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ENTIRE FUNCTIONS SHARING SMALL FUNCTIONS WITH THEIR DIFFERENCE OPERATORS

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Abstract. We study the uniqueness for entire functions that share small functions of finite order with difference operators applied to the entire functions. In particular, we generalize of a result in [2].

1. Introduction and Main Results

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna's value distribution theory [7, 9, 12]. In addition, we will use $\rho(f)$ to denote the order of growth of f and $\tau(f)$ to denote the type of growth of f, we say that a meromorphic function $a(z)$ is a small function of $f(z)$ if $T(r,a) = S(r,f)$, where $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to $f(z)$. For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z+c)$ (Resp. $f_0(z) = f(z)$) and its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z), \quad \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N},\ n \geq 2.$$

In particular, $\Delta_c^n f(z) = \Delta^n f(z)$ for the case $c = 1$.

Let $f(z)$ and $g(z)$ be two meromorphic functions, and let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ counting multiplicity (for short CM), provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros including multiplicities.

The problem of meromorphic functions sharing small functions with their differences is an important topic of uniqueness theory of meromorphic functions (see [1, 4, 5, 6]). In 1986, Jank, Mues and Volkmann [8] proved the following result.

Theorem 1.1. *Let f be a nonconstant meromorphic function, and let $a \neq 0$ be a finite constant. If f, f' and f'' share the value a CM, then $f \equiv f'$.*

Li and Yang [11] gave the following generalization of Theorem 1.1.

Theorem 1.2. *Let f be a nonconstant entire function, let a be a finite nonzero constant, and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, then $f \equiv f'$.*

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Chen et al [2] proved a difference analogue of result of Theorem 1.1 and obtained the following results.

Theorem 1.3. *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f$ and $\Delta_c^2 f$ share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^2 f$.*

Theorem 1.4. *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)$, $b(z)(\not\equiv 0) \in S(f)$ be periodic entire functions with period c. If $f(z) - a(z)$,$\Delta_c f(z) - b(z)$ and $\Delta_c^2 f(z) - b(z)$ share 0 CM, then $\Delta_c f \equiv \Delta_c^2 f$.*

Recently Chen and Li [3] generalized Theorem 1.3 and proved the following results.

Theorem 1.5. *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f$ and $\Delta_c^n f$ $(n \geq 2)$ share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^n f$.*

Theorem 1.6. *Let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^n f(z)$ share 0 CM, then $\Delta_c^n f(z) = C\Delta_c f(z)$, where C is a nonzero constant.*

It is interesting to see what happen when $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM. The aim of this article is to give a difference analogue of result of Theorem 1.2. In fact, we prove that the conclusion of Theorems 1.5 and 1.6 remain valid when we replace $\Delta_c f(z)$ by $\Delta_c^{n+1} f(z)$. We obtain the following results.

Theorem 1.7. *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM, then $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$.*

Example 1.8. Let $f(z) = e^{z \ln 2}$ and $c = 1$. Then, for any $a \in \mathbb{C}$, we notice that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share a CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$. This example satisfies Theorem 1.7.

Remark 1.9. In Example 1.8, we have $\Delta_c^m f(z) \equiv \Delta_c^n f(z)$ for any integer $m > n + 1$. However, it remains open when $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^m f(z)$ $(m > n + 1)$ share $a(z)$ CM, the claim $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$ in Theorem 1.7 can be replaced by $\Delta_c^m f(z) \equiv \Delta_c^n f(z)$ in general.

Theorem 1.10. *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)$, $b(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, then $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$.*

Theorem 1.11. *Let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c^{n+1} f(z) \equiv C\Delta_c^n f(z)$, where C is a nonzero constant.*

Example 1.12. *Let $f(z) = e^{az}$ and $c = 1$ where $a \neq 2k\pi i$ $(k \in \mathbb{Z})$, it is clear that $\Delta_c^n f(z) = (e^a - 1)^n e^{az}$ for any integer $n \geq 1$. So, $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c^{n+1} f(z) \equiv C\Delta_c^n f(z)$ where $C = e^a - 1$. This example satisfies Theorem 1.11.*

2. Some lemmas

Lemma 2.1 ([10]). *Let f and g be meromorphic functions such that $0 < \rho(f)$, $\rho(g) < \infty$ and $0 < \tau(f), \tau(g) < \infty$. Then we have*

(i) *If $\rho(f) > \rho(g)$, then we obtain*
$$\tau(f+g) = \tau(fg) = \tau(f).$$
(ii) *If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then*
$$\rho(f+g) = \rho(fg) = \rho(f) = \rho(g).$$

Lemma 2.2 ([12]). *Suppose $f_j(z)$ $(j = 1, 2, \dots, n+1)$ and $g_j(z)$ $(j = 1, 2, \dots, n)$ $(n \geq 1)$ are entire functions satisfying the following two conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$;
(ii) *The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1 \leq j \leq n+1$, $1 \leq k \leq n$. Furthermore, the order of $f_j(z)$is less than the order of $e^{g_h(z)-g_k(z)}$ for $n \geq 2$ and $1 \leq j \leq n+1$, $1 \leq h < k \leq n$.*

Then $f_j(z) \equiv 0$, $(j = 1, 2, \dots n+1)$.

