GLOBAL EXISTENCE FOR SEAWATER INTRUSION MODELS: COMPARISON BETWEEN SHARP INTERFACE AND SHARP-DIFFUSE INTERFACE APPROACHES

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Abstract. We study seawater intrusion problems in confined and unconfined aquifers. We compare from a mathematical point of view the sharp interface approach with the sharp-diffuse interface approach. We demonstrate that, if the diffuse interface allows to establish a more efficient and logical maximum principle in the unconfined case, this advantage fails in the confined case. Problems can be formulated as strongly coupled systems of partial differential equations which include elliptic and parabolic equations (that can be degenerate), the degeneracy appearing only in the sharp interface case. Global in time existence results of weak solutions are established under realistic assumptions on the data.

1. Introduction

In coastal zones, which are densely populated areas, the intensive extraction of freshwater yields to local water table depression causing sea intrusion problems. In order to get an optimal exploitation of fresh water and also to control seawater intrusion in coastal aquifers, we need to develop efficient and accurate models for simulating the transport of salt water front in coastal aquifer. We refer to the textbooks [4, 5, 7] for general information about seawater intrusion problems.

We distinguish two important cases: the case of free aquifer and the one of confined aquifer. In each case, the aquifer is bounded by two layers with lower layer always supposed to be impermeable. The upper surface is assumed to be impermeable in the confined case and permeable in the unconfined case (the interface between the saturated and unsaturated zones is thus free).

The basis of the modeling is the mass conservation law for each species (fresh and salt water) combined with the classical Darcy law for porous media. In the present work we essentially have chosen to adopt the simplicity of a sharp interface approach. This approach is based on the assumption that the two fluids are immiscible. We assume that each fluid is confined to a well defined portion of the flow domain with a smooth interface separating them called sharp interface. No mass transfer occurs between the fresh and the salt area and capillary pressure's

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type effects are neglected. This approximation is often reasonable (see e.g. [4] and below). Of course, this type of model does not describe the behavior of the real transition zone but gives information concerning the movement of the saltwater front.

Following [10], we can mix this abrupt interface approach with a phase field approach (here an Allen–Cahn type model in fluid-fluid context see e.g. [1, 2, 8, 12]) for re-including the existence of a diffuse interface between fresh and salt water where mass exchanges occur. We thus combine the advantage of respecting the physics of the problem and that of the computational efficiency. The same process is applied to model the transition between the saturated and unsaturated zones in unconfined aquifers.

From a theoretical point of view, in the unconfined case, two advantages resulting from the addition of diffuse areas compared to the sharp interface approximation are stated in [11]:

• If diffuse interfaces are both present, the system has a parabolic structure, so it is not necessary to introduce viscous terms in a preliminary fixed point for treating degeneracy as in the case of sharp interface approach.

• The main advantage is that we can now demonstrate a more efficient maximum principle and logical from the point of view of physics, which can not be established in the case of sharp interface approximation. (see for instance [13, 16, 19]).

However the latter is no longer valid in the confined case. Indeed, we need to assume a freshwater thickness strictly positive in the interior of the aquifer to ensure uniform estimation in the L^2 space of the gradient of the freshwater hydraulic head. This artificial condition is always necessary in the case of diffuse interface. The maximum principle is then identical in both cases (sharp interface and sharp-diffuse interface).

The outline of this article is as follows. Section 2 is devoted to models and their derivation: we model the evolution of the depth h of the interface between freshwater and saltwater and of the freshwater hydraulic head (in the confined case) and of depths h and h_1, the interface between the saturated and unsaturated zone (in the unconfined case). The resulting models consist in a system of strongly and nonlinearly coupled PDEs of parabolic type in the case of free aquifer and a system of strongly and nonlinearly coupled PDEs of elliptic-parabolic type in the case of confined aquifer. In section 3 all mathematical notations are stated and global in time existence results are established in the two following cases: the confined case with sharp-diffuse interface approach and the unconfined case with sharp interface approach. The section 4 is devoted to the proof of the existence results: we apply a Schauder fixed point strategy to a regularized and truncated system then we establish uniform estimates allowing us to turn back to the original problem.

2. Modeling

Introducing specific index for the fresh (f) and salt (s) waters, we write the mass conservation law for each species (fresh and salt water) combined with the classical Darcy law for porous media. Hydraulic heads Φ_i, $i = f, s$ are defined at elevation z by

$$\Phi_i = \frac{P_i}{\rho_i g} + z,$$

where P_i denotes the pressure. The Darcy law relating together the effective velocity q_i of the flow and the hydraulic head Φ_i reads:

$$q_i = -K_i \nabla(\Phi_i), \quad K_i = \frac{\kappa \rho_i g}{\mu_i}. \tag{2.1}$$

Characteristics ρ_i and μ_i are respectively the density and the viscosity of the fluid, κ is the permeability of the soil and g the gravitational acceleration constant. The matrix K_i is the hydraulic conductivity. It expresses the ability of the ground to conduct water, K_i is proportional to κ the permeability of the ground which only depends on the characteristics of the porous medium and not on the fluid.

At this point, using (2.1), we derive from the mass conservation law for each species (fresh and salt water) the following model:

$$S_i \partial_t \Phi_i + \nabla \cdot q_i = Q_i, \quad q_i = -K_i \nabla \Phi_i, \quad K_i = kg\rho_i/\mu_i.$$

The coefficient of water storage S_i $(i = f, s)$ characterizes the workable water volume. It accounts for the rock and fluid compressibility. In general, this coefficient is extremely small because of the weak compressibility of the fluid and of the rock. In the present work, we choose to neglect it but we emphasize that, in the case of free aquifer, $S_f \partial_t \Phi_f$ is of order of $\phi \partial_t \Phi_f$, with ϕ the porosity of the medium. Let us now exploit Dupuit approximation which legitimates the upscaling of the $3D$ problem to a $2D$ model by vertical averaging. We integrate the mass conservation law between the interfaces depths h and h_1 in the fresh layer and between h and the lower topography h_2, in the salty zone. The averaged mass conservation laws for the fresh and salt water thus read

$$S_f B_f \partial_t \tilde{\Phi}_f = \nabla' \cdot (B_f \tilde{K}_f \nabla' \tilde{\Phi}_f) - q_f\big|_{z=h_1} \cdot \nabla(z - h_1) + q_f\big|_{z=h} \cdot \nabla(z - h) + B_f \tilde{Q}_f, \tag{2.2}$$

$$S_s B_s \partial_t \tilde{\Phi}_s = \nabla' \cdot (B_s \tilde{K}_s \nabla' \tilde{\Phi}_s) + q_s\big|_{z=h_2} \cdot \nabla(z - h_2) - q_s\big|_{z=h} \cdot \nabla(z - h) + B_s \tilde{Q}_s, \tag{2.3}$$

where $\nabla' = (\partial_{x_1}, \partial_{x_2})$. The coefficients $B_f = h_1 - h$ and $B_s = h - h_2$ denote the thickness of the fresh and salt water zones and $\tilde{\Phi}_i$, $i = f, s$, the vertically averaged hydraulic heads

$$\tilde{\Phi}_f = \frac{1}{B_f} \int_h^{h_1} \Phi_f dz \quad \text{and} \quad \tilde{\Phi}_s = \frac{1}{B_s} \int_{h_2}^{h} \Phi_s dz.$$

The source terms $\tilde{Q}_i$, $i = f, s$ represent distributed surface supplies of fresh and salt water into the aquifer. Besides sharp interface assumption implies the continuity of the pressure at the interface between salt and fresh water, it follows that

$$(1 + \alpha) \tilde{\Phi}_s = \tilde{\Phi}_f + \alpha h, \quad \alpha = \frac{\rho_s}{\rho_f} - 1. \tag{2.4}$$

Here the parameter α characterizes the densities contrast. Equation (2.4) allows us to avoid $\tilde{\Phi}_s$ in the final system.

Our aim is now to include in the model the continuity properties across interfaces in view of expressing the four flux terms in (2.2)-(2.3). First, since the lower layer is impermeable, there is no flux across the boundary $z = h_2$:

$$q_{s|z=h_2} \cdot \nabla(z - h_2) = 0. \tag{2.5}$$

In the same way, in the case of confined aquifer, the upper layer is impermeable, thus

$$q_{f|z=h_1} \cdot \nabla(z - h_1) = 0. \tag{2.6}$$

At the interface between fresh and salt water, we present the two following approaches:

• Sharp interface approach. With the traditional sharp interface characterization, there is no mass transfer across the interface between fresh and salt water, i.e. the normal component of the effective velocity $\vec{v}$ is continue at the interface $z = h$,

$$\left(\frac{q_{f|z=h}}{\phi} - \vec{v} \right) \cdot \vec{n} = \left(\frac{q_{s|z=h}}{\phi} - \vec{v} \right) \cdot \vec{n} = 0,$$

where $\vec{n}$ denotes the normal unit vector to the interface. Thus we obtain

$$q_{f|z=h} \cdot \nabla(z - h) = q_{s|z=h} \cdot \nabla(z - h) = \phi \partial_t h \tag{2.7}$$

• Sharp-diffuse interface between fresh and salt water. This approach includes now existence of miscible zone, taking the form of diffuse interface of characteristic thickness δ between fresh and salt water. Upscaling the 3D-dynamics of the diffuse interface assumed ruled by a phase field model, we obtain the following continuity equation instead of (2.7) (see [10] for more details about the derivation of this equation):

$$q_{f|z=h} \cdot \nabla(z - h) = q_{s|z=h} \cdot \nabla(z - h) = \phi(\partial_t h - \delta \Delta' h) \tag{2.8}$$

The same approach for the capillary fringe in the unconfined case yields

$$q_{f|z=h_1} \cdot \nabla(z - h_1) = \phi(\partial_t h_1 - \delta \Delta' h_1) \tag{2.9}$$

Finally, the following assumptions are introduced for sake of simplicity in the notation. The medium is assumed to be isotrope and the viscosity the same for the salt and fresh water, then

$$\tilde{K}_s = (1 + \alpha)\tilde{K}_f. \tag{2.10}$$

We re-write models with some notational simplifications. The 'primes' are suppressed in the differentiation operators in $\mathbb{R}^2$ and source terms are denoted without 'tildes'. We also reverse the vertical axis thus changing h_1 into $-h_1$, h into $-h$, h_2 into $-h_2$, z into $-z$ (bearing in mind that now $B_s = h_2 - h$, $B_f = h - h_1$).

In the case of confined aquifer, the well adapted unknowns are the interface depth h and the freshwater hydraulic head Φ_f. We set $\alpha \tilde{K}_f = K$ and $\tilde{\Phi}_f = \alpha f$. The final model then reads

$$-\nabla \cdot (K(h_2 - h_1)\nabla f) + \nabla \cdot (K(h_2 - h)\nabla h) = B_f Q_f + B_s Q_s,$$

$$\phi \frac{\partial h}{\partial t} + \nabla \cdot (K(h_2 - h)\nabla f) - \nabla \cdot (K(h_2 - h)\nabla h) - \beta \delta_h \nabla \cdot (\phi \nabla h) = -B_s Q_s.$$

The coefficient β is equal to 0 in the case of sharp interface and to 1 in the case of sharp-diffuse interface.

In the case of an unconfined aquifer, the unknowns are the interfaces depths h and h_1. Since quantities h and h_1 are only meaningful inside the aquifer, we introduce in the final model $h^+ = \sup(0, h)$ and $h_1^+ = \sup(0, h_1)$. Neglecting the storage coefficient S_f and introducing the characteristic function $\mathcal{X}_0$ on the interval $(0, +\infty)$, the sharp-diffuse interface model reads

$$\phi \mathcal{X}_0(h_1)\partial_t h_1 - \nabla \cdot (\tilde{K}_f \mathcal{X}_0(h_1)((h - h_1) + (h_2 - h))\nabla h_1)$$
$$- \beta \nabla \cdot (\delta \phi \tilde{K}_f \mathcal{X}_0(h_1)\nabla h_1) - \nabla \cdot (\tilde{K}_f \alpha (h_2 - h)\mathcal{X}_0(h)\nabla h) = -B_f Q_f - B_s Q_s,$$

$$\phi \mathcal{X}_0(h)\partial_t h - \nabla \cdot (\alpha \tilde{K}_f(h_2 - h)\mathcal{X}_0(h_1)\nabla h) - \beta \nabla \cdot (\delta \phi \mathcal{X}_0(h)\nabla h)$$
$$- \nabla \cdot (\tilde{K}_f \mathcal{X}_0(h_1)(h_2 - h)\nabla h_1) = -B_s Q_s.$$

Again the coefficient β is equal to 0 in the case of sharp interfaces and to 1 in the case of sharp-diffuse interfaces.

In the previous two systems, the first equation models the conservation of total mass of water, while the second is modeling the mass conservation of fresh water. This is a 2D model, the third dimension being preserved by the upscaling process via the depth information h and h_1.

