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STABILIZATION OF LAMINATED BEAMS WITH INTERFACIAL SLIP

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Abstract. We study a laminated beam consisting of two identical beams of uniform thickness, which is modeled as Timoshenko beams. An adhesive of small thickness is bonding the two layers and creating a restoring force producing a damping. It has been shown that the interfacial slip between the layers alone is not enough to stabilize the system exponentially to its equilibrium state. Some boundary control has been used in the literature for that purpose. In this paper, we show that for viscoelastic material there is no need for any kind of internal or boundary control.

1. Introduction

Many structures in mechanical engineering, electrical engineering, civil engineering and aerospace engineering are formed by a single beam or a number of beams. We can cite for instance, robot arms, rotor turbine and helicopter blades, turbomachineries, electronic equipment, antennas, missiles, panels, pipelines, buildings, bridges, etc. There are mainly three important theories. The first one is named after Euler and Bernoulli and the second one after Rayleigh. To alleviate the shortcomings in these two theories, Timoshenko came up with a new theory which is better suited for engineering practice and is nowadays widely used for moderately thick beams. Both, rotatory inertia and the effect of shear forces are taken into account. In his theory, Timoshenko also assumed that the plane cross-sections perpendicular to the beam centerline remain plane but could become oblique after deformation. An additional kinematics variable is added in the displacement assumptions. Internal and external forces like the weight of the beam, heavy loads, wind, earthquakes and interaction with other bodies or materials are examples of some sources causing high stresses accompanying unwanted vibration. These stresses not only bring some discomfort, reduce the fatigue-life of the material and produce annoying noise but also are harmful to the structure as they may cause significant damage or complete destruction of the machine or equipment. Therefore, some ways and devices capable of enhancing dynamic stability must accompany these structures. To this end various devices and energy dissipation mechanisms have been designed either in the material itself such as smart materials (piezoelectric, pietzoceramic, viscoelastic),

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on its surface (viscoelastic layers, sandwich plates,...) or at the boundary (or part of the boundary). Some well-known dampers are: friction dampers, sensors and actuators, special loads, viscoelastic dampers, tuned mass dampers, tuned liquid dampers and tuned mass liquid dampers. Sometimes they are classified into active, semi-active and passive control methods. In this paper, we would like to investigate the case of two identical beams with an adhesive layer in the interface creating a restoring force. It has been already shown that when this restoring force is proportional to the amount of slip the created frictional damping is unable by itself to stabilize the system exponentially. The first investigators have been forced to control the system by an additional boundary feedback. We intend to seek other ways and means, preferably less costly, less demanding and easy to implement, to stabilize the system exponentially.

Statement of the problem. The original structure consists of a two-layered beam with an adhesive layer bonding the two adjoining surfaces. The adhesive layer creates a restoring force which is assumed proportional to the amount of slip. Therefore, we are in the presence of a structural damping due to interfacial slip. Moreover, we assume that the adhesive layer is of negligible thickness and mass so that the contribution of its mass to the kinetic energy of the structure can be ignored. The equations of motion modeling the system are derived using Timoshenko theory and a third equation is coupled with the first two describing the dynamic of the slip and containing the internal frictional (Kelvin-Voigt) damping. Namely, we have the system

$$\begin{gathered}
\rho w_{tt} + G(\psi - w_x)_x = 0,\\
I_\rho(3s_{tt} - \psi_{tt}) - G(\psi - w_x) - D(3s_{xx} - \psi_{xx}) = 0,\\
3I_\rho s_{tt} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t - 3Ds_{xx} = 0,
\end{gathered}$$

supplemented by the initial data

$$(w,\psi,s)(x,0) = (w_0,\psi_0,s_0), \quad (w_t,\psi_t,s_t)(x,0) = (w_1,\psi_1,s_1)$$

and cantilever boundary conditions.

Here $w, \psi, \rho, G,\ I_\rho, D, \gamma, \beta$ are transverse displacement, rotation angle, density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, adhesive damping parameter and s is proportional to the amount of slip along the interface. The expression $\xi := 3s - \psi$ is the effective rotation angle.

It has been shown in [31] that the frictional damping created by the interfacial slip alone is not enough to stabilize the system exponentially to its equilibrium state. Therefore, a natural question that can be asked is: what are the possible additional damping that can ensure the exponential stability and other kinds of stability of the system? We suggest investigating the case of an additional viscoelastic damping that acts on the effective rotation angle without resorting to any boundary control. Viscoelastic material is very efficient in case there is no considerable change of frequency or temperature in the structure [2]. The viscoelastic damping is (according to the Boltzmann Principle) represented by a memory term in the form of a convolution which arises in the constitutive equation between the stress and the strain

$$\int_0^t h(t-r)(3s-\psi)_{xx}(r)dr.$$

There are basically three main papers in this subject [3, 10, 31]. In [10], the problem has been derived in details. The authors assumed that the adhesive layer is of negligible thickness and mass and that the restoring force created by this layer is proportional to the amount of slip at the interface.