Lemma 2.3 ([5]). *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c, with respect to $f(z)$,*
$$m(r, \frac{\Delta_c^n f}{f-a}) = S(r, f),$$
where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

3. Proof of the Theorems

Proof of the Theorem 1.7. Suppose on the contrary to the assertion that $\Delta_c^n f(z) \not\equiv \Delta_c^{n+1} f(z)$. Note that $f(z)$ is a nonconstant entire function of finite order. By Lemma 2.3, for $n \geq 1$, we have
$$T(r, \Delta_c^n f) = m(r, \Delta_c^n f) \leq m\big(r, \frac{\Delta_c^n f}{f}\big) + m(r, f) \leq T(r, f) + S(r, f).$$
Since $f(z)$, $\Delta^n f(z)$ and $\Delta^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM, then
$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = e^{P(z)}, \tag{3.1}$$
$$\frac{\Delta_c^{n+1} f(z) - a(z)}{f(z) - a(z)} = e^{Q(z)}, \tag{3.2}$$
where P and Q are polynomials. Set
$$\varphi(z) = \frac{\Delta_c^{n+1} f(z) - \Delta_c^n f(z)}{f(z) - a(z)}. \tag{3.3}$$
From (3.1) and (3.2), we obtain $\varphi(z) = e^{Q(z)} - e^{P(z)}$. Then, by supposition and (3.3), we see that $\varphi(z) \not\equiv 0$. By Lemma 2.3, we deduce that
$$T(r, \varphi) = m\big(r, \varphi\big) \leq m\big(r, \frac{\Delta_c^{n+1} f}{f-a}\big) + m\big(r, \frac{\Delta_c^n f}{f-a}\big) + O(1) = S(r, f). \tag{3.4}$$

Note that $\frac{e^{Q(z)}}{\varphi(z)} - \frac{e^{P(z)}}{\varphi(z)} = 1$. By using the second main theorem and (3.4), we have

$$\begin{aligned} T(r, \frac{e^Q}{\varphi}) &\le \overline{N}\big(r, \frac{e^Q}{\varphi}\big) + \overline{N}\big(r, \frac{\varphi}{e^Q}\big) + \overline{N}\big(r, \frac{1}{\frac{e^Q}{\varphi} - 1}\big) + S\big(r, \frac{e^Q}{\varphi}\big) \\ &= \overline{N}\big(r, \frac{e^Q}{\varphi}\big) + \overline{N}\big(r, \frac{\varphi}{e^Q}\big) + \overline{N}\big(r, \frac{\varphi}{e^P}\big) + S\big(r, \frac{e^Q}{\varphi}\big) \\ &= S(r, f) + S\big(r, \frac{e^Q}{\varphi}\big). \end{aligned} \tag{3.5}$$

Thus, by (3.4) and (3.5), we have $T(r, e^Q) = S(r, f)$. Similarly, $T(r, e^P) = S(r, f)$. Setting now $g(z) = f(z) - a(z)$, from (3.1) and (3.2) we have

$$\Delta_c^n g(z) = g(z)e^{P(z)} + a(z), \tag{3.6}$$

$$\Delta_c^{n+1} g(z) = g(z)e^{Q(z)} + a(z). \tag{3.7}$$

By (3.6) and (3.7), we have

$$g(z)e^{Q(z)} + a(z) = \Delta_c(\Delta_c^n g(z)) = \Delta_c(g(z)e^{P(z)} + a(z)).$$

Thus

$$g(z)e^{Q(z)} + a(z) = g_c(z)e^{P_c(z)} - g(z)e^{P(z)},$$

which implies

$$g_c(z) = M(z)g(z) + N(z), \tag{3.8}$$

where $M(z) = e^{-P_c(z)}(e^{P(z)} + e^{Q(z)})$ and $N(z) = a(z)e^{-P_c(z)}$. From (3.8), we have

$$g_{2c}(z) = M_c(z)g_c(z) + N_c(z) = M_c(z)(M(z)g(z) + N(z)) + N_c(z),$$

hence

$$g_{2c}(z) = M_c(z)M_0(z)g(z) + N^1(z),$$

where $N^1(z) = M_c(z)N_0(z) + N_c(z)$. By the same method, we can deduce that

$$g_{ic}(z) = \Big(\prod_{k=0}^{i-1} M_{kc}(z)\Big)g(z) + N^{i-1}(z) \quad (i \ge 1), \tag{3.9}$$

where $N^{i-1}(z)$ $(i \ge 1)$ is an entire function depending on $a(z), e^{P(z)}, e^{Q(z)}$ and their differences. Now, we can rewrite (3.6) as

$$\sum_{i=1}^{n} C_n^i (-1)^{n-i} g_{ic}(z) = (e^{P(z)} - (-1)^n)g(z) + a(z). \tag{3.10}$$

