplicity results of their solutions shall be obtained.

2. Preliminaries

In this section, we shall study the properties of the functionals $\Phi(u), I(u), J(u)$ defined in (1.2)-(1.3).

With the hypotheses of Theorem 1.1, for all $u \in X$,we can write $u = \sum_{j=1}^\infty \alpha_j v_j$, thus $\|u\|^2 = \sum_{j=1}^\infty \alpha_j^2$, and

$$I(u) = \frac{1}{2} \sum_{j=1}^\infty \alpha_j^2 [1 - \mu \int_0^L |v_j|^2 dt] = \frac{1}{2} \sum_{j=1}^\infty (1 - \frac{\mu}{\lambda_j}) \alpha_j^2. \tag{2.1}$$

Case (i). If $\mu = \lambda_k$, then we set

$$u^+ = \sum_{j=k+1}^{\infty} \alpha_j v_j, \quad u^0 = \alpha_k v_k, \quad u^- = \sum_{j=1}^{k-1} \alpha_j v_j, \tag{2.2}$$

$$\begin{gathered} X^+ = \operatorname{span}\{v_j : j \geq k+1\}, \quad X^- = \operatorname{span}\{v_j : 1 \leq j \leq k-1\}, \\ X^0 = \mathcal{N}_k = \operatorname{span}\{v_k\}. \end{gathered} \tag{2.3}$$

Thus, we have $u = u^+ + u^0 + u^-, X = X^- \oplus X^0 \oplus X^+$.

Case (ii). If $\lambda_k < \mu < \lambda_{k+1}$, then we let

$$u^+ = \sum_{j=k+1}^{\infty} \alpha_j v_j, \quad u^- = \sum_{j=1}^{k} \alpha_j v_j, \tag{2.4}$$

$$X^+ = \operatorname{span}\{v_j : j \geq k+1\}, \quad X^- = \operatorname{span}\{v_j : 1 \leq j \leq k\}, \tag{2.5}$$

so we have $u = u^+ + u^-$, $X = X^+ \oplus X^-$.

Lemma 2.1. *Under the assumptions of Theorem 1.1, there exists a norm $\|\cdot\|_*$ of X, equivalent with $\|\cdot\|$, such that*

$$I(u) = \frac{1}{2}(\|u^+\|_*^2 - \|u^-\|_*^2).$$

Proof. Without loss of generality, we only consider the case $\mu = \lambda_k$ in the following. Thus

$$\begin{aligned} \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big)\|u^+\|^2 &= \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big) \sum_{j=k+1}^{\infty} \alpha_j^2 \\ &\leq \sum_{j=k+1}^{\infty} \Big(1 - \frac{\lambda_k}{\lambda_j}\Big)\alpha_j^2 \\ &\leq \sum_{j=k+1}^{\infty} \alpha_j^2 = \|u^+\|^2, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \Big(\frac{\lambda_k}{\lambda_{k-1}} - 1\Big)\|u^-\|^2 &= \Big(\frac{\lambda_k}{\lambda_{k-1}} - 1\Big) \sum_{j=1}^{k-1} \alpha_j^2 \\ &\leq \sum_{j=1}^{k-1} \Big(\frac{\lambda_k}{\lambda_j} - 1\Big)\alpha_j^2 \\ &\leq \frac{\lambda_k}{\lambda_1} \sum_{j=1}^{k-1} \alpha_j^2 = \frac{\lambda_k}{\lambda_1}\|u^-\|^2. \end{aligned} \tag{2.7}$$

Let

$$\|u\|_*^2 = \sum_{j=1}^{k-1} \Big(\frac{\lambda_k}{\lambda_j} - 1\Big)\alpha_j^2 + \sum_{j=k+1}^{\infty} \Big(1 - \frac{\lambda_k}{\lambda_j}\Big)\alpha_j^2 + \alpha_k^2\,. \tag{2.8}$$