In [31], the system is studied assuming that $\sqrt{G/\rho}$ and $\sqrt{D/I_\rho}$ are two different wave speeds. Putting $\xi = 3s - \psi$, they transformed the original system into

$$
\begin{gathered}
\rho w_{tt} + G(3s - \xi - w_x)_x = 0, \\
I_\rho \xi_{tt} - G(3s - \xi - w_x) - D\xi_{xx} = 0, \\
3I_\rho s_{tt} + 3G(3s - \xi - w_x) + 4\gamma s + 4\beta s_t - 3Ds_{xx} = 0
\end{gathered}
$$

where $0 < x < 1$ and $t > 0$. In addition to the well-posedness, the authors pointed out that the frictional damping is enough to asymptotically stabilize the system. However, it is not possible to have exponential stability. They justified their claim by the fact that the eigenvalues of two branches are very close to the imaginary axis as their moduli go to infinity. To achieve exponential decay of solutions they implemented an additional boundary control

$$
\begin{gathered}
w(0,t) = \xi(0,t) = s(0,t) = 0, \\
\xi_x(1,t) = u_1(t) := -k_1\xi_t(1,t), \quad s_x(1,t) = 0, \\
3s(1,t) - \xi(1,t) - w_x(1,t) = u_2(t) := k_2 w_t(1,t)
\end{gathered}
$$

where $t > 0$. The same system but with the boundary control

$$
\begin{gathered}
\psi(0,t) - w_x(0,t) = u_1(t) := -k_1 w_t(0,t) - w(0,t), \\
3s_x(1,t) - \psi_x(1,t) = u_2(t) := -k_2\xi_t(1,t) - \xi(1,t),
\end{gathered}
$$

has been studied in [3]. The authors proved an exponential stabilization result in case $k_1 \neq \sqrt{\rho/G}$, $k_2 \neq \sqrt{I_\rho/D}$ and the dominant part of the system is itself exponentially stable.

For the case of a single viscoelastic Timoshenko beam (therefore without interfacial slip) there exist many papers in the literature. We can cite a few of them [1, 8, 11, 15, 16, 19, 20, 21, 22, 23, 24, 28, 29, 30, 32, 33].

Here, we shall consider the system

$$
\begin{gathered}
\rho w_{tt} + G(\psi - w_x)_x = 0, \\
I_\rho(3s_{tt} - \psi_{tt}) - G(\psi - w_x) - (3s - \psi)_{xx} + \int_0^t h(t-r)(3s-\psi)_{xx}(r)dr = 0, \\
I_\rho s_{tt} + G(\psi - w_x) + \frac{4}{3}\gamma s + \frac{4}{3}\alpha s_t - s_{xx} = 0,
\end{gathered}
\tag{1.1}
$$

where $0 < x < 1$ and $t > 0$, with the boundary conditions

$$
\begin{gathered}
\psi(0,t) = s(0,t) = 0, \\
s_x(1,t) = \psi_x(1,t) = 0, \\
w_x(0,t) = 0, \quad w(1,t) = 0.
\end{gathered}
\tag{1.2}
$$

The well-posedness of the system has been addressed in [3,31] (see [4,5,7,9,17] for the viscoelastic term). We have weak solutions in $(V_*^1 \times L^2)^3$ and strong solutions in $(V_*^2 \times H^1)^3$ where

$$
V_*^k = \{v : v \in H^k(0,1) : v(0) = 0\}, \quad k = 1, 2.
$$

We shall discuss the case where the relaxation function $h:\mathbb{R}_+\to\mathbb{R}_+$ is a bounded differentiable function satisfying the standard conditions (as we shall not be concerned about finding the largest class of admissible kernels, see [6, 12, 13, 14, 18, 25, 26, 27, 28, 29, 30] for this matter)

$$-\beta_0 h\le h'\le -\beta_1 h, \tag{1.3}$$

for some positive constants β_0 and β_1. Moreover we assume that

$$\varsigma := 1-\int_0^\infty h(r)dr>0. \tag{1.4}$$

For G we shall use the following assumption

(H1) If $\varsigma\rho<\frac{\gamma}{12}$, then $G<\min\{\varsigma\rho, \frac{3\varsigma}{2}, \frac{2\gamma-2\sqrt{\gamma^2-9\gamma\varsigma\rho}}{9}\}$, and if $\frac{\gamma}{12}<\varsigma\rho<\frac{\gamma}{9}$ then assume $G<\min\big\{\varsigma\rho, \frac{3\varsigma}{2}\big\}$.

2. Uniform stabilization

The 'modified' energy of the system (1.1)–(1.2) is given by

$$\begin{aligned} E(t)=\frac12\Big[&\rho\|w_t\|^2+I_\rho\|3s_t-\psi_t\|^2+3I_\rho\|s_t\|^2+G\|\psi-w_x\|^2\\ &+(1-\int_0^t h(r)dr)\|3s_x-\psi_x\|^2+3\|s_x\|^2+4\gamma\|s\|^2\\ &+\int_0^1 (h\Box(3s-\psi)_x)dx\Big], \end{aligned}\tag{2.1}$$

for $t\ge 0$, where $\|\cdot\|$ denotes the norm in $L^2(0,1)$ and

$$(g\Box h)(t):=\int_0^t g(t-s)|h(s)-h(t)|^2ds,\quad t\ge 0.$$

Our result reads as follows.

Theorem 2.1. *For the energy $E(t)$ defined above, if $\rho=GI_\rho$ and* (H1) *holds, then there exist two positive constants K and κ_0 such that*

$$E(t)\le Ke^{-\kappa_0 t},\quad t>0.$$

We first give some lemmas that will serve as a support for the proof of this theorem.

Lemma 2.2. *If k and ϕ are two differentiable functions then*

$$\begin{aligned}(k*\phi)(t)\phi'(t)&=\frac12(k'\Box\phi)(t)+\frac12\frac{d}{dt}\Big[\Big(\int_0^t k(s)ds\Big)\phi^2(t)-(k\Box\phi)(t)\Big]\\ &\quad-\frac12 k(t)\ \phi^2(t),\quad t>0\end{aligned}$$

where $*$ stands for the usual convolution.