3. Mathematical setting and main results

We consider a bounded and open domain Ω of $\mathbb{R}^2$ describing the projection of the aquifer on the horizontal plane. The boundary of Ω, assumed $\mathcal{C}^1$, is denoted by Γ. The time interval of interest is $(0, T)$, T being any nonnegative real number, and we set $\Omega_T = (0, T) \times \Omega$.

3.1. **Some auxiliary results.** For any $n \in N^*$ and any $p \in (1, +\infty)$, let $W^{n,p}(\Omega)$ be the usual Sobolev space, with the norm $\|\phi\|_{W^{n,p}(\Omega)} = \sum_{\alpha \in \mathbb{N}^2, \alpha \leq n} \|\partial^\alpha \phi\|_{L^p(\Omega)}$. For the sake of brevity we shall write $H^1(\Omega) = W^{1,2}(\Omega)$ and

$$V = H_0^1(\Omega), \quad E = H_0^1(\Omega) \cap L^\infty(\Omega), \quad H = L^2(\Omega).$$

The embeddings $V \subset H = H' \subset V'$ are dense and compact. For any $T > 0$, let $W(0, T)$ denote the space

$$W(0, T) := \{\omega \in L^2(0, T; V), \ \partial_t \omega \in L^2(0, T; V')\}$$

endowed with the Hilbertian norm $\|\omega\|_{W(0,T)} = \left(\|\omega\|_{L^2(0,T;V)}^2 + \|\partial_t \omega\|_{L^2(0,T;V')}^2\right)^{1/2}$. The following embeddings are continuous [15, prop. 2.1 and thm 3.1, chapter 1]

$$W(0, T) \subset \mathcal{C}([0, T]; [V, V']_{1/2}) = \mathcal{C}([0, T]; H)$$

while the embedding

$$W(0, T) \subset L^2(0, T; H) \tag{3.1}$$

is compact (Aubin's Lemma, see [18]). The following result by Mignot [14] is used in the sequel.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function such that $\limsup_{|\lambda| \to +\infty} |f(\lambda)/\lambda| < +\infty$. Let $\omega \in L^2(0, T; H)$ be such that $\partial_t \omega \in L^2(0, T; V')$ and $f(\omega) \in L^2(0, T; V)$. Then*

$$\langle \partial_t \omega, f(\omega)\rangle_{V',V} = \frac{d}{dt} \int_\Omega \left(\int_0^{\omega(\cdot,y)} f(r)\, dr\right) dy \ \text{in } \mathcal{D}'(0, T).$$

Hence for all $0 \leq t_1 < t_2 \leq T$,

$$\int_{t_1}^{t_2} <\partial_t \omega, f(\omega)>_{V',V} dt = \int_\Omega \left(\int_{\omega(t_1,y)}^{\omega(t_2,y)} f(r)\, dr\right) dy.$$

3.2. **Main results.**

3.2.1. *Case of confined aquifer.* We focus here on models in confined case. We aim giving existence results of physically admissible weak solutions for these models completed by initial and boundary conditions. We consider that the confined aquifer is bounded by two layers, the lower surface corresponds to $z = h_1$ and the upper surface $z = h_2$. Quantity $h_2 - h_1$ is the thickness of the groundwater, we assume that depths h_1, h_2 are constant, such that $h_2 > \delta_1 > 0$ and without lost of generality we can set $h_1 = 0$. We introduce functions T_s and T_f defined by

$$T_s(u) = h_2 - u \quad \forall u \in (\delta_1, h_2) \quad \text{and} \quad T_f(u) = \begin{cases} u & u \in (\delta_1, h_2) \\ 0 & u \leq \delta_1 \end{cases}$$

Functions T_s and T_f are extended continuously and constantly outside (δ_1, h_2) for T_s and for $u \geq h_2$ for T_f. $T_s(h)$ represents the thickness of the salt water zone, the previous extension of T_s for $h \leq \delta_1$ enables us to ensure a thickness of freshwater zone always $\geq \delta_1$ in the aquifer. We also emphasize that the function T_f only acts on the source term Q_f for avoiding the pumping when the thickness of freshwater zone is smaller than δ_1. Then we consider the following set of equations in Ω_T:

$$\phi\partial_t h - \nabla \cdot \Big(KT_s(h)\nabla h\Big) - \nabla \cdot \Big(\beta\delta\phi\nabla h\Big) + \nabla \cdot \Big(KT_s(h)\nabla f\Big) = -Q_sT_s(h), \quad (3.2)$$

$$-\nabla \cdot \Big(h_2K\nabla f\Big) + \nabla \cdot \Big(KT_s(h)\nabla h\Big) = Q_fT_f(h) + Q_sT_s(h). \quad (3.3)$$

This system is complemented with the boundary and initial conditions:

$$h = h_D, \quad f = f_D \quad \text{in } \Gamma \times (0, T), \quad (3.4)$$

$$h(0, x) = h_0(x), \quad \text{in } \Omega, \quad (3.5)$$

with the compatibility conditions

$$h_0(x) = h_D(0, x), \quad x \in \Gamma.$$

Let us now detail the mathematical assumptions. We begin with the characteristics of the porous structure. We assume the existence of two positive real numbers K_- and K_+ such that the hydraulic conductivity tensor is a bounded elliptic and uniformly positive definite tensor:

$$0 < K_-|\xi|^2 \leq \sum_{i,j=1,2} K_{i,j}(x)\xi_i\xi_j \leq K_+|\xi|^2 < \infty \quad x \in \Omega, \ \xi \in \mathbb{R}^2, \ \xi \neq 0.$$

We assume that porosity is constant in the aquifer. Indeed, in the field envisaged here, the effects due to variations in ϕ are negligible compared with those due to density contrasts. From a mathematical point of view, these assumptions do not change the complexity of the analysis but rather avoid complicated computations.

The source terms Q_f and Q_s are given functions in $L^2(0, T; H)$ such that $Q_s \leq 0$. Notice for instance that pumping of freshwater corresponds to assumption $Q_f \leq 0$ a.e. in $\Omega \times (0, T)$. Functions h_D and f_D belong to the space $\big(L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))')\big) \times L^2(0, T; H^1(\Omega))$ while function h_0 is in $H^1(\Omega)$. Finally, we assume that the boundary and initial data satisfy conditions on the hierarchy of interfaces depths:

$$0 < \delta_1 \leq h_D \leq h_2 \quad \text{a.e. in } \Gamma \times (0, T), \quad 0 < \delta_1 \leq h_0 \leq h_2 \quad \text{a.e. in } \Omega.$$

We state and prove the following existence result.

Theorem 3.2. *Assume a low spatial heterogeneity for the hydraulic conductivity tensor:*

$$K_- \leq K_+ \leq \frac{3}{2}K_-.$$

Then for any $T > 0$, problem (3.2)-(3.5) admits a weak solution (h, f) satisfying

$$(h - h_D, f - f_D) \in W(0, T) \times L^2(0, T; H_0^1(\Omega)).$$

Furthermore the following maximum principle holds

$$0 < \delta_1 \leq h(t, x) \leq h_2 \quad \text{for a.e. } x \in \Omega \text{ and for any } t \in (0, T).$$

Theorem 3.2 is proven in [16] in the degenerated case $\beta = 0$. The main difficulty is the handling of the degeneracy since the classical Aubin's Lemma can not be applied. Furthermore, we need to assume the thickness of freshwater zone $\geq \delta_1 > 0$ inside the aquifer to ensure an uniform estimate in L^2 space of the gradient of fresh water hydraulic head f.

With the additional diffuse interface (corresponding to the case $\beta = 1$), the system has a parabolic structure, it is thus no longer necessary to introduce viscous terms in a preliminary fixed point step for avoiding degeneracy . But we still need to impose a freshwater thickness strictly positive inside the aquifer to prove an uniform estimate of the gradient of f since the presence of the diffuse interface does not allow us to get this estimate. We can then establish the same maximum principle for the sharp interface approximation than for that of the diffuse interface.

3.2.2. *Case of unconfined aquifer.* We focus now on the unconfined case. $\tilde{K}_f$ is now denoted by K and we set $\alpha = 1$. We assume that depth h_2 is constant, $h_2 > 0$. We distinguish the two approaches as follows :

• $\beta = 0$. We define functions T_s and T_f by

$$T_s(u) = \begin{cases} h_2 - u & u \in (0, h_2) \\ 0 & u \leq 0 \,. \end{cases} \quad T_f(u) = u, \quad \forall u \in (\delta_1, h_2).$$

Function T_s is extended continuously and constantly for $u \geq h_2$ and T_f is extended continuously and constantly outside (δ_1, h_2). This condition on T_f imposes a thickness of freshwater always $\geq \delta_1$ inside the aquifer.

• $\beta = 1$. We define functions T_s and T_f by

$$T_s(u) = h_2 - u, \quad T_f(u) = u, \quad \text{for } u \in (0, h_2)$$

and T_s and T_f are extended continuously and constant outside $(0, h_2)$.

Then we consider the following set of equations in Ω_T,

$$\begin{aligned} &\phi \partial_t h - \nabla \cdot \left(K T_s(h) \nabla h \right) - \nabla \cdot \left(\beta \delta \phi \nabla h \right) - \nabla \cdot \left(K T_s(h) \mathcal{X}_0(h_1) \nabla h_1 \right) \\ &= -Q_s T_s(h), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\phi \partial_t h_1 - \nabla \cdot \left(K \left(T_f(h - h_1) + T_s(h) \right) \nabla h_1 \right) - \nabla \cdot \left(\beta \delta \phi \nabla h_1 \right) \\ &\quad - \nabla \cdot \left(K T_s(h) \mathcal{X}_0(h_1) \nabla h \right) \\ &= -\mathcal{X}_0(h_1) \left(Q_f T_f(h - h_1) + Q_s T_s(h) \right). \end{aligned} \tag{3.7}$$

Notice that we do not use $h^+ = \sup(0, h)$ and $h_1^+ = \sup(0, h_1)$ in functions T_s and T_f because a maximum principle will ensure that these supremums are useless.

Likewise, we have canceled the terms $\mathcal{X}_0(h)$ (resp. $\mathcal{X}_0(h_1)$) in front of $\partial_t h$ and ∇h (resp. $\partial_t h_1$). System (3.7) is completed by the following boundary and initial conditions:

$$h = h_D, \quad h_1 = h_{1,D} \quad \text{in } \Gamma \times (0,T), \tag{3.8}$$

$$h(0,x) = h_0(x), \quad h_1(0,x) = h_{1,0}(x) \quad \text{in } \Omega, \tag{3.9}$$

with the compatibility conditions

$$h_0(x) = h_D(0,x), \quad h_{1,0}(x) = h_{1,D}(0,x), \quad x \in \Gamma.$$

We make the same mathematical assumptions than above for the porosity and the hydraulic conductivity tensor K but we do not make any assumptions on the sign of the source terms Q_f and Q_s. Functions h_D and $h_{1,D}$ belong to the space $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$ while functions h_0 and $h_{1,0}$ are in $H^1(\Omega)$. Finally, we assume that the boundary and initial data satisfy physically realistic conditions on the hierarchy of interfaces depths:

$$0 \le h_{1,D} \le h_D \le h_2 \quad \text{a.e. in } \Gamma \times (0,T), \quad 0 \le h_{1,0} \le h_0 \le h_2 \quad \text{a.e. in } \Omega.$$

Now we state and prove the following existence result.

Theorem 3.3. *Assume a spatial heterogeneity for the hydraulic conductivity tensor:*

$$K_+ \le 2\sqrt{\gamma} K_-, \quad 0 < \gamma < \frac{8}{9}.$$

Then for any $T > 0$, problem (3.6)-(3.9) admits a weak solution (h, h_1) satisfying

$$(h - h_D, h_1 - h_{1,D}) \in \left(L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;(H_0^1(\Omega))') \right)^2$$

Furthermore the following maximum principle holds,

- *If $\beta = 0$, $0 \le h_1(t,x)$ and $0 \le h(t,x) \le h_2$ a.e. $x \in \Omega$ and for any $t \in (0,T)$.*
- *If $\beta = 1$, $0 \le h_1(t,x) \le h(t,x) \le h_2$ a.e. $x \in \Omega$ and for any $t \in (0,T)$.*

Theorem 3.3 is proven in [11] in the non degenerated case $\beta = 1$, with condition $K_- \le K_+ \le \frac{3}{2}K_-$ on the spatial heterogeneity for the hydraulic conductivity. We aim to give an existence result of weak solutions for this model when $\beta = 0$. We introduce a viscous term depending on a parameter ϵ in the preliminary fixed point step for avoiding degeneracy. We again suppose the thickness of freshwater zone $\ge \delta_1 > 0$ inside the aquifer to ensure an uniform estimate in L^2 of the gradient of h_1. But, since ϵ is expected to tend to zero, we only can establish a weaker maximum principle without hierarchy between h_1 and h .

Remark 3.4. We can prove Theorem 3.2 without any restrictions on the sign of the source terms Q_f and Q_s, but in this case, we have to impose assumptions on additional leakage terms q_{Lf} and q_{Ls} like in [11].