Clearly, $\|\cdot\|_*$ is a norm on X, and is equivalent with the norm $\|\cdot\|$. The corresponding inner product is

$$\langle u, w\rangle_* = \sum_{j=1}^{k-1} \Big(\frac{\lambda_k}{\lambda_j} - 1\Big)\alpha_j\beta_j + \sum_{j=k+1}^{\infty} \Big(1 - \frac{\lambda_k}{\lambda_j}\Big)\alpha_j\beta_j + \alpha_k\beta_k, \tag{2.9}$$

where $u=\sum_{j=1}^{\infty}\alpha_j v_j$, $w=\sum_{j=1}^{\infty}\beta_j v_j\in X$. Consequently, according to (2.1) and (2.8), one obtains

$$I(u)=\frac{1}{2}(\|u^+\|_*^2-\|u^-\|_*^2)\,. \tag{2.10}$$

Then

$$\Phi'(u)w=\langle u^+,w\rangle_*-\langle u^-,w\rangle_*-\int_0^L W_u(t,u)w\,\mathrm{d}t,\quad \forall u,w\in X. \tag{2.11}$$

Finally, we point out that, in the nonresonant case of $\lambda_k<\mu<\lambda_{k+1}$, (2.8) and (2.9) should be replaced by

$$\|u\|_*^2=\sum_{j=1}^{k}\Big(\frac{\mu}{\lambda_j}-1\Big)\alpha_j^2+\sum_{j=k+1}^{\infty}\Big(1-\frac{\mu}{\lambda_j}\Big)\alpha_j^2, \tag{2.12}$$

$$\langle u,w\rangle_*=\sum_{j=1}^{k}\Big(\frac{\mu}{\lambda_j}-1\Big)\alpha_j\beta_j+\sum_{j=k+1}^{\infty}\Big(1-\frac{\mu}{\lambda_j}\Big)\alpha_j\beta_j, \tag{2.13}$$

respectively. The proof is complete. □

Lemma 2.2. *Under the assumptions of Theorem 1.1, the functional $\Phi(u)$ satisfies the Palais-Smale condition on X.*

Proof. We shall use the idea given by Rabinowitz [11, Theorem 4.12] and Costa [2, Proposition 3.2] for a PDE existence problem. Let $\{u_j\}\subset X$ be such that $\Phi(u_j)$ is bounded, and $\Phi'(u_j)\to 0$. We shall prove $\{u_j\}$ has a convergent subsequence.

Setting $u_j=u_j^++u_j^0+u_j^-$ with $u_j^+\in X^+$, $u_j^0\in X^0$, $u_j^-\in X^-$ for all $j\geq 1$. For j sufficiently large, we have

$$\|u_j^\pm\|_*\geq \Phi'(u_j)u_j^\pm=\langle u_j^+,u_j^\pm\rangle_*-<u_j^-,u_j^\pm\rangle_*-J'(u_j)u_j^\pm. \tag{2.14}$$

From (J2), it follows that

$$|J'(u_j)u_j^\pm|\leq K\|u_j^\pm\|_{L^1}\leq K_1\|u_j^\pm\|_* \tag{2.15}$$

with $K_1>0$ coming from the continuous embedding $L^1\to(X,\|\cdot\|)\to(X,\|\cdot\|_*)$. Combining (2.14) with + in the exponents, and (2.15) with + in the exponents, we obtain

$$\|u_j^+\|_*\geq\|u_j^+\|_*^2-K_1\|u_j^+\|_*, \tag{2.16}$$

thus, $\{u_j^+\}$ is bounded on X. Similarly, we also deduce that $\{u_j^-\}$ is bounded. Therefore, there exists $d>0$ such that