Proof. The statement of the this follows from the identity

$$\begin{aligned}\frac{d}{dt}(k\Box\phi)(t)&=(k'\Box\phi)(t)+2\Big(\int_0^t k(s)ds\Big)\ \phi_t(t)\phi(t)-2(k*\phi)(t)\phi_t(t)\\ &=(k'\Box\phi)(t)+\frac{d}{dt}\Big[\Big(\int_0^t k(s)ds\Big)\phi^2(t)\Big]-k(t)\ \phi^2(t)\\ &\quad-2(k*\phi)(t)\phi_t(t),\quad t>0.\end{aligned}$$

□

Lemma 2.3. *The energy $E(t)$ given by (2.1) satisfies*

$$\frac{d}{dt}E(t) = -\frac{h(t)}{2}\|(3s-\psi)_x\|^2 - 4\alpha\|s_t\|^2 + \frac{1}{2}\int_0^1 (h'\square(3s-\psi)_x)dx, \quad t>0.$$

Proof. Multiplying the first equation of (1.1) by w_t and integrating over $(0,1)$ we obtain

$$\frac{\rho}{2}\frac{d}{dt}\left[\|w_t\|^2\right] + G\int_0^1 (\psi - w_x)_x w_t dx = 0$$

or

$$\frac{\rho}{2}\frac{d}{dt}\left[\|w_t\|^2\right] - G\int_0^1 (\psi - w_x) w_{xt} dx + [G(\psi - w_x)w_t]_0^1 = 0$$

and by our boundary conditions (1.2)

$$\frac{\rho}{2}\frac{d}{dt}[\|w_t\|^2] - G\int_0^1 (\psi - w_x) w_{xt} dx = 0, \quad t>0.$$

Note that

$$\begin{aligned} G\int_0^1 (\psi - w_x) w_{xt} dx &= -G\int_0^1 (\psi - w_x)(\psi - w_x - \psi)_t dx \\ &= -\frac{G}{2}\frac{d}{dt}[\|\psi - w_x\|^2] + G\int_0^1 (\psi - w_x)\psi_t dx. \end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}[\rho\|w_t\|^2 + G\|\psi - w_x\|^2] - G\int_0^1 (\psi - w_x)\psi_t dx = 0, \quad t>0. \tag{2.2}$$

Similarly multiplying the second equation of (1.1) by $3s_t - \psi_t$ and integrating over $(0,1)$ we obtain

$$\begin{aligned} &\frac{I_\rho}{2}\frac{d}{dt}[\|3s_t - \psi_t\|^2] - G\int_0^1 (\psi - w_x)(3s_t - \psi_t)dx \\ &- \int_0^1 (3s-\psi)_{xx}(3s_t - \psi_t)dx + \int_0^1 (3s_t - \psi_t)\int_0^t h(t-r)(3s-\psi)_{xx}(r)\,dr\,dx = 0 \end{aligned}$$

or, using integration by parts and the boundary conditions (1.2)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[I_\rho\|3s_t - \psi_t\|^2 + \|3s_x - \psi_x\|^2\right] - G\int_0^1 (\psi - w_x)(3s_t - \psi_t)dx \\ &- \int_0^1 (3s_t - \psi_t)_x \int_0^t h(t-r)(3s-\psi)_x(r)\,dr\,dx = 0, \quad t>0. \end{aligned} \tag{2.3}$$

By using Lemma 2.3 we see that

$$\begin{aligned} &\int_0^1 (3s_t - \psi_t)_x \int_0^t h(t-r)(3s-\psi)_x(r)\,dr\,dx \\ &= \frac{1}{2}(h'\,\square(3s-\psi)_x)(t) - \frac{h(t)}{2}\|3s_x - \psi_x\|^2 \\ &\quad + \frac{1}{2}\frac{d}{dt}\Big[\Big(\int_0^t h(s)ds\Big)\|3s_x - \psi_x\|^2 - (h\square(3s-\psi)_x)(t)\Big], \quad t>0. \end{aligned} \tag{2.4}$$

Likewise, multiplying the third equation of (1.1) by s_t and integrating over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\big[I_\rho\|s_t\|^2+\frac{4\gamma}{3}\|s\|^2+\|s_x\|^2\big]+G\int_0^1(\psi-w_x)s_t dx+\frac{4\alpha}{3}\|s_t\|^2=0, \tag{2.5}$$

for $t>0$. Now it is clear from (2.2)–(2.5) that

$$E'(t)=-4\alpha\|s_t\|^2-\frac{h(t)}{2}\|(3s-\psi)_x\|^2+\frac{1}{2}\int_0^1(h'\Box(3s-\psi)_x)dx, \quad t>0.$$

This completes the proof. $\square$

As $h'(t)\le 0$, we see that $E'(t)\le 0$ for all $t>0$. Therefore the energy is non-increasing and uniformly bounded above by $E(0)$.

Next we shall construct a Lyapunov functional F satisfying the inequalities

$$\lambda_1 E(t)\le F(t)\le \lambda_2 E(t) \quad \text{and} \quad \frac{d}{dt}F(t)\le -\kappa F(t)$$

for some positive constants λ_1, λ_2 and κ. The first two inequalities show that $E(t)$ and $F(t)$ are equivalent. The second one gives the exponential decay of $F(t)$ (and therefore the exponential decay of $E(t)$ as well). To this end, we define

$$F(t)=E(t)+\sum_{i=1}^5\delta_i G_i(t), \quad \delta_i>0,\ i=1,\dots,5,\ t\ge 0,$$

where

$$G_1(t)=I_\rho(s_t,s), \quad G_2(t)=-\rho(w_t,w), \quad G_3(t)=I_\rho(3s_t-\psi_t,3s-\psi), \quad t\ge 0,$$
$$G_4(t)=-\frac{4\gamma\rho}{G}(w_t,\Theta)-\frac{3\rho}{G}(s_x,w_t)+3I_\rho(s_t,\psi-w_x), \quad t\ge 0,$$

with $\Theta(x,t)=\int_x^1 s(\xi,t)d\xi$ and

$$G_5(t)=-I_\rho\Big(3s_t-\psi_t,\int_0^t h(t-r)\,[(3s-\psi)(t)-(3s-\psi)(r)]\,dr\Big), \quad t\ge 0.$$

Using the Cauchy-Schwarz inequality and the Poincaré inequality, one can easily see that all the $G_i(t)$, $i=1,\dots,5$ are bounded (above and below) by an expression containing the existing terms in the energy $E(t)$. This leads to the equivalence of $F(t)$ and $E(t)$.