Depths h_1 and h_2 are assumed to be constant for sake of simplicity but the proof extends directly to $h_i \in L^\infty(\Omega)$, $i = 1, 2$.

Next section is devoted to proofs of Theorem 3.2 for $\beta = 1$ and of Theorem 3.3 for $\beta = 0$. Let us sketch our strategy. First step consists in using a Schauder fixed point theorem for proving an existence result for an auxiliary regularized and truncated problem. More precisely, in the unconfined case, we regularize the equations by adding a viscous term and we also regularize the step function $\mathcal{X}_0$ with a parameter $\epsilon > 0$. Furthermore we introduce a weight based on the velocity of

the fresh front in the two equations. We show that the regularized solution satisfies the maximum principles announced in Theorem 3.2 and in Theorem 3.3. We then prove that we have sufficient control on the velocity of the fresh front to ignore the latter weight. We finally show sufficient uniform estimates to let the regularization ϵ tends to zero.

4. Proofs

4.1. Proof of Theorem 3.2.

Step 1: Existence for the truncated system. Let M be a positive constant to be determine later. For $x \in \mathbb{R}_+^*$, we set

$$L_M(x) = \min\left(1, \frac{M}{x}\right).$$

Such a truncation L_M was originally introduced in [17]. It allows to use the following point in the estimates hereafter.

For any $(g, g_1) \in (L^2(0, T; H^1(\Omega)))^2$, setting

$$d(g, g_1) = -T_s(g)L_M\big(\|\nabla g_1\|_{L^2(\Omega_T)^2}\big)\nabla g_1,$$

we have

$$\|d(g, g_1)\|_{L^2(0,T;H)} = \|T_s(g)L_M\big(\|\nabla g_1\|_{L^2(\Omega_T)^2}\big)\nabla g_1\|_{L^2(\Omega_T)^2} \leq Mh_2.$$

Now, we denote $L_M\big(\|\nabla g_1\|_{L^2(\Omega_T)^2}\big)$ by $L_M\big(\|\nabla g_1\|_{L^2}\big)$. The variational formulation of the problem under consideration involves the two following integral equations:

$$
\begin{aligned}
&\int_0^T \phi\langle\partial_t h, w\rangle_{V,V'} + \int_{\Omega_T} \delta\phi\nabla h \cdot \nabla w \\
&+ \int_{\Omega_T} T_s(h)\big(K\nabla h \cdot \nabla w - L_M(\|\nabla f\|_{L^2(\Omega_T)^2})K\nabla f \cdot \nabla w\big)\, dx\, dt \\
&+ \int_{\Omega_T} Q_s T_s(h) w\, dx\, dt = 0\,,
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
&\int_{\Omega_T} h_2 K\nabla f \cdot \nabla w\, dx\, dt - \int_{\Omega_T} T_s(h)K\nabla h \cdot \nabla w\, dx\, dt \\
&- \int_{\Omega_T} (Q_s T_s(h) + Q_f T_f(h)) w\, dx\, dt = 0.
\end{aligned}
\tag{4.2}
$$

For the fixed point strategy, we define the application

$$\mathcal{F} : L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega)) \to L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega))$$

$$(\bar{h}, \bar{f}) \to \mathcal{F}(\bar{h}, \bar{f}) = (\mathcal{F}_1(\bar{h}, \bar{f}) = h, \mathcal{F}_2(\bar{h}, \bar{f}) = f),$$

where the pair (h,f) is a solution of next variational problem: for all $w \in V$,

$$
\begin{aligned}
&\int_0^T \phi\langle\partial_t h, w\rangle_{V,V'} + \int_{\Omega_T} \delta\phi\nabla h \cdot \nabla w + \int_{\Omega_T} T_s(\bar{h})\big(K\nabla h \cdot \nabla w \\
&- L_M(\|\nabla \bar{f}\|_{L^2})K\nabla \bar{f} \cdot \nabla w\big)\, dx\, dt + \int_{\Omega_T} Q_s T_s(\bar{h}) w\, dx\, dt = 0\,,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&\int_{\Omega_T} h_2 K\nabla f \cdot \nabla w\, dx\, dt - \int_{\Omega_T} T_s(\bar{h})K\nabla h \cdot \nabla w\, dx\, dt \\
&- \int_{\Omega_T} (Q_s T_s(\bar{h}) + Q_f T_f(\bar{h})) w\, dx\, dt = 0.
\end{aligned}
\tag{4.4}
$$

Indeed we know from classical parabolic theory (see *e.g.* [15]) that the linear variational system (4.3)-(4.4) admits an unique solution. The end of the present subsection is devoted to the proof a fixed point property for application $\mathcal{F}$.

Continuity of $\mathcal{F}_1$. Let $(\bar{h^n}, \bar{f^n})$ be a sequence of functions of $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ and $(\bar{h}, \bar{f})$ be a function of $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ such that

$$(\bar{h^n}, \bar{f^n}) \to (\bar{h}, \bar{f}) \quad \text{in } L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)).$$

We set $h_n = \mathcal{F}_1(\bar{h^n}, \bar{f^n})$ and $h = \mathcal{F}_1(\bar{h}, \bar{f})$. We aim showing that $h_n \to h$ in $L^2(0,T;H^1(\Omega))$.

For all $n \in \mathbb{N}$, h_n satisfies (4.3). Choosing $w = h_n - h_D$ in the n-dependent counterpart of (4.3) yields

$$\int_0^T \phi \langle \partial_t(h_n - h_D), h_n - h_D \rangle_{V',V} dt + \int_{\Omega_T} (\delta\phi + KT_s(\bar{h^n})) \nabla h_n \cdot \nabla h_n \, dx \, dt$$

$$= \int_{\Omega_T} \left(T_s(\bar{h^n}) L_M(\|\nabla \bar{f^n}\|_{L^2}) K \nabla \bar{f^n} \cdot \nabla(h_n - h_D) \right) dx \, dt,$$

$$\int_{\Omega_T} -Q_s T_s(\bar{h^n})(h_n - h_D) \, dx \, dt - \int_0^T \langle \partial_t h_D, h_n - h_D \rangle_{V',V} dt$$

$$+ \int_{\Omega_T} (\delta\phi + K T_s(\bar{h^n})) \nabla h_n \cdot \nabla h_D \, dx \, dt$$

Function $h_n - h_D$ belongs to $L^2(0,T;V) \cap H^1(0,T;V')$ and then to $\mathcal{C}(0,T;L^2(\Omega))$. Thus, thanks moreover to Lemma 3.1, we write

$$\int_0^T \phi \langle \partial_t(h_n - h_D), (h_n - h_D) \rangle_{V',V} dt = \frac{\phi}{2}\|h_n(\cdot,T) - h_D\|_H^2 - \frac{\phi}{2}\|h_0 - h_{D|t=0}\|_H^2.$$

Also

$$\int_{\Omega_T} (\delta\phi + KT_s(\bar{h^n})) \nabla h_n \cdot \nabla h_n \, dx \, dt \geq \delta\phi \|\nabla h_n\|_{L^2(0,T;H)}^2.$$

Then applying Cauchy-Schwarz and Young inequalities, for all $\epsilon_1 > 0$ we obtain

$$\left| \int_{\Omega_T} (\delta\phi + KT_s(\bar{h^n})) \nabla h_n \cdot \nabla h_D \, dx \, dt \right|$$

$$\leq (\delta\phi + K_+ h_2) \|\nabla h_n\|_{L^2(0,T;H)} \|\nabla h_D\|_{L^2(0,T;H)}$$

$$\leq \frac{\varepsilon_1}{2} \|\nabla h_n\|_{L^2(0,T;H)}^2 + \frac{(\delta\phi + K_+ h_2)^2}{2\varepsilon_1} \|\nabla h_D\|_{L^2(0,T;H)}^2,$$

$$\left| - \int_{\Omega_T} KT_s(\bar{h^n}) L_M(\|\nabla \bar{f^n}\|_{L^2}) \nabla \bar{f^n} \cdot \nabla h_n \, dx \, dt \right|$$

$$\leq K_+ \|d(\bar{h^n}, \bar{f^n})\|_{L^2(0,T;H)} \|\nabla h_n\|_{L^2(0,T;H)}$$

$$\leq M K_+ h_2 \|\nabla h_n\|_{L^2(0,T;H)}$$

$$\leq \frac{K_+^2 M^2}{2\varepsilon_1} h_2^2 + \frac{\varepsilon_1}{2} \|\nabla h_n\|_{L^2(0,T;H)}^2.$$

Since it depends on h_D, the next term is simply estimated by

$$\left| \int_{\Omega_T} KT_s(\bar{h^n}) L_M(\|\nabla \bar{f^n}\|_{L^2}) \nabla \bar{f^n} \cdot \nabla h_D \, dx \, dt \right|$$

$$\leq K_+ \|d(\bar{h^n}, \bar{f^n})\|_{L^2(0,T;H)} \|h_D\|_{L^2(0,T;H^1)}$$

$$\leq MK_+ h_2 \|h_D\|_{L^2(0,T;H^1)}.$$

Finally we have

$$\Big| - \int_0^T \phi \langle \partial_t h_D, (h_n - h_D) \rangle_{V',V} dt \Big|$$
$$\leq \frac{\phi}{\delta} \|\partial_t h_D\|_{L^2(0,T;(H^1(\Omega))')}^2 + \frac{\delta\phi}{2} \|h_n\|_{L^2(0,T;V)}^2 + \frac{\delta\,\phi}{2} \|h_D\|_{L^2(0,T;V)}^2,$$

and

$$\Big| - \int_{\Omega_T} Q_s T_s(\bar{h}^n)(h_n - h_D)\, dx\, dt \Big| \leq \frac{\|Q_s\|_{L^2(0,T;H)}^2}{2\phi} h_2^2 + \frac{\phi}{2} \|h_n - h_D\|_{L^2(0,T;H)}^2.$$

Summing up all these estimates, after simplifications, we obtain

$$\frac{\phi}{2} \|h_n(\cdot,T) - h_D\|_H^2 + (\frac{\delta\phi}{2} - \varepsilon_1)\|\nabla h_n\|_{L^2(0,T;H)}^2$$
$$\leq \frac{\phi}{2} \|h_0 - h_{D|t=0}\|_H^2 + \frac{\phi}{2} \int_0^T \|h_n - h_D\|_H^2\, dt + \frac{\phi\delta}{2} \|h_D\|_{L^2(0,T;V)}^2$$
$$+ \Big(\frac{\|Q_s\|_{L^2(\Omega_T)}^2}{2\phi} + \frac{K_+^2 M^2}{2\varepsilon_1} \Big) h_2^2 + \frac{(\delta\phi + K_+ h_2)^2}{2\varepsilon_1} \|h_D\|_{L^2(0,T;V)}^2 \tag{4.5}$$
$$+ \frac{\phi}{\delta} \|\partial_t h_D\|_{L^2(0,T;(H^1(\Omega))')}^2 + MK_+ h_2 \|h_D\|_{L^2(0,T;H^1)}.$$

We choose ε_1 such that $\delta\phi/2 - \varepsilon_1 \geq \epsilon_0 > 0$ for some $\epsilon_0 > 0$. Relation (4.5) with Gronwall lemma enables to conclude that there exists real numbers $A_M = A_M(\phi, \delta, K, h_0, h_D, h_2, Q_s, M, T)$ and $B_M = B_M(\phi, \delta, K, h_0, h_D, h_2, Q_s, M, T)$ depending only on the data of the problem such that

$$\|h_n\|_{L^\infty(0,T;H)} \leq A_M, \quad \|h_n\|_{L^2(0,T;H^1)} \leq B_M. \tag{4.6}$$

Hence sequence $(h_n)_n$ is uniformly bounded in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;H)$. Notice that the estimate in $L^\infty(0,T;H)$ is justified by the fact that we could make the same computations replacing T by any $\tau \leq T$ in the time integration. In the sequel, we set

$$C_M = \max(A_M, B_M).$$

Now we prove that $(\partial_t(h_n - h_D))_n$ is bounded in $L^2(0,T;V')$.

$$\|\partial_t(h_n - h_D)\|_{L^2(0,T;V')}$$
$$= \sup_{\|w\|_{L^2(0,T;V)} \leq 1} \Big| \int_0^T \langle \partial_t(h_n - h_D), w \rangle_{V',V} dt \Big|$$
$$= \sup_{\|w\|_{L^2(0,T;V)} \leq 1} \Big| \int_0^T -\langle \partial_t h_D, w \rangle_{V',V} dt - \frac{1}{\phi} \Big(\int_{\Omega_T} \big(\delta\phi + KT_s(\bar{h}^n) \big) \nabla h_n \cdot \nabla w\, dx\, dt$$
$$+ \int_{\Omega_T} KT_s(\bar{h}^n) L_M \big(\|\nabla \bar{f}^n\|_{L^2} \big) \nabla \bar{f}^n \cdot \nabla w\, dx\, dt - \int_{\Omega_T} Q_s T_s(\bar{h}^n) w\, dx\, dt \Big) \Big|.$$