$$\|u_j-u_j^0\|_*=\|u_j^++u_j^-\|_*\leq d, \tag{2.17}$$

$$\begin{aligned}\Big|J(u_j)-J(u_j^0)\Big|&=\Big|\int_0^1\frac{d}{dt}J((1-t)u_j^0+tu_j)dt\Big|\\&=\Big|\int_0^1 J'((1-t)u_j^0+tu_j)(u_j-u_j^0)dt\Big|\\&\leq K\|u_j-u_j^0\|_{L^1}\leq K_1\|u_j-u_j^0\|_*\\&\leq K_1 d,\end{aligned} \tag{2.18}$$

which together with

$$J(u_j^0)=\frac{1}{2}(\|u_j^+\|_*^2-\|u_j^-\|_*^2)-\Phi(u_j)-[J(u_j)-J(u_0)]$$

yields $J(u_j^0)$ is bounded. By (J4), we get $\{u_j^0\}$ is bounded. Thus $\{u_j\}$ is bounded on X.

It should be noted that the gradient of $\Phi(u)$, $\nabla\Phi(u) : X \to X$ satisifes

$$\nabla\Phi(u) = u - G(u) \tag{2.19}$$

with $G(u) : X \to X$ being a compact operator defined by

$$\langle G(u), z\rangle = \mu \int_0^L u(t)z(t)dt + J'(u)z, \quad u, z \in X.$$

From the boundedness of $\{u_j\}$ and (2.19), we infer that $\{u_j\}$ has at least one convergent subsequence on X. So the Palais-Smale condition holds. □

Lemma 2.3. *Under the hypotheses of Theorem 1.1, the functional $\Phi(u)$ is bounded from below on X^+.*

Proof. From (J2), we have the estimate

$$J(u) = \int_0^1 \frac{d}{dt} J(tu)dt = \int_0^1 J'(tu)u\,dt \le K\|u\|_{L^1} \le K_1\|u\|_*, \quad \forall u \in X. \tag{2.20}$$

Then for $u \in X^+$, we infer that

$$\Phi(u) = \frac{1}{2}\|u\|_*^2 - J(u) \ge \frac{1}{2}\|u\|_*^2 - K_1\|u\|_* \to \infty \quad (\|u\|_* \to \infty). \tag{2.21}$$

Namely, $\Phi(u)$ is coercive and bounded from below on X^+. □

Lemma 2.4. *Under the assumptions of Theorem 1.1, there exists a subspace V of X with $\dim V = k + p$ and $\widetilde{\rho} > 0$ such that $\sup_{u\in V, \|u\|=\widetilde{\rho}} \Phi(u) < 0$.*

Proof. Put

$$V = \Big\{u = \sum_{j=1}^{k+p} \alpha_j v_j : \alpha_j \in \mathbb{R}\ (1 \le j \le k + p)\Big\}, \tag{2.22}$$

$$Z = \Big\{u \in V : \sum_{j=1}^{k+p} \alpha_j^2 = \widetilde{\rho}^2\Big\}, \tag{2.23}$$

where $\widetilde{\rho} = \rho/(c_\infty\sqrt{k + p})$, c_∞ satisfies $\|z\|_{L^\infty} \le c_\infty\|z\|$ for all $z \in X$.

For each $u(t) = \sum_{j=1}^{k+p} \alpha_j v_j(t) \in Z$, we have by Cauchy-Schwarz inequality

$$|u(t)|^2 \le \Big(\sum_{j=1}^{k+p} |v_j(t)|^2\Big)\Big(\sum_{j=1}^{k+p} \alpha_j^2\Big) \le (k + p)c_\infty^2\widetilde{\rho}^2 = \rho^2, \tag{2.24}$$

using $\|v_j\|_{L^\infty} \le c_\infty\|v_j\| = c_\infty$ for all $j \ge 1$. Hence