We shall now prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy in the estimations of the derivatives of the above functionals.

Lemma 2.4. *Along the solutions of* (1.1)–(1.2)*, we have*

$$G_1'(t)\le -\|s_x\|^2+\big(\frac{G}{4\varepsilon_0}+\varepsilon-\frac{4}{3}\gamma\big)\|s\|^2+\varepsilon_0 G\|\psi-w_x\|^2+\big(I_\rho+\frac{4\alpha^2}{9\varepsilon}\big)\|s_t\|^2,$$

for all $t>0$ and some $\varepsilon_0,\varepsilon>0$.

Proof. Clearly,

$$G_1'(t)=I_\rho\|s_t\|^2+I_\rho(s_{tt},s), \quad t>0$$

and by the third equation in (1.1) we obtain that for $t>0$,

$$G_1'(t)=I_\rho\|s_t\|^2-\|s_x\|^2-\frac{4\gamma}{3}\|s\|^2-\frac{4\alpha}{3}(s_t,s)-G(\psi-w_x,s)$$

$$
\begin{aligned}
&\le I_\rho \|s_t\|^2 - \|s_x\|^2 + \big(\frac{G}{4\varepsilon_0} - \frac{4\gamma}{3}\big)\|s\|^2 + \varepsilon_0 G\|\psi - w_x\|^2 + \varepsilon\|s\|^2 + \frac{4\alpha^2}{9\varepsilon}\|s_t\|^2 \\
&\le -\|s_x\|^2 + \big(\frac{G}{4\varepsilon_0} + \varepsilon - \frac{4\gamma}{3}\big)\|s\|^2 + \varepsilon_0 G\|\psi - w_x\|^2 + \big(I_\rho + \frac{4\alpha^2}{9\varepsilon}\big)\|s_t\|^2.
\end{aligned}
$$

□

Lemma 2.5. *The derivative of $G_2(t)$ along solutions of (1.1)–(1.2) satisfies*

$$
G_2'(t) \le -\rho\|w_t\|^2 + (G+\varepsilon_1)\|\psi - w_x\|^2 + \frac{G}{2\varepsilon_1}\|\psi_x - 3s_x\|^2 + \frac{9G}{2\varepsilon_1}\|s_x\|^2,
$$

for all $t > 0$ and some $\varepsilon_1 > 0$.

Proof. Using the first equation in (1.1) and the boundary conditions (1.2), we have that for $t > 0$,

$$
\begin{aligned}
G_2'(t) &= -\rho\|w_t\|^2 - \rho(w_{tt}, w) \\
&= -\rho\|w_t\|^2 + G((\psi - w_x)_x, w) \\
&= -\rho\|w_t\|^2 - G(\psi - w_x, w_x) + G\left[(\psi - w_x)w\right]_0^1 \\
&= -\rho\|w_t\|^2 + G(\psi - w_x, \psi - w_x) - G(\psi - w_x, \psi) \\
&\le -\rho\|w_t\|^2 + G\|\psi - w_x\|^2 + \varepsilon_1 G\|\psi - w_x\|^2 + \frac{G}{4\varepsilon_1}\|\psi_x\|^2 \\
&\le -\rho\|w_t\|^2 + (G+\varepsilon_1)\|\psi - w_x\|^2 + \frac{G}{2\varepsilon_1}\|\psi_x - 3s_x\|^2 + \frac{9G}{2\varepsilon_1}\|s_x\|^2.
\end{aligned}
$$

□

Lemma 2.6. *The derivative of $G_3(t)$ along solutions of (1.1)–(1.2) satisfies*

$$
\begin{aligned}
G_3'(t) &\le I_\rho\|3s_t - \psi_t\|^2 - (\varsigma - \frac{G}{4\varepsilon_2} - \varepsilon)\|3s_x - \psi_x\|^2 + \varepsilon_2 G\|\psi - w_x\|^2 \\
&\quad + \frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\Box(3s_x - \psi_x))dx, \quad t > 0
\end{aligned}
$$

for $\varepsilon_2 > 0$, $\varepsilon > 0$.