Since

$$\Big| \int_{\Omega_T} \big(\delta\phi + KT_s(\bar{h}^n) \big) \nabla h_n . \nabla w\, dx\, dt \Big| \leq (\delta\phi + K_+ h_2) \|h_n\|_{L^2(0,T;H^1(\Omega))} \|w\|_{L^2(0,T;V)},$$

and since h_n is uniformly bounded in $L^2(0,T;H^1(\Omega))$, we write

$$\Big| \int_{\Omega_T} \big(\delta\phi + KT_s(\bar{h}^n)\big)\nabla h_n \cdot \nabla w \, dx \, dt \Big| \le \big(\delta\phi + K_+ h_2\big)C_M \|w\|_{L^2(0,T;V)}. \quad (4.7)$$

Furthermore,

$$\Big| \int_{\Omega_T} T_s(\bar{h}^n)L_M\big(\|\nabla\bar{f}^n\|_{L^2}\big)\nabla\bar{f}^n \cdot \nabla w \, dx \, dt \Big| \le M h_2 \|w\|_{L^2(0,T;V)}, \quad (4.8)$$

$$\Big| \int_{\Omega_T} Q_s T_s(\bar{h}^n) w \, dx \, dt \Big| \le \|Q_s\|_{L^2(\Omega_T)} h_2 \|w\|_{L^2(0,T;V)}. \quad (4.9)$$

Summing up (4.7)–(4.9), we conclude that

$$\begin{aligned}
\|\partial_t h_n\|_{L^2(0,T;V')} \le \frac{1}{\phi}\Big(&\|\partial_t h_D\|^2_{L^2(0,T;(H^1(\Omega))')} + \delta\phi C_M \\
&+ h_2(K_+ C_M + M + \|Q_s\|_{L^2(\Omega_T)})\Big) := D_M.
\end{aligned} \quad (4.10)$$

We have proved that $(h_n)_n$ is uniformly bounded in $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')$. Using Aubin's lemma, we extract a subsequence, not relabeled for convenience, $(h_n)_n$, converging strongly in $L^2(\Omega_T)$ and weakly in the space $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')$ to some limit denoted by ℓ. Using in particular the strong convergence in $L^2(\Omega_T)$ and thus the convergence a.e. in Ω_T, we check that ℓ is a solution of (4.3). The solution of (4.3) being unique, we actually have $\ell = h$.

It remains to prove that $(h_n)_n$ actually tends to h strongly in $L^2(0,T;H^1(\Omega))$. Subtracting the weak formulation (4.3) to its n-dependent counterpart for the test function $w = h_n - h$, we obtain

$$\begin{aligned}
&\int_0^T \phi\langle\partial_t(h_n - h), h_n - h\rangle_{V',V} \, dt \\
&+ \int_{\Omega_T} \big(\delta\phi + KT_s(\bar{h}^n)\big)\nabla(h_n - h) \cdot \nabla(h_n - h) \, dx \, dt \\
&- \int_{\Omega_T} K\big(T_s(\bar{h}^n) - T_s(\bar{h})\big)\nabla(h_n - h) \cdot \nabla h \, dx \, dt \\
&+ \int_{\Omega_T} K\big(T_s(\bar{h}^n)L_M\big(\|\nabla\bar{f}^n\|_{L^2}\big)\nabla\bar{f}^n - T_s(\bar{h})L_M\big(\|\nabla\bar{f}\|_{L^2}\big)\nabla\bar{f}\big) \cdot \nabla(h_n - h) \, dx \, dt \\
&+ \int_{\Omega_T} Q_s\big(T_s(\bar{h}^n) - T_s(\bar{h})\big)(h_n - h) \, dx \, dt = 0.
\end{aligned} \quad (4.11)$$

Using assumption $(\bar{h}^n, \bar{f}^n) \to (\bar{h}, \bar{f})$ in $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ and the above results of convergence for h_n, the limit as $n \to \infty$ in (4.11) reduces to

$$\lim_{n\to\infty} \Big(\int_{\Omega_T} \big(\delta\phi + KT_s(\bar{h}^n)\big)\nabla(h_n - h) \cdot \nabla(h_n - h) \, dx \, dt \Big) = 0.$$

Due to the positiveness of K, we infer from the latter relation that

$$\lim_{n\to\infty} \Big(\int_{\Omega_T} \delta\phi|\nabla(h_n - h)|^2 \, dx \, dt + \int_{\Omega_T} K_- T_s(\bar{h}^n)|\nabla(h_n - h)|^2 \, dx \, dt \Big) \le 0.$$

Hence $\nabla h_n \to \nabla h$ strongly in $L^2(0,T;H)$. Continuity of $\mathcal{F}_1$ for the strong topology of $L^2(0,T;H^1(\Omega))$ is proved.

Continuity of $\mathcal{F}_2$. Likewise, we prove the continuity of $\mathcal{F}_2$ by setting $f_n = \mathcal{F}_2(\bar{h}^n, \bar{f}^n)$ and $f = \mathcal{F}_2(\bar{h}, \bar{f})$ and showing that $f_n \to f$ in $L^2(0,T;H^1(\Omega))$. The key estimates are obtained using the same type of arguments than in the proof of the continuity of $\mathcal{F}_1$. We thus do not detail the computations. Let us only emphasize that we can now use the estimate (4.6) previously derived for h^n, thus the dependence with regard to C_M in the estimate

$$\|f_n\|_{L^2(0,T;H^1)} \le E_M = F_M\big(\phi, \delta, K, f_D, h_2, Q_s, Q_f, M, C_M, T\big). \qquad (4.12)$$

Conclusion. $\mathcal{F}$ is continuous in $(L^2(0,T;H^1(\Omega)))^2$ because its two components $\mathcal{F}_1$ and $\mathcal{F}_2$ are. Furthermore, let $A \in \mathbb{R}_+^*$ be the real number defined by

$$A = \max(C_M, D_M, E_M),$$

and W be the nonempty (strongly) closed convex bounded set in $(L^2(0,T;H^1(\Omega)))^2$ defined by

$$W = \Big\{ (g, g_1) \in \big(L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')\big)^2 ; (g(0), g_1(0)) = (h_0, f_0),$$

$$(g|_\Gamma, g_1|_\Gamma) = (h_D, f_D); \|(g, g_1)\|_{(L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V')) \times L^2(0,T;H^1(\Omega))} \le A \Big\}.$$

We have shown that $\mathcal{F}(W) \subset W$. It follows from Schauder theorem [20, cor. 9.7] that there exists $(h, f) \in W$ such that $\mathcal{F}(h, f) = (h, f)$. This fixed point for $\mathcal{F}$ is a weak solution of truncated problem (4.1)-(4.2).

Step 2: Maximum Principles. We are going to prove that for almost every $x \in \Omega$ and for all $t \in (0, T)$,

$$\delta_1 \le h(t,x) \le h_2.$$

First we show that $h(t,x) \le h_2$ a.e. $x \in \Omega$ and for all $t \in (0, T)$. We set

$$h_m = \big(h - h_2\big)^+ = \sup(0, h - h_2) \in L^2(0,T;V).$$

It satisfies $\nabla h_m = \chi_{\{h>h_2\}} \nabla h$ and $h_m(t,x) \neq 0$ if and only if $h(t,x) > h_2$, where χ denotes the characteristic function. Let $\tau \in (0, T)$. Taking $w(t,x) = h_m(t,x)\chi_{(0,\tau)}(t)$ in (4.1) yields

$$\int_0^\tau \phi \langle \partial_t h, h_m \chi_{(0,\tau)} \rangle_{V',V}\, dt + \int_0^\tau \int_\Omega \delta\phi \nabla h \cdot \nabla h_m + \int_0^\tau \int_\Omega KT_s(h)\nabla h \cdot \nabla h_m\, dx\, dt$$

$$+ \int_0^\tau \int_\Omega KT_s(h)L_M\big(\|\nabla f\|_{L^2}\big)\nabla f \cdot \nabla h_m\, dx\, dt + \int_0^\tau \int_\Omega Q_s T_s(h)h_m\, dx\, dt = 0;$$

$$(4.13)$$

that is,

$$\int_0^\tau \phi \langle \partial_t h, h_m \rangle_{V',V}\, dt + \int_0^\tau \int_\Omega \delta\phi \chi_{\{h>h_2\}}|\nabla h|^2\, dx\, dt$$

$$+ \int_0^\tau \int_\Omega KT_s(h)\chi_{\{h>h_2\}}|\nabla h|^2\, dx\, dt$$

$$+ \int_0^\tau \int_\Omega KT_s(h)L_M\big(\|\nabla f\|_{L^2}\big)\nabla f \cdot \nabla h_m(x,t)\, dx\, dt$$

$$+ \int_0^\tau \int_\Omega Q_s T_s(h)h_m(x,t)\, dx\, dt = 0.$$

$$(4.14)$$

To evaluate the first term in the left-hand side of (4.14), we apply Lemma 1 with function f defined by $f(\lambda) = \lambda - h_2$, $\lambda \in \mathbb{R}$. We write

$$\int_0^\tau \phi\langle\partial_t h, h_m\rangle_{V',V}\, dt = \frac{\phi}{2}\int_\Omega \left(h_m^2(\tau, x) - h_m^2(0, x)\right) dx = \frac{\phi}{2}\int_\Omega h_m^2(\tau, x)\, dx,$$

since $h_m(0, \cdot) = \left(h_0(\cdot) - h_2(\cdot)\right)^+ = 0$. Moreover $T_s(h)\chi_{\{h>h_2\}} = 0$ by definition of T_s, the three last terms in the left-hand side of (4.14) are null. Hence (4.14) becomes

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau, x)\, dx \le -\int_0^\tau\int_\Omega \delta\phi\chi_{\{h>h_2\}}|\nabla h|^2\, dx\, dt \le 0.$$

Then $h_m = 0$ a.e. in Ω_T.

Now we claim that $\delta_1 \le h(t, x)$ a.e. $x \in \Omega$ and for all $t \in (0, T)$. We set

$$h_m = \left(h - \delta_1\right)^- \in L^2(0, T; V).$$

Let $\tau \in (0, T)$. We recall that $h_m(0, \cdot) = 0$ a.e. in Ω thanks to the maximum principle satisfied by the initial data h_0. Moreover, $\nabla(h - \delta_1) \cdot \nabla h_m = \chi_{\{\delta_1 - h > 0\}}|\nabla(h - \delta_1)|^2$. Thus, taking $w(t, x) = h_m(x, t)\chi_{(0,\tau)}(t)$ in (4.1) and

$$w(t, x) = \frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})h_m(x, t)\chi_{(0,\tau)}(t)$$

in (4.2) and adding the two equations gives

$$\int_0^\tau \phi\langle\partial_t h, h_m(x, t)\rangle_{V',V}\, dt + \int_{\Omega_\tau}(\delta\phi + KT_s(h))\nabla h \cdot \nabla h_m\, dx\, dt$$

$$-\int_{\Omega_\tau K} T_s(h)L_M(\|\nabla f\|_{L^2})\nabla f \cdot \nabla h_m\, dx\, dt$$

$$+\int_{\Omega_T}(h_2 - \delta_1)L_M(\|\nabla f\|_{L^2})K\nabla f \cdot \nabla h_m\, dx\, dt$$

$$-\int_{\Omega_\tau} T_s(h)\frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})K\nabla h \cdot \nabla h_m\, dx\, dt$$

$$+\int_{\Omega_\tau}\left(Q_s T_s(h)\left(1 - \frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})\right)h_m\right.$$

$$\left. - Q_f T_f(h)\frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})h_m\right) dx\, dt = 0.$$

By definition of $T_s(h)$, $T_s(h)\chi_{\{h<\delta_1\}} = h_2 - \delta_1$, we can simplify the above equation as follows

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau, x)dx + \int_{\Omega_\tau}\chi_{\{h<\delta_1\}}\delta\phi\nabla h \cdot \nabla h\, dx\, dt$$

$$+\int_{\Omega_\tau} Q_f T_f(h)\chi_{\{h<\delta_1\}}(\delta_1 - h)\frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})\, dx\, dt$$

$$+\int_{\Omega_\tau}(h_2 - \delta_1)\left(1 - \frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})\right)\chi_{\{h<\delta_1\}}K\nabla h \cdot \nabla h\, dx\, dt$$

$$+\int_{\Omega_\tau}\chi_{\{h<\delta_1\}}(h - \delta_1)Q_s(h_2 - \delta_1)\left(1 - \frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})\right) dx\, dt = 0.$$

We first note that

$$\left(1 - \frac{h_2 - \delta_1}{h_2}L_M(\|\nabla f\|_{L^2})\right) \ge 0.$$

Moreover, since $T_f(h)\chi_{\{h<\delta_1\}} = 0$ by definition of T_f and $Q_s \leq 0$, the previous equation leads to

$$\frac{1}{2}\int_\Omega h_m^2(\tau, x)dx \leq 0$$

and then $h_m = 0$ a.e. in Ω_T.