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\|u\|_{L^2}^2 - J(u)\\
&\le \frac{1}{2}\|u\|^2 - \frac{\mu + M}{2}\|u\|_{L^2}^2\\
&= \frac{1}{2}\sum_{j=1}^{k+p} \alpha_j^2 - \frac{\mu + M}{2}\sum_{j=1}^{k+p} \frac{1}{\lambda_j}\alpha_j^2
\end{aligned}$$

$$= \frac{1}{2}\sum_{j=1}^{k+p} \frac{\lambda_j - \mu - M}{\lambda_j}\alpha_j^2$$
$$\le \frac{1}{2}(\lambda_{k+p} - \lambda_k - M)\sum_{j=1}^{k+p} \frac{\alpha_j^2}{\lambda_j} < 0,$$

which implies $\sup\{\Phi(u) : u \in Z\} < 0$. □

3. Proof and extension of Theorem 1.1

Proof of Theorem 1.1. With the aid of Lemmas 2.2-2.4 , by Theorem 1.2, we conclude that $\Phi(u)$ (1.2) possesses at least p distinct pairs $(u_i, -u_i)$ of critical points.

Corollary 3.1. *Under the assumptions of Theorem 1.1, if condition* (J3) *is replaced by*

(J3') $\lim_{|u|\to 0} \frac{W(t,u)}{|u|^2} = \infty$ *uniformly in* $t \in [0, L]$,

then the functional $\Phi(u)$ *defined in* (1.2) *has infinitely many distinct pairs* $(u, -u)$ *of critical points.*

Proof. For any fixed $p \in N$, we may take M large enough such that $M > \lambda_{k+p} - \lambda_k$. By (J3'), there exists ρ sufficiently small satisfying

$$W(t, w) \ge \frac{1}{2}M|w|^2, \quad \forall w \in \mathbb{R}^n,\ |w| \le \rho \tag{3.1}$$

uniformly in $t \in [0, L]$. Thus, if $u = u(t) \in X$ with $\|u\|_{L^\infty} \le \rho$, then

$$W(t, u(t)) \ge \frac{1}{2}M|u(t)|^2 \tag{3.2}$$

uniformly in $t \in [0, L]$, and one obtains

$$J(u) \ge \frac{1}{2}M\|u\|_{L^2}^2. \tag{3.3}$$

Therefore, in view of Theorem 1.1, the functional $\Phi(u)$ has at least p distinct pairs $(u_i, -u_i)$ of critical points $(1 \le i \le p)$. Since p is arbitrary, there exist infinitely many distinct pairs $(u_i, -u_i)$ of critical points of $\Phi(u)$ $(i = 1, 2, 3, \dots)$. □

Remark 3.2. For all $\beta \in (0, 1/2)$, $\gamma \in (0, 1)$, we can take a function $H(s) \in C^1([0, \infty), R)$ such that

$$s^{1+2\beta} \le H(s) \le s^{1+\beta}, \quad \forall s \in [0, 1], \tag{3.4}$$
$$-\frac{1}{8}s^{\gamma-1} \le H'(s) \le \frac{1}{8}s^{\gamma-1}\ quad \forall s \in [2, \infty), \tag{3.5}$$
$$H(s) \to \pm\infty \quad \text{as } s \to \infty. \tag{3.6}$$

Define $W(t, u) = H(|u|)((\sin t)^{2m} + 2), m \ge 1$. A straightforward computation shows that (3.4) and (3.5) imply (J1)–(J3). In addition, (J4) can be easily deduced by (3.6), see [11, Lemma 4.21].

From a carefully analyzing the constructions of V and Z in (2.22)-(2.23), we have the following result which is more general than Theorem 1.1.

Theorem 3.3. *Suppose that* $(X, \|\cdot\|) \subset L^2$ *is a Hilbert space, continuously embedded in* $L^q, \forall q \in [1, \infty]$. *Let* $n_j = dim\, \mathcal{N}_j$ *and* $\{v_{j1}, v_{j2}, \dots, v_{jn_j}\}$ *be an orthogonal basis of* $\mathcal{N}_j (\forall j \geq 1)$ *such that* $\{v_{ji}(t) : j \geq 1, 1 \leq i \leq n_j\}$ *is an orthogonal basis in* X *and* L^2 *with*

$$\|v_{ji}(x)\|^2 = 1 = \lambda_j \|v_{ji}(x)\|_{L^2}^2, \quad \forall j \geq 1, 1 \leq i \leq n_j.$$