Proof. Using the second equation in (1.1) we find that

$$
\begin{aligned}
&I_\rho \frac{d}{dt}(3s_t - \psi_t, 3s - \psi) = I_\rho\|3s_t - \psi_t\|^2 - \|3s_x - \psi_x\|^2 \\
&+ \left[(3s_x - \psi_x)(3s - \psi)\right]_0^1 + G((\psi - w_x), (3s - \psi)) \\
&+ \Big(\int_0^t h(t-r)(3s_x - \psi_x)(r)dr, 3s_x - \psi_x\Big), \quad t > 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
G_3'(t) &\le I_\rho\|3s_t - \psi_t\|^2 - \|3s_x - \psi_x\|^2 + \varepsilon_2 G\|\psi - w_x\|^2 + \frac{G}{4\varepsilon_2}\|3s_x - \psi_x\|^2 \\
&\quad + \Big(\int_0^t h(t-r)\left[(3s_x - \psi_x)(r) - (3s_x - \psi_x)(t)\right]dr, 3s_x - \psi_x\Big) \\
&\quad + \Big(\int_0^t h(r)dr\Big)((3s_x - \psi_x, 3s_x - \psi_x)
\end{aligned}
$$

for $\varepsilon_2 > 0$, or

$$\begin{aligned} G_3'(t) &\le I_\rho\|3s_t-\psi_t\|^2 - \|3s_x-\psi_x\|^2 + \varepsilon_2 G\|\psi-w_x\|^2 \\ &\quad + \frac{G}{4\varepsilon_2}\|3s_x-\psi_x\|^2 + \varepsilon\|3s_x-\psi_x\|^2 + \frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\square(3s_x-\psi_x))dx \\ &\quad + (1-\varsigma)\|(3s_x-\psi_x)\|^2, \end{aligned}$$

for $\varepsilon > 0$. Hence

$$\begin{aligned} G_3'(t) &\le I_\rho\|3s_t-\psi_t\|^2 - (\varsigma - \frac{G}{4\varepsilon_2} - \varepsilon)\|3s_x-\psi_x\|^2 + \varepsilon_2 G\|\psi-w_x\|^2 \\ &\quad + \frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\square(3s_x-\psi_x))dx,\ t>0. \end{aligned}$$

$\square$

Lemma 2.7. *The derivative of $G_4(t)$ is estimated as follows*

$$\begin{aligned} G_4'(t) &\le -(3G-\varepsilon_1)\|\psi-w_x\|^2 + \varepsilon_1(1+\varepsilon)I_\rho\|3s_t-\psi_t\|^2 + \varepsilon_1\|w_t\|^2 \\ &\quad + \big[\frac{4\gamma^2\rho^2}{\varepsilon_1 G^2} + \frac{4\alpha^2}{\varepsilon_1} + (9+\frac{1}{\varepsilon}+\frac{9}{4\varepsilon_1})I_\rho\big]\|s_t\|^2, \quad t>0, \end{aligned}$$

for $\varepsilon_1, \varepsilon > 0$ provided that $I_\rho = \frac{\rho}{G}$.

Proof. Using the first and third equations in (1.1),

$$\begin{aligned} G_4'(t) &= -\frac{4\gamma\rho}{G}(w_{tt},\Theta) - \frac{4\gamma\rho}{G}(w_t,\Theta_t) - \frac{3\rho}{G}(s_{xt},w_t) - \frac{3\rho}{G}(s_x,w_{tt}) \\ &\quad + 3I_\rho(s_{tt},\psi-w_x) + 3I_\rho(s_t,\psi_t-w_{xt}). \end{aligned}$$

Then we find that

$$\begin{aligned} G_4'(t) &= 4\gamma((\psi-w_x)_x,\Theta) - \frac{4\gamma\rho}{G}(w_t,\Theta_t) - \frac{3\rho}{G}(s_{xt},w_t) + 3(s_x,(\psi-w_x)_x) \\ &\quad + 3(-G(\psi-w_x) - \frac{4\gamma}{3}s - \frac{4\alpha}{3}s_t + s_{xx},\psi-w_x) + 3I_\rho(s_t,\psi_t-w_{xt}), \end{aligned}$$

for $t>0$. Next, by the definition of Θ and the assumption $I_\rho = \frac{\rho}{G}$, we obtain

$$G_4'(t) = -\frac{4\gamma\rho}{G}(w_t,\Theta_t) - 3G\|\psi-w_x\|^2 - 4\alpha(s_t,\psi-w_x) + 3I_\rho(s_t,\psi_t),$$

for $t>0$. Now, clearly

$$\begin{gathered} \frac{4\gamma\rho}{G}(w_t,\Theta_t) \le \varepsilon_1\|w_t\|^2 + \frac{4\gamma^2\rho^2}{\varepsilon_1 G^2}\|s_t\|^2, \\ 4\alpha(s_t,\psi-w_x) \le \varepsilon_1\|\psi-w_x\|^2 + \frac{4\alpha^2}{\varepsilon_1}\|s_t\|^2, \end{gathered}$$

$$\begin{aligned} 3(s_t,\psi_t) &\le \varepsilon_1\|\psi_t\|^2 + \frac{9}{4\varepsilon_1}\|s_t\|^2 \\ &\le \varepsilon_1(1+\varepsilon)\|3s_t-\psi_t\|^2 + (9+\frac{1}{\varepsilon}+\frac{9}{4\varepsilon_1})\|s_t\|^2 \end{aligned}$$

lead to

$$\begin{aligned} G_4'(t) &\le \varepsilon_1\|w_t\|^2 + \frac{4\gamma^2\rho^2}{\varepsilon_1 G^2}\|s_t\|^2 - 3G\|\psi-w_x\|^2 + \varepsilon_1\|\psi-w_x\|^2 + \frac{4\alpha^2}{\varepsilon_1}\|s_t\|^2 \\ &\quad + \varepsilon_1(1+\varepsilon)I_\rho\|3s_t-\psi_t\|^2 + (9+\frac{1}{\varepsilon}+\frac{9}{4\varepsilon_1})I_\rho\|s_t\|^2 \end{aligned}$$

or

$$\begin{aligned}G_4'(t) &\le -(3G-\varepsilon_1)\|\psi - w_x\|^2 + \varepsilon_1(1+\varepsilon)I_\rho\|3s_t-\psi_t\|^2 + \varepsilon_1\|w_t\|^2 \\ &\quad + \big[\frac{4\gamma^2\rho^2}{\varepsilon_1 G^2} + \frac{4\alpha^2}{\varepsilon_1} + \big(9 + \frac{1}{\varepsilon} + \frac{9}{4\varepsilon_1}\big)I_\rho\big]\|s_t\|^2, \quad t \ge 0.\end{aligned}$$

□

For the next lemma we need to get away from zero to ensure strict positivity of $\int_0^t h(r)dr$. So for that $t \ge t_0 > 0$ we have $\int_0^t h(r)dr \ge \int_0^{t_0} h(r)dr = h_0 > 0$.