Step 3: Elimination of the auxiliary function L_M. We now claim that there exist a real number $B > 0$, not depending on M, such that any weak solution $(h, f) \in W$ of problem (4.1)-(4.2) satisfies

$$\|\nabla h\|_{L^2(0,T;H)} \leq B \quad \text{and} \quad \|\nabla f\|_{L^2(0,T;H)} \leq B. \tag{4.15}$$

Taking $w = h - h_D$ (resp. $w = f - f_D$) in (4.1) (resp. (4.2)) leads to

$$\int_0^T \phi\langle\partial_t h, h - h_D\rangle_{V',V}\, dt + \int_{\Omega_T} (\delta\,\phi + T_s(h)K)\nabla h \cdot \nabla(h - h_D)\, dx\, dt$$

$$= \int_{\Omega_T} \left(T_s(h)L_M(\|\nabla f\|_{L^2}))K\nabla f \cdot \nabla(h - h_D) - Q_s T_s(h)(h - h_D) \right) dx\, dt$$

and

$$\int_{\Omega_T} h_2 K\nabla f \cdot \nabla(f - f_D)\, dx dt - \int_{\Omega_T} T_s(h)K\nabla h \cdot \nabla(f - f_D)\, dx\, dt$$

$$= \int_{\Omega_T} (Q_s T_s(h)(f - f_D) + Q_f T_f(h)(f - f_D))\, dx\, dt.$$

Summing up the previous equations yields

$$\frac{\phi}{2}\int_\Omega [(h - h_D)^2(T, x) - (h - h_D)^2(0, x)]dx + \int_{\Omega_T} \delta\,\phi|\nabla h|^2\, dx\, dt$$

$$+ \int_{\Omega_T} K_- T_s(h)|\nabla(h - f)|^2\, dx\, dt + \int_{\Omega_T} K_-(h_2 - T_s(h))|\nabla f|^2\, dx\, dt$$

$$+ \int_{\Omega_T} K_- T_s(h)(1 - L_M(\|\nabla f\|_{L^2}))|\nabla h|^2\, dx\, dt$$

$$\leq -\int_0^T \phi\langle\partial_t h_D, h - h_D\rangle_{V',V} dt$$

$$+ \int_{\Omega_T} T_s(h)(1 - L_M(\|\nabla f\|_{L^2}))K\nabla h \cdot \nabla(h - f)\, dx\, dt$$

$$+ \int_{\Omega_T} \delta\phi\nabla h \cdot \nabla h_D\, dx\, dt + \int_{\Omega_T} T_s(h)K\nabla(h - f) \cdot \nabla h_D\, dx\, dt$$

$$+ \int_{\Omega_T} T_s(h)(1 - L_M(\|\nabla f\|_{L^2}))K\nabla f \cdot \nabla h_D\, dx\, dt$$

$$- \int_{\Omega_T} (T_s(h) - h_2)K\nabla f \cdot \nabla f_D\, dx\, dt$$

$$- \int_{\Omega_T} T_s(h)K\nabla(h - f) \cdot \nabla f_D\, dx\, dt + \int_{\Omega_T} T_s(h)K\nabla h \cdot \nabla f_D\, dx\, dt$$

$$+ \int_{\Omega_T} Q_s T_s(h)[(f - h) + (h_D - f_D)]\, dx\, dt + \int_{\Omega_T} Q_f h(f - f_D)\, dx\, dt$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}.$$

Using the Cauchy-Schwarz and Young inequalities, we obtain

$$|I_1| \leq \phi \|\partial_t h_D\|_{L^2(0,T,V')} \|h - h_D\|_{L^2(0,T,V)}$$
$$\leq \phi \|\partial_t h_D\|_{L^2(0,T,V')} (\|\nabla h\|_{L^2(0,T;L^2(\Omega))} + \|\nabla h_D\|_{L^2(0,T;L^2(\Omega))})$$
$$\leq \frac{\delta\phi}{6} \|\nabla h\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{3\phi}{2\delta} \|\partial_t h_D\|_{L^2(0,T,V')}^2 + \frac{\phi}{2} \|\partial_t h_D\|_{L^2(0,T,V')}^2$$
$$+ \frac{\phi}{2} \|\nabla h_D\|_{L^2(0,T,V')}^2,$$

$$|I_2| \leq K_+ \Big(\int_{\Omega_T} T_s(h)(1 - L_M(\|\nabla f\|_{L^2})) |\nabla h|^2 \, dx \, dt \Big)^{1/2}$$
$$\times \Big(\int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt \Big)^{1/2}$$
$$\leq \frac{K_+}{2} \Big(\int_{\Omega_T} T_s(h)(1 - L_M(\|\nabla f\|_{L^2})) |\nabla h|^2 \, dx \, dt \Big)$$
$$+ \frac{K_+}{2} \Big(\int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt \Big),$$

$$|I_3| \leq \phi \, \delta \Big(\int_{\Omega_T} |\nabla h|^2 \, dx \, dt \Big)^{1/2} \Big(\int_{\Omega_T} |\nabla h_D|^2 \, dx \, dt \Big)^{1/2}$$
$$\leq \frac{\delta\phi}{6} \int_{\Omega_T} |\nabla h|^2 \, dx \, dt + \frac{3\delta\phi}{2} \int_{\Omega_T} |\nabla h_D|^2 \, dx \, dt,$$

$$|I_4| \leq K_+ \Big(\int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt \Big)^{1/2} \Big(\int_{\Omega_T} h_2 |\nabla h_D|^2 \, dx \, dt \Big)^{1/2}$$
$$\leq \frac{K_-}{12} \int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt + \frac{6K_+^2 h_2}{2K_-} \int_{\Omega_T} |\nabla h_D|^2 \, dx \, dt,$$

$$|I_5| \leq K_+ \sqrt{h_2} \Big(\Big(\int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt \Big)^{1/2}$$
$$+ \Big(\int_{\Omega_T} T_s(h)(1 - L_M(\|\nabla f\|_{L^2})) |\nabla h|^2 \, dx \, dt \Big)^{1/2} \Big) \Big(\int_{\Omega_T} |\nabla h_D|^2 \, dx \, dt \Big)^{1/2}$$
$$\leq \frac{K_-}{12} \int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt$$
$$+ \frac{K_-}{12} \int_{\Omega_T} T_s(h)(1 - L_M(\|\nabla f\|_{L^2})) |\nabla h|^2 \, dx \, dt + \frac{6h_2 K_+^2}{K_-} \int_{\Omega_T} |\nabla h_D|^2 \, dx \, dt,$$

$$|I_6| \leq K_+ \sqrt{h_2} \Big(\int_{\Omega_T} (h_2 - T_s(h)) |\nabla f|^2 \, dx \, dt \Big)^{1/2} \Big(\int_{\Omega_T} |\nabla f_D|^2 \, dx \, dt \Big)^{1/2}$$
$$\leq \frac{K_-}{6} \int_{\Omega_T} (h_2 - T_s(h)) |\nabla f|^2 \, dx \, dt + \frac{3h_2 K_+^2}{2K_-} \int_{\Omega_T} |\nabla f_D|^2 \, dx \, dt,$$

$$|I_7| \leq K_+ \sqrt{h_2} \Big(\int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt \Big)^{1/2} \Big(\int_{\Omega_T} |\nabla f_D|^2 \, dx \, dt \Big)^{1/2}$$
$$\leq \frac{K_-}{12} \int_{\Omega_T} T_s(h) |\nabla(h - f)|^2 \, dx \, dt + \frac{6h_2 K_+^2}{2K_-} \int_{\Omega_T} |\nabla f_D|^2 \, dx \, dt,$$

$$|I_8| \leq K_+ h_2 \Big(\int_{\Omega_T} |\nabla h|^2 \, dx \, dt \Big)^{1/2} \Big(\int_{\Omega_T} |\nabla f_D|^2 \, dx \, dt \Big)^{1/2}$$

$$\leq \frac{\delta \, \phi}{6} \int_{\Omega_T} |\nabla h|^2 \, dx \, dt + \frac{3 h_2^2 K_+^2}{2 \delta \, \phi} \int_{\Omega_T} |\nabla f_D|^2 \, dx \, dt,$$

$$|I_9| \leq h_2 \|Q_s\|_{L^2(0,T;H)} (\|h - h_D\|_{L^2(0,T;H)} + \|f - f_D\|_{L^2(0,T;H)})$$

$$\leq h_2 \|Q_s\|_{L^2(0,T;H)} (\|h - h_D\|_{L^2(0,T;H)} + C_p \|\nabla(f - f_D)\|_{L^2(0,T;H)})$$

$$\leq \frac{\phi}{2} \int_{\Omega_T} (h - h_D)^2 \, dx \, dt + \frac{h_2^2}{2\phi} \|Q_s\|_{L^2(0,T;H)}^2$$

$$+ \frac{\delta_1 K_-}{6} \int_{\Omega_T} |\nabla f|^2 \, dx \, dt + \frac{3 C_p^2 h_2^2}{2 \delta_1 K_-} \|Q_s\|_{L^2(0,T;H)}$$

$$+ h_2 \|Q_s\|_{L^2(0,T;H)} C_p \|\nabla f_D\|_{L^2(0,T;H)},$$

$$|I_{10}| \leq h_2 \|Q_s\|_{L^2(0,T;H)} \|f - f_D\|_{L^2(0,T;H)}$$

$$\leq \frac{\delta_1 K_-}{6} \int_{\Omega_T} |\nabla f|^2 \, dx \, dt + \frac{3 C_p^2 h_2^2}{2 \delta_1 K_-} \|Q_s\|_{L^2(0,T;H)}$$

$$+ h_2 \|Q_s\|_{L^2(0,T;H)} C_p \|\nabla f_D\|_{L^2(0,T;H)}.$$

Since $\delta_1 \leq h \leq h_2$ a.e. in Ω_T, it follows that $T_s(h) - h_2 = h \geq \delta_1$ and then

$$\frac{\phi}{2} \int_{\Omega_T} (h - h_D)^2(\tau, x) dx + \frac{\delta \phi}{2} \int_{\Omega_T} |\nabla h|^2 \, dx \, dt + \frac{K_- \delta_1}{2} \int_{\Omega_T} |\nabla f|^2 \, dx \, dt$$

$$+ \int_{\Omega_T} \frac{(3K_- - 2K_+)}{4} T_s(h) |\nabla(h - f)|^2 \, dx \, dt$$

$$+ \int_{\Omega_T} \frac{11 K_- - 6 K_+}{12} T_s(h) (1 - L_M(\|\nabla f\|_{L^2})) |\nabla h|^2 \, dx \, dt$$

$$\leq \frac{\phi}{2} \int_{\Omega_T} (h - h_D)^2 \, dx \, dt + C(h_2, \delta, \delta_1, h_0, h_D, f_D, Q_f, Q_s)$$

Now we apply the Gronwall lemma and we deduce that there exists a real number B, that does not depend on M, such that

$$\|h\|_{L^\infty(0,T;H) \cap L^2(0,T;H^1(\Omega))} \leq B \quad \text{and} \quad \|f\|_{L^2(0,T;H^1(\Omega))} \leq B.$$

In particular, $\|\nabla f\|_{L^2(\Omega_T)^2} \leq B$ and this estimate does not depend on the choice of the real number M that defines function L_M. Hence if we choose $M = B$, any weak solution of the system

$$\phi \partial_t h - \nabla \cdot (\delta \, \phi \nabla h) - \nabla \cdot \big(K T_s(h) \nabla h \big) + \nabla \cdot \big(K T_s(h) L_B(\|\nabla f\|_{L^2}) \nabla f \big)$$

$$= -Q_s T_s(h),$$

$$-\nabla \cdot \big(K h_2 \nabla f \big) + \nabla \cdot \big(K T_s(h) \nabla h \big) = Q_f T_f(h - h_1) + Q_s T_s(h)$$

in Ω_T, with the initial and boundary conditions $h = h_D$ and $f = f_D$ on Γ, $h(0, x) = h_0$ a.e. in Ω, satisfies $L_B \big(\|\nabla f\|_{L^2} \big) = 1$. Then the term $L_B \big(\|\nabla f\|_{L^2} \big) = 1$ may be dropped. The proof of Theorem 3.2 is complete.