Furthermore, assume that the functional $J(u) \in C^1(X, R)$ *satisfies* $J(0) = 0$, $J'(u)$ *is a compact operator, and* (J1)-(J4) *hold. Then, there exist at least* $\sum_{j=k+1}^{k+p} n_j$ *distinct pairs* $(u, -u)$ *of critical points of* $\Phi(u)$ *(If* $\mu \in (\lambda_k, \lambda_{k+1})$*, then* (J4) *can be omitted).*

To prove this theorem, we need only changes in Lemmas 2.1 and 2.4. Especially, V, Z in (2.22)-(2.23) shall be replaced by

$$\widetilde{V} = \{u = \sum_{j=1}^{k+p} \sum_{i=1}^{n_j} \alpha_{ji} v_{ji} : \alpha_{ji} \in \mathbb{R}\ (1 \leq j \leq k+p, 1 \leq i \leq n_j)\}, \tag{3.7}$$

$$\widetilde{Z} = \{u \in \widetilde{V} : \sum_{j=1}^{k+p} \sum_{i=1}^{n_j} \alpha_{ji}^2 = \widetilde{\rho}^2\}, \tag{3.8}$$

respectively.

4. Applications

Application i. Given $T > 0$, we discuss the existence of T-periodic solutions to the second-order Hamiltonian system

$$\ddot{u}(t) + \mu u(t) + W_u(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{4.1}$$

where $W(t, u) \in C^1(R \times \mathbb{R}^n, R)$ is a T-periodic function in the variable t and $W(t, 0) \equiv 0$.

Since 1973, many authors studied periodic solutions for Hamiltonian systems via critical point theory. Clarke and Ekeland [3] studied a family of convex sublinear Hamiltonian systems where $W(t, u) = W(u)$ satisfies $\lim_{|u| \to 0} \frac{W(t,u)}{|u|^2} = \infty$, and they used the dual variational method to obtain the first variational result on periodic solutions having a prescribed minimal period. Later, Mawhin and Willem [8] made a good improvement. Rabinowitz [9, 10], Tang [13] and others proved the existence under the sublinear condition $uW_u(t, u) \leq \alpha W(t, u) (0 < \alpha < 2)$, which plays an important role. Schechter [12] assumed that $W(t, u)$ is sublinear, and $2W(t, u) - uW_u(t, u) \to -\infty (|u| \to \infty)$ or $2W(t, u) - uW_u(t, u) \leq W_0(t)$, then he proved that (4.1) has one non-constant periodic solution. Long [7] also studied this problem for bi-even sublinear potentials, and got the existence of one odd periodic solution. Li-Wang-Xiao [6] considered the existence and multiplicity of odd periodic solution for bi-even sublinear (4.1) in the case of $\mu < \lambda_1$.

Motivated by the above papers, using Theorem 3.3, we shall give a multiplicity result for (4.1) with sublinear potentials in the case of $\lambda_k \leq \mu < \lambda_{k+1}$.

Theorem 4.1. *Assume that* $L = T/2$, *and there exists some* $k \in N$ *such that* $(\frac{k\pi}{L})^2 \leq \mu < (\frac{(k+1)\pi}{L})^2$. *Let* $W(t, u) \in C^1(R \times \mathbb{R}^n, R)$ *be* T*-periodic in* t, *and bi-even, namely*

$$W_u(t, u) = -W_u(-t, -u), \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^n.$$

Suppose that

(W11) $W(t,u) = W(t,-u)$ *for all* $t \in \mathbb{R}$, $u \in \mathbb{R}^n$;

(W12) *there exists* $K > 0$ *such that* $|W_u(t,u)| \le K$ *for all* $t \in \mathbb{R}$, $u \in \mathbb{R}^n$;