Lemma 2.8. *For the functional $G_5(t)$ we have*

$$\begin{aligned}G_5'(t) &\le G\varepsilon\|\psi - w_x\|^2 + (G + 4\varepsilon + 2 - \varsigma)\frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\,\square(3s-\psi)_x)dx \\ &\quad + (2-\varsigma)\varepsilon\|3s_x - \psi_x\|^2 + I_\rho(\varepsilon - h_0)\|3s_t - \psi_t\|^2 \\ &\quad + \frac{I_\rho h(0)}{4\varepsilon}\int_0^1 (|h'|\square(3s-\psi)_x)dx, \quad t \ge t_0 > 0\end{aligned}$$

for $\varepsilon > 0$.

Proof. We recall that

$$G_5(t) = -I_\rho\Big(3s_t - \psi_t, \int_0^t h(t-r)[(3s-\psi)(t) - (3s-\psi)(r)]dr\Big), \quad t > 0$$

and therefore

$$\begin{aligned}G_5'(t) &= -I_\rho(3s_{tt} - \psi_{tt}, \int_0^t h(t-r)[(3s-\psi)(t) - (3s-\psi)(r)]dr) \\ &\quad - I_\rho\Big(3s_t - \psi_t, \int_0^t h'(t-r)[(3s-\psi)(t) - (3s-\psi)(r)]dr\Big) \\ &\quad - I_\rho\Big(\int_0^t h(r)dr\Big)\|3s_t - \psi_t\|^2, \quad t > 0.\end{aligned}$$

In view of the second equation in (1.1) and the boundary conditions (1.2) we write

$$\begin{aligned}G_5'(t) &= -\Big(G(\psi - w_x) + (3s-\psi)_{xx}, \int_0^t h(t-r)[(3s-\psi)(t) - (3s-\psi)(r)]dr\Big) \\ &\quad + \Big(\int_0^t h(t-r)(3s-\psi)_{xx}(r)dr, \int_0^t h(t-r)[(3s-\psi)(t) - (3s-\psi)(r)]dr\Big) \\ &\quad - I_\rho\Big(3s_t - \psi_t, \int_0^t h'(t-r)[(3s-\psi)(t) - (3s-\psi)(r)]dr\Big) \\ &\quad - I_\rho\Big(\int_0^t h(r)dr\Big)\|3s_t - \psi_t\|^2, \quad t > 0.\end{aligned} \tag{2.6}$$

It is easy to see that for $t > 0$,

$$\begin{aligned}&- G\Big(\psi - w_x, \int_0^t h(t-r)\,[(3s-\psi)(t) - (3s-\psi)(r)]\,dr\Big) \\ &\le G\varepsilon\|\psi - w_x\|^2 + \frac{G(1-\varsigma)}{4\varepsilon}\int_0^1 (h\,\square(3s-\psi)_x)dx,\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\Big((3s-\psi)_{xx}, \int_0^t h(t-r)\,[(3s-\psi)(t)-(3s-\psi)(r)]\,dr\Big)\\
&= -\Big((3s-\psi)_x, \int_0^t h(t-r)\,[(3s-\psi)_x(t)-(3s-\psi)_x(r)]\,dr\Big) \\
&\le \varepsilon\|3s_x-\psi_x\|^2 + \frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\Box(3s-\psi)_x)dx,
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\Big(\int_0^t h(t-r)(3s-\psi)_{xx}(r)dr, \int_0^t h(t-r)[(3s-\psi)(t)-(3s-\psi)(r)]dr\Big)\\
&= \|\int_0^t h(t-r)[(3s-\psi)_x(t)-(3s-\psi)_x(r)]dr\|^2\\
&\quad - \Big(\int_0^t h(r)dr\Big)\Big((3s-\psi)_x, \int_0^t h(t-r)[(3s-\psi)_x(t)-(3s-\psi)_x(r)]dr\Big)\\
&\le \|\int_0^t h(t-r)[(3s-\psi)_x(t)-(3s-\psi)_x(r)]dr\|^2 + (1-\varsigma)\Big\{\varepsilon\|3s_x-\psi_x\|^2\\
&\quad + \frac{1}{4\varepsilon}\|\int_0^t h(t-r)[(3s-\psi)_x(t)-(3s-\psi)_x(r)]dr\|^2\Big\}\\
&\le (1+\frac{1-\varsigma}{4\varepsilon})(1-\varsigma)\int_0^1 (h\,\Box(3s-\psi)_x)dx + \varepsilon(1-\varsigma)\|3s_x-\psi_x\|^2,
\end{aligned} \tag{2.9}$$

for $t>0$. Further

$$\begin{aligned}
&I_\rho(3s_t-\psi_t, \int_0^t h'(t-r)[(3s-\psi)(t)-(3s-\psi)(r)]dr)\\
&\le \varepsilon I_\rho\|3s_t-\psi_t\|^2 + \frac{I_\rho h(0)}{4\varepsilon}\int_0^1 (|h'|\Box(3s-\psi)_x)dx, \quad t>0.
\end{aligned} \tag{2.10}$$