4.2. **Proof of Theorem 3.3.** Let $\epsilon > 0$ and let M be a positive constant to be determined later. For $x \in \mathbb{R}_+^*$, we set

$$L_M(x) = \min\left(1, \frac{M}{x}\right).$$

Such a truncation L_M allows again to use the following point in the estimates hereafter. For any $(g, g_1) \in (L^2(0, T; H^1(\Omega)))^2$, setting

$$d(g, g_1) = -T_s(g)L_M\big(\|\nabla g_1\|_{L^2(\Omega_T)}\big)\nabla g_1,$$

we have

$$\|d(g, g_1)\|_{L^2(0,T;H)} = \|T_s(g)L_M\big(\|\nabla g_1\|_{L^2(\Omega_T)}\big)\nabla g_1\|_H \leq M h_2.$$

We also define a regularized step function for $\mathcal{X}_0$ by

$$\mathcal{X}_0(h_1) = \begin{cases} 0 & \text{if } h_1 \leq 0 \\ 1 & \text{if } h_1 > 0, \end{cases} \qquad \mathcal{X}_0^\epsilon(h_1) = \begin{cases} 0 & \text{if } h_1 \leq 0 \\ h_1/(h_1^2 + \epsilon)^{1/2} & \text{if } h_1 > 0. \end{cases}$$

Then $0 \leq \mathcal{X}_0^\epsilon \leq 1$ and $\mathcal{X}_0^\epsilon \to \mathcal{X}_0$ as $\epsilon \to 0$. Introducing the regularization $\mathcal{X}_0^\epsilon$ of $\mathcal{X}_0$, we replace system (3.7) by the system

$$\phi \partial_t h^\epsilon - \nabla \cdot (\epsilon \nabla h^\epsilon) - \nabla \cdot \big(KT_s(h^\epsilon)\nabla h^\epsilon\big)$$
$$- \nabla \cdot \big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)L_M\big(\|\nabla h_1^\epsilon\|_{L^2}\big)\nabla h_1^\epsilon\big)$$
$$= -Q_s T_s(h^\epsilon),$$

$$\phi \partial_t h_1^\epsilon - \nabla \cdot (\epsilon \nabla h_1^\epsilon) - \nabla \cdot \big(K\big(T_f(h^\epsilon - h_1^\epsilon) + T_s(h^\epsilon)\big)\nabla h_1^\epsilon\big)$$
$$- \nabla \cdot \big(KT_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon)\nabla h^\epsilon\big)$$
$$= -Q_f T_f(h^\epsilon - h_1^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon) - Q_s T_s(h^\epsilon)\mathcal{X}_0^\epsilon(h_1^\epsilon).$$

The proof is outlined as follows: In the first step, using Schauder theorem, we prove that for every $T > 0$ and every $\epsilon > 0$, the above regularized system completed by the initial and boundary conditions

$$h^\epsilon = h_D, \quad h_1^\epsilon = h_{1,D} \quad \text{in } \Gamma \times (0, T),$$
$$h^\epsilon(0, x) = h_0(x), \quad h_1^\epsilon(0, x) = h_{1,0}(x) \quad \text{a.e. in } \Omega,$$

has a solution $(h^\epsilon - h_D, h_1^\epsilon - h_{1,D}) \in W(0, T) \times W(0, T)$. We observe that the sequence $(h^\epsilon - h_D, h_1^\epsilon - h_{1,D})$ is uniformly bounded in $(L^2(0, T; V))^2$ and we show the maximum principle $0 \leq h_1^\epsilon(t, x)$ and $0 \leq h^\epsilon(t, x) \leq h_2$ a.e. in Ω_T for every $\epsilon > 0$ Finally we prove that any (weak) limit (h, h_1) in $(L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V'))^2$ of the sequence $(h^\epsilon, h_1^\epsilon)$ is a solution of the original problem.

Step 1: Existence for the regularized system. We now omit ϵ for the sake of simplicity in the notation. Then the weak formulation of the latter problem reads: for any $w \in V$,

$$\int_0^T \phi \langle \partial_t h, w \rangle_{V',V} dt + \int_{\Omega_T} \epsilon \nabla h \cdot \nabla w \, dx \, dt$$

$$+ \int_{\Omega_T} KT_s(h)\nabla h \cdot \nabla w \, dx \, dt + \int_{\Omega_T} KT_s(h)\mathcal{X}_0^\epsilon(h_1)L_M\big(\|\nabla h_1\|_{L^2}\big)\nabla h_1 \cdot \nabla w \, dx \, dt$$

$$+ \int_{\Omega_T} Q_s T_s(h)w \, dx \, dt = 0,$$

$$(4.16)$$

$$\int_0^T \phi \langle \partial_t h_1, w \rangle_{V',V} dt + \int_{\Omega_T} \epsilon \nabla h_1 \cdot \nabla w \, dx \, dt$$

$$+ \int_{\Omega_T} K \Big(T_s(h) + T_f(h - h_1) \Big) \nabla h_1 \cdot \nabla w \, dx \, dt + \int_{\Omega_T} K \mathcal{X}_0^\epsilon(h_1) T_s(h) \nabla h \cdot \nabla w \, dx \, dt$$

$$+ \int_{\Omega_T} \Big(Q_f T_f(h - h_1) + Q_s T_s(h) \Big) \mathcal{X}_0^\epsilon(h_1^\epsilon) w \, dx \, dt = 0.$$

$$(4.17)$$

For the fixed point strategy, we define the application $\mathcal{F}$ by

$$\mathcal{F} : L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega)) \to L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega))$$

$$(\bar{h}, \bar{h}_1) \longmapsto \mathcal{F}(\bar{h}, \bar{h}_1) = \Big(\mathcal{F}_1(\bar{h}, \bar{h}_1) = h, \mathcal{F}_2(\bar{h}, \bar{h}_1) = h_1 \Big),$$

where (h, h_1) is the solution of

$$\int_0^T \phi \langle \partial_t h, w \rangle_{V',V} dt + \int_{\Omega_T} \epsilon \nabla h \cdot \nabla w \, dx \, dt + \int_{\Omega_T} K T_s(\bar{h}) \nabla h \cdot \nabla w \, dx \, dt$$

$$+ \int_{\Omega_T} K T_s(\bar{h}) L_M \big(\| \nabla \bar{h}_1 \|_{L^2} \big) \mathcal{X}_0^\epsilon(\bar{h}_1) \nabla \bar{h}_1 \cdot \nabla w \, dx \, dt + \int_{\Omega_T} Q_s T_s(\bar{h}) w \, dx \, dt \quad (4.18)$$

$$= 0 \quad \forall w \in V,$$

$$\int_0^T \phi \langle \partial_t h_1, w \rangle_{V',V} dt + \int_{\Omega_T} \epsilon \nabla h_1 \cdot \nabla w \, dx \, dt$$

$$+ \int_{\Omega_T} K \Big(T_s(\bar{h}) + T_f(\bar{h} - \bar{h}_1) \Big) \nabla h_1 \cdot \nabla w \, dx \, dt + \int_{\Omega_T} K T_s(\bar{h}) \mathcal{X}_0^\epsilon(\bar{h}_1) \nabla h \cdot \nabla w \, dx \, dt$$

$$+ \int_{\Omega_T} \Big(Q_f T_f(\bar{h} - \bar{h}_1) + Q_s T_s(\bar{h}) \Big) \mathcal{X}_0^\epsilon(\bar{h}_1) w \, dx \, dt = 0 \quad \forall w \in V.$$

$$(4.19)$$

Indeed we know from classical parabolic theory (see *e.g.* [15]) that the linear variational system (4.18)-(4.19) admits an unique solution. It remains to prove a fixed point property for application $\mathcal{F}$.

Since the proofs of the continuity of $\mathcal{F}_1$ and $\mathcal{F}_2$ are very similar to the previous ones, we do not reproduce here the computations. We also emphasize that this step is proven in [11] by considering that the parameter ϵ plays the same role as δ, the thickness of the diffuse interface, of course the estimates depend on ϵ, but it is sufficient for this step. We can thus conclude that $\mathcal{F}$ is continuous in $(L^2(0, T; H^1(\Omega)))^2$ because its two components $\mathcal{F}_1$ and $\mathcal{F}_2$ are. Furthermore, there exist a real number $A \in \mathbb{R}_+^*$ (depending on the data and on the parameter ϵ) and a non empty (strongly) closed convex bounded set W in $(L^2(0, T; H^1(\Omega)))^2$ defined by

$$W = \Big\{ (g, g_1) \in \big(L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V') \big)^2 ; (g(0), g_1(0)) = (h_0, h_{1,0}),$$

$$(g|_\Gamma, g_1|_\Gamma) = (h_D, h_{1,D}); \| (g, g_1) \|_{(L^2(0,T;H^1(\Omega)) \cap H^1(0,T;V'))^2} \leq A \Big\}.$$

such that $\mathcal{F}(W) \subset W$. It follows from Schauder theorem [20, cor. 9.7] that there exists $(h, h_1) \in W$ such that $\mathcal{F}(h, h_1) = (h, h_1)$. This fixed point for $\mathcal{F}$ is a weak solution of problem (4.16)-(4.17).

Step 2: Maximum Principles. We aim at proving that for almost all $x \in \Omega$ and for all $t \in (0,T)$, $0 \leq h_1(t,x)$ and $0 \leq h(t,x) \leq h_2$.

First we show that $h(t,x) \leq h_2$ a.e. $x \in \Omega$ and all $t \in (0,T)$. We set

$$h_m = (h - h_2)^+ = \sup(0, h - h_2) \in L^2(0,T;V).$$

It satisfies $\nabla h_m = \chi_{\{h>h_2\}} \nabla h$ and $h_m(t,x) \neq 0$ if and only if $h(t,x) > h_2$, where χ denotes the characteristic function. Let $\tau \in (0,T)$. Taking $w(t,x) = h_m(t,x)\chi_{(0,\tau)}(t)$ in (4.16) yields

$$
\begin{aligned}
&\int_0^\tau \phi \langle \partial_t h, h_m \rangle_{V',V} \, dt + \int_0^\tau \int_\Omega \epsilon \chi_{\{h>h_2\}} |\nabla h|^2 \, dx \, dt \\
&+ \int_0^\tau \int_\Omega K_- T_s(h) \chi_{\{h>h_2\}} |\nabla h|^2 \, dx \, dt \\
&+ \int_0^\tau \int_\Omega \mathcal{X}_0^\epsilon(h_1^\epsilon) \chi_{\{h>h_2\}} K T_s(h) L_M\big(\|\nabla h_1\|_{L^2}\big) \nabla h_1 \cdot \nabla h(x,t) \, dx \, dt \\
&+ \int_0^\tau \int_\Omega Q_s T_s(h) h_m(x,t) \, dx \, dt \leq 0.
\end{aligned}
\tag{4.20}
$$

Lemma 1 applied to (4.20), gives

$$\int_0^\tau \phi \langle \partial_t h, h_m \rangle_{V',V} \, dt = \frac{\phi}{2} \int_\Omega \big(h_m^2(\tau,x) - h_m^2(0,x)\big) \, dx = \frac{\phi}{2} \int_\Omega h_m^2(\tau,x) \, dx,$$

taking into account $h_m(0,\cdot) = \big(h_0(\cdot) - h_2(\cdot)\big)^+ = 0$. Since $T_s(h)\chi_{\{h>h_2\}} = 0$ by definition of T_s, the three last terms of (4.20) are null. Hence (4.20) becomes

$$\frac{\phi}{2} \int_\Omega h_m^2(\tau,x) \, dx \leq -\int_0^\tau \int_\Omega \epsilon \chi_{\{h>h_2\}} |\nabla h|^2 \, dx \, dt \leq 0.$$

Then $h_m = 0$ a.e. in Ω_T.

Now we claim that $0 \leq h(t,x)$ a.e. $x \in \Omega$ and all $t \in (0,T)$. We set

$$h_m = (-h)^+ \in L^2(0,T;V) \quad \text{since } h_D(\cdot,\cdot) \geq 0.$$

Let $\tau \in (0,T)$. We recall that $h_m(0,\cdot) = 0$ a.e. in Ω thanks to the maximum principle satisfied by the initial data h_0. Moreover, $\nabla h_m = -\chi_{\{h<0\}} \nabla h$. Thus, taking $w(t,x) = h_m(x,t)\chi_{(0,\tau)}(t)$ in (4.16) yields

$$
\begin{aligned}
&\int_0^\tau \phi \langle \partial_t h, h_m \rangle_{V',V} \, dt - \int_0^\tau \int_\Omega \epsilon \chi_{\{h<0\}} |\nabla h|^2 \, dx \, dt \\
&= \int_0^\tau \int_\Omega T_s(h) \chi_{\{h<0\}} K \nabla h \cdot \nabla h \, dx \, dt \\
&+ \int_0^\tau \int_\Omega \mathcal{X}_0^\epsilon(h_1) \chi_{\{h<0\}} K T_s(h) L_M\big(\|\nabla h_1\|_{L^2}\big) \nabla h_1 \cdot \nabla h(x,t) \, dx \, dt \\
&- \int_0^\tau \int_\Omega Q_s T_s(h) h_m(x,t) \, dx \, dt.
\end{aligned}
\tag{4.21}
$$

Applying Lemma 1 to the first term of (4.21) and taking into account $h_m(0,\cdot) = (-h_0)^+ = 0$, gives

$$\int_0^\tau \phi \langle \partial_t h, h_m \rangle_{V',V} \, dt = \frac{-\phi}{2} \int_\Omega \big(h_m^2(\tau,x) - h_m^2(0,x)\big) \, dx = -\frac{\phi}{2} \int_\Omega h_m^2(\tau,x) \, dx.$$

Since $T_s(h)\chi_{\{h<0\}} = 0$ by definition of T_s, the three last terms in the left-hand side of (4.21) are null. Hence (4.21) becomes

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx \leq -\int_0^\tau \int_\Omega \epsilon\chi_{\{h>h_2\}}|\nabla h|^2\,dx\,dt \leq 0.$$

Then $h_m = 0$ a.e. in Ω_T.