(W13) *there exist* $p \in N$, $M > 0$, $\rho > 0$ *such that if* $M > \frac{p(p+2k)}{L^2}\pi^2$ *then*

$$W(t,u) \ge \frac{1}{2}M|u|^2 \quad \forall t \in \mathbb{R}, |u| \le \rho;$$

(W14) *for* $u = c\sin\frac{j\pi t}{L}\theta_i$ *with* $\theta_i = (0,0,\dots,0,1,0,\dots,0) \in \mathbb{R}^n$ *(the* $i-th$ *element is 1,* $1 \le i \le n$*), for all* $j \ge 1$, $\int_0^L W(t,u(t))dt \to \pm\infty$ *as* $|c| \to \infty$.

Then, (4.1) *has* $np-$ *distinct pairs* $(u(t),-u(t))$ *of odd* T*-periodic solutions. If* $(\frac{k\pi}{L})^2 < \mu < (\frac{(k+1)\pi}{L})^2$*, then* (W14) *can be omitted.*

Remark 4.2. If $W(t,u)$ satisfies

$$W(t,u) = W(t,-u) = W(-t,-u),$$

then $W(t,u)$ is bi-even, and (W_{11}) holds. For this, a typical example is, $W(t,u) = b(t)\widetilde{W}(u)$, where $b(t)$ and $\widetilde{W}(u)$ are even in the variable t,u, respectively.

Proof of Theorem 4.1. Firstly, consider the boundary value problem

$$\begin{gathered} -\ddot{u}(t) = \mu u(t) + W_u(t,u(t)), \quad 0 < t < L, \\ u(0) = u(L) = 0. \end{gathered} \tag{4.2}$$

If $u = u(t)$ is a solution of (4.2), then we define

$$\overline{u} = \overline{u}(t) = \begin{cases} u(t), & 0 \le t \le L, \\ -u(-t), & -L \le t \le 0. \end{cases} \tag{4.3}$$

By the bi-even condition, $\overline{u} = \overline{u}(t)$ is a solution of (4.1) restricted on $[-L,L]$, so its odd extension in $(-\infty,\infty)$ is an odd T-periodic solution of (4.1).

Secondly, let $X = H_0^1([0,L],\mathbb{R}^n)$ be the usual Hilbert space with the inner product $(x,y) = \int_0^L \dot{x}(t)\cdot\dot{y}(t)dt$ and the norm $\|x\| = (\int_0^L |\dot{x}(t)|^2dt)^{1/2}$. Set

$$\Phi(u) = \frac{1}{2}\int_0^L [|\dot{u}(t)|^2 - \mu|u(t)|^2]\,\mathrm{d}t - \int_0^L W(t,u(t))\,\mathrm{d}t, \tag{4.4}$$

then $\Phi(u) \in C^1(X,R)$, and its critical points are the classical solutions of (4.2).

By direct computations, we know that the problem

$$-\ddot{u}(t) = \lambda u(t), \quad u(0) = u(L) = 0$$

possesses eigenvalues $\lambda_j = (\frac{j\pi}{L})^2, j \ge 1$, and the corresponding eigenfunctions are $u_{ji} = c\theta_i \sin\frac{j\pi t}{L}, 1 \le i \le n, c \in \mathbb{R}$. Furthermore,

$$\big\{\theta_i\sin\frac{\pi t}{L}, \theta_i\sin\frac{2\pi t}{L}, \theta_i\sin\frac{3\pi t}{L}, \dots, 1 \le i \le n\big\} \tag{4.5}$$

is an orthogonal basis on both X and L^2 . Since

$$\int_0^L |\dot{u}_{ji}(t)|^2\,\mathrm{d}t = \lambda_j\frac{L}{2} = \lambda_j\int_0^L |u_{ji}(t)|^2\,\mathrm{d}t, \tag{4.6}$$

writing $v_{ji} = \sqrt{\frac{2}{L\lambda_j}}u_{ji}$, we have $\|v_{ji}\|^2 = \int_0^L |\dot{v}_{ji}|^2\,\mathrm{d}t = 1 = \lambda_j\int_0^L |v_{ji}|^2\,\mathrm{d}t$.