Taking into account estimates (2.7)–(2.10), in (2.6) and considering $t \ge t_0 > 0$, we obtain

$$\begin{aligned}
G_5'(t) &\le G\varepsilon\|\psi-w_x\|^2 + \frac{G(1-\varsigma)}{4\varepsilon}\int_0^1 (h\,\Box(3s-\psi)_x)dx + \varepsilon\|3s_x-\psi_x\|^2\\
&\quad + \frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\Box(3s-\psi)_x)dx + (1+\frac{1-\varsigma}{4\varepsilon})(1-\varsigma)\int_0^1 (h\,\Box(3s-\psi)_x)dx\\
&\quad + \varepsilon(1-\varsigma)\|3s_x-\psi_x\|^2 + \varepsilon I_\rho\|3s_t-\psi_t\|^2\\
&\quad + \frac{I_\rho h(0)}{4\varepsilon}\int_0^1 (|h'|\Box(3s-\psi)_x)dx - I_\rho h_0\|3s_t-\psi_t\|^2
\end{aligned}$$

or, for $t \ge t_0 > 0$

$$\begin{aligned}
G_5'(t) &\le G\varepsilon\|\psi-w_x\|^2 + (G+4\varepsilon+2-\varsigma)\frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\,\Box(3s-\psi)_x)dx\\
&\quad + (2-\varsigma)\varepsilon\|3s_x-\psi_x\|^2 + I_\rho(\varepsilon-h_0)\|3s_t-\psi_t\|^2\\
&\quad + \frac{I_\rho h(0)}{4\varepsilon}\int_0^1 (|h'|\Box(3s-\psi)_x)dx.
\end{aligned}$$

The proof is complete. $\square$

Using the previous lemmas we now give the proof of our main result.

Proof of Theorem 2.1. Gathering the estimates in the previous lemmas we find that

$$
\begin{aligned}
F'(t) = E'(t) + \sum\nolimits_{i=1}^{5} \delta_i G_i'(t) &\le -4\alpha\|s_t\|^2 - \frac{h(t)}{2}\|(3s-\psi)_x\|^2 \\
&+ \frac{1}{2}\int_0^1 (h'\Box(3s-\psi)_x)dx - \delta_1\|s_x\|^2 + \delta_1(\frac{G}{4\varepsilon_0} + \varepsilon - \frac{4}{3}\gamma)\|s\|^2 \\
&+ \delta_1\varepsilon_0 G\|\psi - w_x\|^2 + \delta_1(I_\rho + \frac{4\alpha^2}{9\varepsilon})\|s_t\|^2 - \delta_2\rho\|w_t\|^2 \\
&+ \delta_2(G+\varepsilon_1)\|\psi - w_x\|^2 + \frac{G\delta_2}{2\varepsilon_1}\|3s_x - \psi_x\|^2 + \frac{9G\delta_2}{2\varepsilon_1}\|s_x\|^2 \\
&+ \delta_3 I_\rho\|3s_t - \psi_t\|^2 - \delta_3(\varsigma - \frac{G}{4\varepsilon_2} - \varepsilon)\|3s_x - \psi_x\|^2 + \delta_3\varepsilon_2 G\|\psi - w_x\|^2 \\
&+ \delta_3\frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\Box(3s_x - \psi_x))dx - \delta_4(3G - \varepsilon_1)\|\psi - w_x\|^2 \\
&+ \delta_4\varepsilon_1(1+\varepsilon)I_\rho\|3s_t - \psi_t\|^2 + \delta_4[\frac{4\gamma^2\rho^2}{\varepsilon_2 G^2} + \frac{4\alpha^2}{\varepsilon_1} + (9 + \frac{1}{\varepsilon} + \frac{9}{4\varepsilon_1})I_\rho]\|s_t\|^2 \\
&+ \varepsilon_1\delta_4\|w_t\|^2 + \delta_5 G\varepsilon\|\psi - w_x\|^2 + \delta_5\varepsilon(2-\varsigma)\|3s_x - \psi_x\|^2 \\
&+ \delta_5(G + 4\varepsilon + 2 - \varsigma)\frac{1-\varsigma}{4\varepsilon}\int_0^1 (h\,\Box(3s-\psi)_x)dx \\
&+ \delta_5 I_\rho(\varepsilon - h_0)\|3s_t - \psi_t\|^2 + \delta_5\frac{I_\rho h(0)}{4\varepsilon}\int_0^1 (|h'|\Box(3s-\psi)_x)dx, \quad t \ge t_0 > 0
\end{aligned}
$$

or

$$
\begin{aligned}
F'(t) \le &-\Big\{4\alpha - \delta_1(I_\rho + \frac{4\alpha^2}{9\varepsilon}) - \delta_4\Big[(9 + \frac{1}{\varepsilon} + \frac{9}{4\varepsilon_1})I_\rho + \frac{4\alpha^2}{\varepsilon_1} + \frac{4\gamma^2\rho^2}{\varepsilon_2 G^2}\Big]\Big\}\|s_t\|^2 \\
&- (\delta_1 - \frac{9G\delta_2}{2\varepsilon_1})\|s_x\|^2 + \delta_1(\frac{G}{4\varepsilon_0} + \varepsilon - \frac{4}{3}\gamma)\|s\|^2 \\
&- [\delta_4(3G - \varepsilon_1) - \delta_1\varepsilon_0 G - \delta_2(G+\varepsilon_1) - \delta_3\varepsilon_2 G - \delta_5 G\varepsilon]\,\|\psi - w_x\|^2 \\
&- \Big\{\delta_3(\varsigma - \frac{G}{4\varepsilon_2} - \varepsilon) - \frac{G\delta_2}{2\varepsilon_1} - \delta_5\varepsilon(2-\varsigma)\Big\}\|3s_x - \psi_x\|^2 \\
&- (\delta_2\rho - \varepsilon_2\delta_4)\|w_t\|^2 + I_\rho\,[\delta_3 + \delta_4\varepsilon_1(1+\varepsilon) + \delta_5(\varepsilon - h_0)]\,\|3s_t - \psi_t\|^2 \\
&- \Big\{\frac{\beta_1}{2} - \delta_3\frac{1-\varsigma}{4\varepsilon} - \delta_5(G + 4\varepsilon + 2 - \varsigma)\frac{1-\varsigma}{4\varepsilon} \\
&- \delta_5\frac{\beta_0 I_\rho h(0)}{4\varepsilon}\Big\}\int_0^1 (h\,\Box(3s-\psi)_x)dx.
\end{aligned}
\tag{2.11}
$$