We finally prove that $0 \leq h_1(t,x)$ a.e. $x \in \Omega$ and all $t \in (0,T)$. We set

$$h_m = \big(-h_1\big)^+ \in L^2(0,T;V) \quad \text{since } h_{1,D}(\cdot,\cdot) \geq 0.$$

Let $\tau \in (0,T)$. We recall that $h_m(0,\cdot) = 0$ a.e. in Ω thanks to the maximum principle satisfied by the initial data $h_{1,0}$. Moreover, $\nabla h_m = -\chi_{\{h_1<0\}}\nabla h_1$. Thus, taking $w(t,x) = h_m(x,t)\chi_{(0,\tau)}(t)$ in (4.17) yields

$$\int_0^\tau \phi\langle \partial_t h_1, h_m\rangle_{V',V}\,dt - \int_{\Omega_\tau} \chi_{\{h_1<0\}}(\epsilon + T_s(h) + T_f(h-h_1))\nabla h_1 \cdot \nabla h_1\,dx\,dt$$

$$- \int_{\Omega_\tau} \mathcal{X}_0^\epsilon(h_1)T_s(h)\chi_{\{h_1<0\}}\nabla h \cdot \nabla h_1\,dx\,dt$$

$$- \int_{\Omega_\tau} \big(Q_s T_s(h) + Q_f T_f(h-h_1)\big)\mathcal{X}_0^\epsilon(h_1)\chi_{\{h_1<0\}}h_1\,dx\,dt = 0.$$
$$(4.22)$$

Applying Lemma 1 to (4.22) and taking into account $h_m(0,\cdot) = \big(-h_0\big)^+ = 0$, gives

$$\int_0^\tau \phi\langle \partial_t h_1, h_m\rangle_{V',V}\,dt = -\frac{\phi}{2}\int_\Omega \Big(h_m^2(\tau,x) - h_m^2(0,x)\Big)\,dx = -\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx.$$

Since $\mathcal{X}_0^\epsilon(h_1)\chi_{\{h_1<0\}} = 0$ by definition of $\mathcal{X}_0^\epsilon(\cdot)$, the two last terms of (4.22) are null. Hence (4.22) becomes (since T_s and T_f are nonnegative functions)

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx \leq -\int_0^\tau \int_\Omega \epsilon\chi_{\{h>h_2\}}|\nabla h|^2\,dx\,dt \leq 0.$$

Then $h_m = 0$ a.e. in Ω_T.

Step 3: Elimination of the auxiliary function L_M. We now claim that there exist a real number $B > 0$, not depending on ϵ neither on M, such that any weak solution $(h,h_1) \in W$ of problem (4.16)-(4.17) satisfies

$$\|\nabla h\|_{L^2(0,T;H)} \leq B \quad \text{and} \quad \|\nabla h_1\|_{L^2(0,T;H)} \leq B. \tag{4.23}$$

Taking $w = h - h_D$ (resp. $w = h_1 - h_{1,D}$) in (4.16) (resp. (4.17)) leads to

$$\int_0^T \phi\langle \partial_t h, h - h_D\rangle_{V',V}\,dt + \int_{\Omega_T} \epsilon\nabla h \cdot \nabla(h - h_D)\,dx\,dt$$

$$+ \int_{\Omega_T} KT_s(h)\nabla h \cdot \nabla(h - h_D)\,dx\,dt$$

$$= -\int_{\Omega_T} KT_s(h)\mathcal{X}_0^\epsilon(h_1)L_M\big(\|\nabla h_1\|_{L^2}\big)\nabla h_1 \cdot \nabla(h - h_D)\,dx\,dt$$

$$- \int_{\Omega_T} Q_s T_s(h)(h - h_D)\,dx\,dt$$
$$(4.24)$$

and

$$
\begin{aligned}
&\int_0^T \phi\langle\partial_t h_1, h_1 - h_{1,D}\rangle_{V',V}\,dt + \int_{\Omega_T} \epsilon\nabla h_1 \cdot \nabla(h_1 - h_{1,D})\,dx\,dt \\
&+ \int_{\Omega_T} K\big(T_s(h) + T_f(h - h_1)\big)\nabla h_1 \cdot \nabla(h_1 - h_{1,D})\,dx\,dt \\
&= -\int_{\Omega_T} K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h \cdot \nabla(h_1 - h_{1,D})\,dx\,dt \\
&\quad - \int_{\Omega_T} \mathcal{X}_0^\epsilon(h_1)\big(Q_f T_f(h - h_1) + Q_s T_s(h)\big)(h_1 - h_{1,D})\,dx\,dt.
\end{aligned}
\tag{4.25}
$$

Summing up relations (4.24) and (4.25), and using the decomposition

$$
\begin{aligned}
&K\nabla h \cdot \nabla h + K\mathcal{X}_0^\epsilon(h_1)\big(L_M(\|\nabla h_1\|_{L^2}) + 1\big)\nabla h_1 \cdot \nabla h + K\nabla h_1 \cdot \nabla h_1 \\
&= K\nabla(h + h_1) \cdot \nabla(h + h_1) + K\Big(1 - \mathcal{X}_0^\epsilon(h_1)L_M(\|\nabla h_1\|_{L^2})\Big)\nabla h_1 \cdot \nabla h_1 \\
&\quad - K\Big(1 - \mathcal{X}_0^\epsilon(h_1)L_M(\|\nabla h_1\|_{L^2})\Big)\nabla h_1 \cdot \nabla(h + h_1),
\end{aligned}
\tag{4.26}
$$

we write

$$
\underbrace{\int_0^T \phi\Big(\langle\partial_t(h - h_D), h - h_D\rangle_{V',V}\,dt + \langle\partial_t(h_1 - h_{1,D}), h_1 - h_{1,D}\rangle_{V',V}\Big)\,dt}_{J_1}
$$

$$
+ \underbrace{\int_{\Omega_T} \epsilon\big(\nabla h \cdot \nabla h + \nabla h_1 \cdot \nabla h_1\big)\,dx\,dt}_{J_2} + \underbrace{\int_{\Omega_T} K T_s(h)\nabla(h + h_1) \cdot \nabla(h + h_1)\,dx\,dt}_{J_3}
$$

$$
+ \underbrace{\int_{\Omega_T} K\Big((1 - \mathcal{X}_0^\epsilon(h_1)L_M(\|\nabla h_1\|_{L^2}))T_s(h) + T_f(h - h_1)\Big)\nabla h_1 \cdot \nabla h_1\,dx\,dt}_{J_4}
$$

$$
= \underbrace{\int_{\Omega_T} K\big(1 - \mathcal{X}_0^\epsilon(h_1)L_M(\|\nabla h_1\|_{L^2})\big)T_s(h)\nabla h_1 \cdot \nabla(h + h_1)\,dx\,dt}_{J_5}
$$

$$
+ \underbrace{\int_{\Omega_T} \big(\epsilon + K T_s(h)\big)\nabla h \cdot \nabla h_D\,dx\,dt}_{J_6}
$$

$$
+ \underbrace{\int_{\Omega_T} \big(\epsilon + K T_s(h) + K T_f(h - h_1)\big)\nabla h_1 \cdot \nabla h_{1,D}\,dx\,dt}_{J_7}
$$

$$
+ \underbrace{\int_{\Omega_T} K T_s(h)L_M(\|\nabla h_1\|_{L^2})\mathcal{X}_0^\epsilon(h_1)\nabla h_1 \cdot \nabla h_D\,dx\,dt}_{J_8}
$$

$$
+ \underbrace{\int_{\Omega_T} K T_s(h)\mathcal{X}_0^\epsilon(h_1)\nabla h \cdot \nabla h_{1,D}\,dx\,dt}_{J_9}
$$

$$-\int_{\Omega_T} \big(Q_s T_s(h)(h-h_D) + \mathcal{X}_0^\epsilon(h_1)\big(Q_f T_f(h-h_1) + Q_s T_s(h)\big)(h_1 - h_{1,D})\big)\, dx\, dt$$

$$\underbrace{\hspace{10cm}}_{J_{10}}$$

$$-\int_0^T \phi\big(\langle \partial_t h_D, h - h_D\rangle_{V',V} + \langle \partial_t h_{1,D}, h_1 - h_{1,D}\rangle_{V',V}\big)\, dt\,.$$

$$\underbrace{\hspace{9cm}}_{J_{11}}$$

We now estimate all the terms in the above expression. We recall that

$$|\mathcal{X}_0^\epsilon(h_1)| \le 1, \quad L_M\big(\|\nabla h_1\|_{L^2}\big) \le 1, \quad 0 \le T_s(h) \le h_2, \quad \delta_1 \le T_f(h - h_1) \le h_2.$$

Then, we note that

$$|J_1| = \frac{\phi}{2}\int_\Omega \big((h - h_D)^2(T,x) - (h_0 - h_{0,D})^2(x)\big)\, dx$$

$$+ \frac{\phi}{2}\int_\Omega \big((h_1 - h_{1,D})^2(T,x) - (h_{1,0} - h_{0,D})^2(x)\big)\, dx,$$

$$|J_2| = \int_{\Omega_T} \epsilon |\nabla h|^2\, dx\, dt + \int_{\Omega_T} \epsilon |\nabla h_1|^2\, dx\, dt,$$

$$|J_3| \ge \int_{\Omega_T} K_- T_s(h)|\nabla(h + h_1)|^2\, dx\, dt,$$

$$|J_4| \ge \int_{\Omega_T} K_-\Big(\big(1 - \mathcal{X}_0^\epsilon(h_1)L_M(\|\nabla h_1\|_{L^2})\big)T_s(h) + \delta_1\Big)|\nabla h_1|^2\, dx\, dt.$$

Next, applying the Cauchy-Schwarz and Young inequalities, for some real $\gamma > 0$, we obtain

$$|J_5| \le \int_{\Omega_T} T_s(h)\Big(\frac{1}{4\gamma}\Big(1 - \mathcal{X}_0^\epsilon(h_1)L_M\big(\|\nabla h_1\|_{L^2}\big)\Big)^2$$

$$\times \frac{K_+^2}{K_-}|\nabla h_1|^2 + \gamma K_- |\nabla(h + h_1)|^2\Big)\, dx\, dt,$$

$$|J_6| \le \frac{\epsilon}{2}\int_{\Omega_T} |\nabla h|^2\, dx\, dt + \frac{\epsilon}{2}\int_{\Omega_T} |\nabla h_D|^2\, dx\, dt + \frac{\gamma K_-}{16}\int_{\Omega_T} T_s(h)|\nabla(h + h_1)|^2\, dx\, dt$$

$$+ \frac{4\, h_2 K_+^2}{\gamma K_-}\int_{\Omega_T} |\nabla h_D|^2\, dx\, dt + \frac{\delta_1 K_-}{12}\int_{\Omega_T} |\nabla h_1|^2\, dx\, dt$$

$$+ \frac{3\, h_2^2 K_+^2}{K_- \delta_1}\int_{\Omega_T} |\nabla h_D|^2\, dx\, dt,$$

$$|J_7| \le \frac{\epsilon}{2}\int_{\Omega_T} |\nabla h_1|^2\, dx\, dt + \frac{\delta_1 K_-}{6}\int_{\Omega_T} |\nabla h_1|^2\, dx\, dt$$

$$+ \int_{\Omega_T} \Big(\frac{\epsilon}{2} + \frac{3\, h_2^2 K_+^2}{\delta_1 K_-}\Big)|\nabla h_{1,D}|^2\, dx\, dt$$

$$|J_8| \le \frac{\delta_1 K_-}{12}\int_{\Omega_T} |\nabla h_1|^2\, dx\, dt + \frac{6K_+^2 h_2^2}{2\, \delta_1 K_-}\int_{\Omega_T} |\nabla h_D|^2\, dx\, dt,$$

$$|J_9| \le \frac{\delta_1 K_-}{12}\int_{\Omega_T} |\nabla h_1|^2\, dx\, dt + \frac{\gamma K_-}{16}\int_{\Omega_T} T_s(h)|\nabla(h + h_1)|^2\, dx\, dt$$

$$+ \frac{K_+^2}{K_-}\Big(\frac{3\, h_2^2}{\delta_1} + \frac{4 h_2}{\gamma}\Big)\int_{\Omega_T} |\nabla h_{1,D}|^2\, dx\, dt,$$

$$|J_{10}| \leq \int_{\Omega_T} T_s(h)|Q_s(h - h_D)| \, dx \, dt + \int_{\Omega_T} T_f(h - h_1)|Q_f(h_1 - h_{1,D})| \, dx \, dt$$

$$+ \int_{\Omega_T} T_s(h)|Q_s(h_1 - h_{1,D})| \, dx \, dt$$

$$\leq \frac{3\|Q_s\|^2_{L^2(0,T;H)} + 2\|Q_f\|^2_{L^2(0,T;H)}}{2\phi} h_2^2 + \frac{\phi}{2} \int_{\Omega_T} |h - h_D|^2 \, dx \, dt$$

$$+ \frac{\phi}{2} \int_{\Omega_T} |h_1 - h_{1,D}|^2 \, dx \, dt,$$

$$|J_{11}| \leq \frac{\phi}{2} \int_{\Omega_T} |h - h_D|^2 \, dx \, dt + \frac{\delta_1 K_-}{24} \int_{\Omega_T} |\nabla(h_1 - h_{1,D})|^2 \, dx \, dt$$

$$+ \frac{\phi}{2}\|\partial_t h_D\|^2_{L^2(0,T;H)} + \frac{12\phi^2}{\delta_1 K_-}\|\partial_t h_{1,D}\|^2_{L^2(0,T;V')}$$

$$\leq \frac{\phi}{2} \int_{\Omega_T} |h - h_D|^2 \, dx \, dt + \frac{\delta_1 K_-}{12} \int_{\Omega_T} |\nabla h_1|^2 \, dx \, dt$$

$$+ \frac{\delta_1 K_-}{12} \int_{\Omega_T} |\nabla h_{1,D}|^2 \, dx \, dt + \frac{\phi}{2}\|\partial_t h_D\|^2_{L^2(0,T;H)} + \frac{12\phi^2}{\delta_1 K_-}\|\partial_t h_{1,D}\|^2_{L^2(0,T;V')}.$$