Noticing that

$$\frac{p(p+2k)}{L^2}\pi^2 = \lambda_{k+p} - \lambda_k,$$

the functional (4.4) satisfies all hypotheses of Theorem 3.3, hence it has at least np distinct pairs $(u_i, -u_i)$ of critical points $(1 \leq i \leq np)$. Consequently, in the way of (4.3), the extensions of $\pm\overline{u}_i(t)(1 \leq i \leq np)$ are np distinct pairs of odd T-periodic solutions of (4.1). □

Application ii. We are concerned with a class of Extended Fisher-Kolmogorov type equations (see [4, 5, 14] and their references)

$$u^{(4)}(t) = \mu u(t) + W_u(t, u(t)) \quad 0 \leq t \leq L \tag{4.7}$$

with the boundary condition

$$u(0) = u(L) = u''(0) = u''(L) = 0,$$

which appears in the formation of spatial patterns in bistable systems.

Theorem 4.3. *Assume that there exists some $k \in N$ such that $(\frac{k\pi}{L})^4 \leq \mu < (\frac{(k+1)\pi}{L})^4$. Let $W(t, u) \in C^1([0, L] \times R, R)$ satisfy the following conditions:*

(W21) *$W(t, u) = W(t, -u)$ for all $t \in [0, L]$, $u \in \mathbb{R}$;*

(W22) *there exists $K > 0$ such that $|W_u(t, u)| \leq K$ for all $t \in [0, L], u \in \mathbb{R}$;*

(W23) *there exist $p \in N, M > 0, \rho > 0$ such that if $M > \frac{(p+k)^4 - k^4}{L^4}\pi^4$ then*

$$W(t, u) \geq \frac{1}{2}M|u|^2 \quad \forall t \in [0, L], |u| \leq \rho;$$

(W24) *for $u = c \sin \frac{j\pi t}{L}$, for all $j \geq 1$, $c \in \mathbb{R}$, $\int_0^L W(t, u(t))dt \to \pm\infty$ as $|c| \to \infty$.*

Then, (4.7) *has p distinct pairs $(u(t), -u(t))$ of classical solutions. If $(\frac{k\pi}{L})^4 < \mu < (\frac{(k+1)\pi}{L})^4$, then* (W24) *can be omitted.*

Proof. Similarly to the proof of Theorem 4.1, we sketch it. Set

$$X = H^2(0, L) \cap H_0^1(0, L), \tag{4.8}$$

by [5, Lemma 2.1], $\|u\| = (\int_0^T |\ddot{u}(t)|^2 dt)^{1/2}$ is a norm of X, and

$$v_j(t) = \sin \frac{j\pi t}{L} \Big(\sqrt{\frac{L}{2}}(\frac{j\pi}{L})^2\Big)^{-1} \tag{4.9}$$

is an orthogonal basis on X and L^2 such that

$$\|v_j(t)\|^2 = 1 = (\frac{j\pi}{L})^4 \|v_j(t)\|_{L^2}^2, \quad j \geq 1. \tag{4.10}$$

In addition, the problem

$$u^{(4)}(t) = \lambda u(t)$$

has eigenvalues $\lambda_j = (\frac{j\pi}{L})^4$, $j \geq 1$, and the corresponding eigenfunctions are exactly $v_j(t)$ in (4.9). Define the functional

$$\Phi(u) = \frac{1}{2} \int_0^L |\ddot{u}(t)|^2 dt - \frac{1}{2}\mu \int_0^L |u(t)|^2 \, dt - \int_0^L W(t, u(t)) \, dt, \quad u \in X, \tag{4.11}$$

then the critical points of $\Phi(u)$ in (4.11) are the classical solutions of the problem (4.7). Therefore, by Theorem 1.1, we have the statement in Theorem 4.3. □

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