Our strategy for selecting the different coefficients and parameters is as follows: all the δ_i, $i = 1, \dots 5$ will be determined in terms of only one of them (here δ_1). This δ_1 will be accountable in front of α and β_1 in the coefficients of the first and the last term in (2.11). From the beginning, we have managed in our estimations to balance the largest coefficients (here $1/\varepsilon$) on the terms that appear in the derivative of the energy. This will allow us to ignore ε at the beginning of the process of selection.

Let us ignore for the moment the first and the last terms in (2.11). We shall, at the same time, ignore the terms having coefficients in ε. The focus will be on

$$\begin{gathered}
\delta_1-\frac{9G}{2\varepsilon_1}\delta_2>0,\quad \frac{G}{4\varepsilon_0}-\frac{4}{3}\gamma<0,\\
\delta_4(3G-\varepsilon_1)-\delta_1\varepsilon_0 G-\delta_2(G+\varepsilon_1)-\delta_3\varepsilon_2 G>0,\\
\delta_3(\varsigma-\frac{G}{4\varepsilon_2})-\frac{G}{2\varepsilon_1}\delta_2>0,\\
\delta_2\rho-\varepsilon_2\delta_4>0,\quad \delta_3+\delta_4\varepsilon_1-\delta_5 h_0<0,
\end{gathered}$$

or

$$\begin{gathered}
\frac{9G}{2\varepsilon_1}\delta_2<\delta_1,\quad \frac{G}{4\varepsilon_0}<\frac{4}{3}\gamma,\\
\delta_1\varepsilon_0 G+\delta_2(G+\varepsilon_1)+\delta_3\varepsilon_2 G<\delta_4(3G-\varepsilon_1),\\
\frac{G}{2\varepsilon_1}\delta_2<\delta_3(\varsigma-\frac{G}{4\varepsilon_2}),\\
\varepsilon_2\delta_4<\delta_2\rho,\quad \delta_3+\delta_4\varepsilon_1<\delta_5 h_0.
\end{gathered} \tag{2.12}$$

Let $\varepsilon_0=\frac{G}{4\gamma}$ so that the second inequality in (2.12) is satisfied. Put $\varepsilon_2=\frac{G}{2\varsigma}$, $\varepsilon_1=G$ and ignore the last inequality (we will take δ_5 large enough as it does not appear elsewhere), we will be left with

$$\begin{gathered}
\frac{9}{2}\delta_2<\delta_1,\\
\delta_1\frac{G}{4\gamma}+2\delta_2+\delta_3\frac{G}{2\varsigma}<2\delta_4,\\
\delta_2<\varsigma\delta_3,\quad \frac{G}{2\varsigma}\delta_4<\delta_2\rho.
\end{gathered} \tag{2.13}$$

Note that $2\delta_2<\delta_4<\frac{2\varsigma}{G}\delta_2\rho$ is valid if $G<\varsigma\rho$ and $\delta_4=\frac{G+\varsigma\rho}{G}\delta_2$. Therefore (2.13) reduces to

$$\begin{gathered}
\frac{9}{2}\delta_2<\delta_1,\\
\delta_1\frac{G}{4\gamma}+\delta_3\frac{G}{2\varsigma}<\frac{G+\varsigma\rho}{G}\delta_2,\\
\delta_2<\varsigma\delta_3.
\end{gathered}$$

By assumption (H1) we may have

$$\delta_1\frac{G}{4\gamma}<\frac{G+\varsigma\rho}{2G}\delta_2<\frac{G+\varsigma\rho}{9G}\delta_1,\quad \delta_3\frac{G}{2\varsigma}<\frac{G+\varsigma\rho}{2G}\delta_2<\frac{G+\varsigma\rho}{2G}\varsigma\delta_3.$$

These inequalities ensure the possibility of selecting (for instance) δ_2 and δ_3 in terms of δ_1. It is now possible to select δ_5 (satisfying the last relation in (2.12)) in terms of δ_1 and then ε. Finally, δ_1 is chosen so small that the coefficients of the first and the last terms in (2.11) are satisfied. We end up with an inequality of the form

$$F'(t)\le -CF(t),\quad t\ge t_0>0.$$

This gives the exponential decay of $F(t)$ on $[t_0,\infty)$. The exponential decay of the energy follows from the equivalence with $F(t)$ and the statement of the theorem for $t\ge 0$ is clear. The proof is complete. $\square$

Remark. It would be nice to remove the conditions on G although $\rho = GI_\rho$ (equal wave speeds) seems natural as we have a similar one in the theory of Timoshenko beams. The assumption (H1) looks technical and we believe that it may be improved considerably through a better choice of the functionals and adequate estimations. Investigations on other boundary conditions would also be of great importance.

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