Summing up all these estimates, we obtain

$$\phi \int_\Omega (h - h_D)^2(T, x) \, dx + \phi \int_\Omega (h_1 - h_{1,D})^2(T, x) \, dx + \int_{\Omega_T} \epsilon(|\nabla h|^2 + |\nabla h_1|^2) \, dx \, dt$$

$$+ 2 \int_{\Omega_T} \left((K_- - \frac{1 K_+^2}{4\gamma K_-})(1 - \mathcal{X}_0^\epsilon(h_1) L_M(\|\nabla h_1\|_{L^2})) T_s(h) \right) |\nabla h_1|^2 \, dx \, dt$$

$$+ 2 \int_{\Omega_T} \frac{\delta_1 K_-}{2} |\nabla h_1|^2 \, dx \, dt + 2 \int_{\Omega_T} K_-(1 - \frac{9\gamma}{8}) T_s(h) |\nabla(h + h_1)|^2 \, dx \, dt$$

$$\leq \phi \int_{\Omega_T} |h - h_D|^2 \, dx \, dt + \phi \int_{\Omega_T} |h_1 - h_{1,D}|^2 \, dx \, dt + C,$$

$$(4.27)$$

where $C = C(u_0, v_0, h_D, h_{1,D}, h_2, Q_s, Q_f)$. We now aim at applying the Gronwall lemma in (4.27). By the hypotheses on K_- and K_+, $(K_- - \frac{1 K_+^2}{4\gamma K_-}) \geq 0$ and taking γ such that $0 < \gamma < 8/9$, we obtain that

$$\left(K_- - \frac{1 K_+^2}{4\gamma K_-} \right) \geq 0$$

is always true because $0 \leq 1 - \mathcal{X}_0^\epsilon(h_1) L_M(x) \leq 1$ and $K_+ \leq 2\sqrt{\gamma} K_-$.

Now we apply the Gronwall lemma and we deduce that there exists a real number B, that does not depend on ϵ nor on M, such that

$$\|h\|_{L^\infty(0,T;H) \cap L^2(0,T;(H^1(\Omega))')} \leq B, \quad \|\Psi(h)\|_{L^2(0,T;H^1(\Omega))} \leq B,$$

$$\|h_1\|_{L^\infty(0,T;H) \cap L^2(0,T;H^1(\Omega))} \leq B,$$

$$(4.28)$$

where we set

$$\Psi(x) = \int_0^x (h_2 - t)^{1/2} dt = \frac{2}{3}\left(h_2^{3/2} - (h_2 - x)^{3/2} \right).$$

In particular, $\|\nabla h_1\|_{L^2(0,T;H)} \leq B$ and this estimate does not depend on the choice of the real number M that defines function L_M. Then the term $L_B(\|\nabla h_1\|_{L^2}) = 1$ may be dropped.

Remark on the Maximum Principle. Even if we establish step 3, we can not prove that $h_1(t,x) + \delta_1 \le h(t,x)$ a.e. $x \in \Omega$ and $\forall t \in (0,T)$. We set

$$h_m = \big(\delta_1 + h_1 - h\big)^+ \in L^2(0,T;V) \quad \text{since } h_{1,D}(\cdot,\cdot) + \delta_1 \le h_D(\cdot,\cdot).$$

Likewise, we recall that $h_m(0,\cdot) = 0$ a.e. in Ω thanks to the maximum principle satisfied by the initial data h_0 and $h_{1,0}$. Moreover, $\nabla h_m = \chi_{\{h<\delta_1+h_1\}}\nabla(h_1 - h)$. Let $\tau \in (0,T)$, thus, taking $w(t,x) = h_m(x,t)\chi_{(0,\tau)}(t)$ in (4.17)–(4.16) yields

$$\int_0^\tau \phi\langle\partial_t(h_1 - h), h_m\rangle_{V',V}\,dt + \int_{\Omega_\tau}\chi_{\{h<\delta_1+h_1\}}(\epsilon + T_s(h))|\nabla(h_1 - h)|^2\,dx\,dt$$

$$+ \int_{\Omega_\tau}T_f(h - h_1))\nabla h \cdot \nabla h_m\,dx\,dt$$

$$+ \int_{\Omega_\tau}\mathcal{X}_0^\epsilon(h_1)T_s(h)\big(\nabla h \cdot \nabla h_m - L_M\big(\|\nabla h_1\|_{L^2}\big)\nabla h_1 \cdot \nabla h_m\big)\,dx\,dt$$

$$- \int_{\Omega_\tau}Q_f T_f(h - h_1)\mathcal{X}_0^\epsilon(h_1)h_m\,dx\,dt = 0.$$

Applying Lemma 1 to (4.22) and taking into account $h_m(0,\cdot) = 0$, gives

$$\int_0^\tau \phi\langle\partial_t h_1, h_m\rangle_{V',V}\,dt = \frac{\phi}{2}\int_\Omega\Big(h_m^2(\tau,x) - h_m^2(0,x)\Big)\,dx = \frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx.$$

Moreover, $L_M\big(\|\nabla h_1\|_{L^2}\big) = 1$, then the above equation becomes

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx + \int_{\Omega_\tau}\chi_{\{h<\delta_1+h_1\}}\Big(\epsilon + T_s(h)(1 - \mathcal{X}_0^\epsilon(h_1))\Big)|\nabla(h_1 - h)|^2\,dx\,dt$$

$$+ \int_{\Omega_\tau}T_f(h - h_1))\nabla h \cdot \nabla h_m\,dx\,dt - \int_{\Omega_\tau}Q_f T_f(h - h_1)\mathcal{X}_0^\epsilon(h_1)h_m\,dx\,dt = 0.$$

$$(4.29)$$

Since $T_f(h - h_1)\chi_{\{h<\delta_1+h_1\}} = \delta_1 > 0$ by definition of T_f, if we assume $Q_f \le 0$ hence (4.22) becomes

$$\frac{\phi}{2}\int_\Omega h_m^2(\tau,x)\,dx + \int_{\Omega_\tau}\delta_1\nabla h \cdot \nabla h_m\,dx\,dt$$

$$\le -\int_0^\tau\int_\Omega\chi_{\{h<\delta_1+h_1\}}\Big(\epsilon + T_s(h)(1 - \mathcal{X}_0^\epsilon(h_1))\Big)|\nabla(h_1 - h)|^2\,dx\,dt \le 0.$$

and because of the term $\int_{\Omega_\tau}\delta_1\nabla h \cdot \nabla h_m\,dx\,dt$, we can no more conclude that $h_m = 0$ a.e. on Ω_T.

Step 4: Existence for the initial system. We now proceed to the last step in the proof of Theorem 3.3, namely we let $\epsilon \to 0$. We infer from the above estimates that $(h_1^\epsilon - h_{1,D})_\epsilon$ is uniformly bounded in $W(0,T)$. We deduce thanks to the compactness result of Aubin that $(h_1^\epsilon - h_{1,D})_\epsilon$ is sequentially compact in $L^2(0,T;H)$.

Concerning the sequence $\{h^\epsilon - h_D\}_\epsilon$, we proceed as in the case of the confined aquifer when $\beta = 0$. We first observe that Ψ is a strictly decreasing function of class $\mathcal{C}^1$ on $[0, h_2]$ and Ψ^{-1} is continue on $(0, \frac{2}{3}h_2^{3/2})$. Using the time translate estimates (see e. g. [3]), we deduce that $\{\Psi(h^\epsilon)\}_\epsilon$ is sequentially compact in $L^2(\Omega_T)$. Namely,

(4.28) yields the following time translate estimate in L_2 norm of sequence $\{h^\epsilon\}_\epsilon$

$$\int_\xi^T \langle (h^\epsilon(t,.) - h^\epsilon(t-\xi,.), \Psi(h^\epsilon(t,x)) - \Psi(h^\epsilon(t-\xi,.))\rangle_{V',V}\, dt$$

$$\leq \Big(\int_\xi^T \|h^\epsilon(t,\cdot) - h^\epsilon(t-\xi,\cdot)\|_{V'}^2\, dt \Big)^{1/2} \Big(\int_\xi^T \|\Psi(h^\epsilon(t,\cdot)) - \Psi(h^\epsilon(t-\xi,\cdot)\|_V^2\, dt \Big)^{1/2}$$

$$\leq C \Big(\int_\xi^T \|h^\epsilon(t,\cdot) - h^\epsilon(t-\xi,\cdot)\|_{V'}^2\, dt \Big)^{1/2} \quad \text{thanks to (4.28)}.$$

However, we know that for all $\xi \in (0,T)$ (see [5])

$$\frac{1}{\xi^2} \int_\xi^T \|h^\epsilon(t,\cdot) - h^\epsilon(t-\xi,\cdot)\|_{V'}^2\, dt \leq \int_\xi^T \|\partial_t h^\epsilon\|_{(H^1(\Omega))'}^2\, dt \leq C,$$

Finally we obtain

$$\int_\xi^T \langle (h^\epsilon(t,.) - h^\epsilon(t-\xi,.), \Psi(h^\epsilon(t,x)) - \Psi(h^\epsilon(t-\xi,.))\rangle_{V',V}\, dt \leq C\xi, \quad \forall \xi \in (0,T).$$

Thanks to the regularity of Ψ, we deduce

$$\int_\xi^T \Big(\Psi(h^\epsilon(t,\cdot)) - \Psi(h^\epsilon(t-\xi,\cdot)), \Psi(h^\epsilon(t,\cdot)) - \Psi(h^\epsilon(t-\xi,\cdot)) \Big)_{L^2(\Omega)}\, dt \leq C\xi,$$

for all $\xi \in (0,T)$. Therefore, as in [[9], Lemma 2.6], we deduce that $\{\Psi(h^\epsilon)\}_\epsilon$ converges strongly in $L^2(\Omega_T)$.

Up to the extraction of a subsequence, not relabeled for convenience, we claim that there exists functions h and h_1 such that $(h - h_D, h_1 - h_{1,D}) \in W(0,T)^2$ and

$$h^\epsilon \to h \quad \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega \times (0,T),$$

$$\Psi(h^\epsilon) \rightharpoonup \Psi(h) \quad \text{weakly in } L^2(0,T;H^1(\Omega)),$$

$$\partial_t h^\epsilon \rightharpoonup \partial_t h \quad \text{weakly in } L^2(0,T;V'),$$

$$h_1^\epsilon \to h_1 \quad \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega \times (0,T),$$

$$h_1^\epsilon \rightharpoonup h_1 \quad \text{weakly in } L^2(0,T;H^1(\Omega)),$$

$$\partial_t h_1^\epsilon \rightharpoonup \partial_t h_1 \quad \text{weakly in } L^2(0,T;V').$$

Letting $\epsilon \to 0$ in the weak formulation of the regularized problem and using Lebesgue Theorem (thanks to the uniform estimates 4.28), we obtain at once (3.6)-(3.7). The boundary and initial condition (3.4)-(3.9) holds true since the map $i \in W(0,T) \mapsto i(0) \in H$ is continuous. Furthermore (h, h_1) satisfies a maximum principle which is not consistent with physical reality because we lost the information between h_1 and h:

$$0 \leq h_1(x,t) \quad \text{and} \quad 0 \leq h(x,t) \leq h_2, \quad \forall t \in (0,T), \text{ a.e. } x \in \Omega.$$

The proof of Theorem 3.3 is complete.

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