By (3.9) and (3.10), we have

$$\sum_{i=1}^{n} C_n^i (-1)^{n-i} \Big(\Big(\prod_{k=0}^{i-1} M_{kc}(z)\Big)g(z) + N^{i-1}(z)\Big) - (e^{P(z)} - (-1)^n)g(z) = a(z)$$

which implies

$$A(z)g(z) + B(z) = 0, \tag{3.11}$$

where

$$A(z) = \sum_{i=1}^{n} C_n^i (-1)^{n-i} \prod_{k=0}^{i-1} M_{kc}(z) - e^{P(z)} + (-1)^n,$$

$$B(z)=\sum_{i=1}^{n}C_n^i(-1)^{n-i}N^{i-1}(z)-a(z).$$

It is clear that $A(z)$ and $B(z)$ are small functions with respect to $f(z)$. If $A(z)\not\equiv 0$, then (3.11) yields the contradiction

$$T(r,f)=T(r,g)=T(r,\frac{B}{A})=S(r,f).$$

Suppose now that $A(z)\equiv 0$, rewrite the equation $A(z)\equiv 0$ as

$$\sum_{i=1}^{n}C_n^i(-1)^{n-i}\prod_{k=0}^{i-1}e^{-P_{(k+1)c}}(e^{P_{kc}}+e^{Q_{kc}})=e^P-(-1)^n.$$

We can rewrite the left side of above equality as

$$\begin{aligned}
&\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{-\sum_{k=1}^{i}P_{kc}}\prod_{k=0}^{i-1}(e^{P_{kc}}+e^{Q_{kc}})\\
&=\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{-\sum_{k=1}^{i}P_{kc}}e^{\sum_{k=0}^{i-1}P_{kc}}\prod_{k=0}^{i-1}(1+e^{Q_{kc}-P_{kc}})\\
&=\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{P-P_{ic}}\prod_{k=0}^{i-1}(1+e^{Q_{kc}-P_{kc}}).
\end{aligned}$$

So

$$\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{P-P_{ic}}\prod_{k=0}^{i-1}(1+e^{h_{kc}})=e^P-(-1)^n, \tag{3.12}$$

where $h_{kc}=Q_{kc}-P_{kc}$. On the other hand, let $\Omega_i=\{0,1,\dots,i-1\}$ be a finite set of i elements, and

$$P(\Omega_i)=\{\emptyset,\{0\},\{1\},\dots,\{i-1\},\{0,1\},\{0,2\},\dots,\Omega_i\},$$

where $\emptyset$ is the empty set. It is easy to see that

$$\begin{aligned}
\prod_{k=0}^{i-1}(1+e^{h_{kc}})&=1+\sum_{A\in P(\Omega_i)\setminus\{\emptyset\}}\exp\Big(\sum_{j\in A}h_{jc}\Big)\\
&=1+[e^h+e^{h_c}+\dots+e^{h_{(i-1)c}}]\\
&\quad+[e^{h+h_c}+e^{h+h_{2c}}+\dots]+\dots+[e^{h+h_c+\dots+h_{(i-1)c}}].
\end{aligned}\tag{3.13}$$

We divide the proof into two parts:

Part (1). $h(z)$ is non-constant polynomial. Suppose that $h(z)=a_mz^m+\dots+a_0$ $(a_m\neq 0)$, since $P(\Omega_i)\subset P(\Omega_{i+1})$, then by (3.12) and (3.13) we have

$$\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{P-P_{ic}}+\alpha_1e^{a_mz^m}+\alpha_2e^{2a_mz^m}+\dots+\alpha_ne^{na_mz^m}=e^P-(-1)^n$$

which is equivalent to

$$\alpha_0+\alpha_1e^{a_mz^m}+\alpha_2e^{2a_mz^m}+\dots+\alpha_ne^{na_mz^m}=e^P, \tag{3.14}$$

where α_i $(i = 0, \dots, n)$ are entire functions of order less than m. Moreover,

$$\begin{aligned}\alpha_0 &= \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{P-P_{ic}} + (-1)^n \\ &= e^P (\sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{-P_{ic}} + (-1)^n e^{-P}) \\ &= e^P \Delta_c^n e^{-P}.\end{aligned}$$

(i) If $\deg P > m$, then we obtain from (3.14) that $\deg P \le m$ which is a contradiction.

(ii) If $\deg P < m$, then by using Lemma 2.1 and (3.14) we obtain

$$\deg P = \rho(e^P) = \rho\Big(\alpha_0 + \alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \dots + \alpha_n e^{n a_m z^m}\Big) = m,$$

which is also a contradiction.

(iii) If $\deg P = m$, then we suppose that $P(z) = dz^m + P^*(z)$ where $\deg P^* < m$. We have to study two subcases:

$(*)$ If $d \ne i a_m$ $(i = 1, \dots, n)$, then

$$\alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \dots + \alpha_n e^{n a_m z^m} - e^{P^*} e^{dz^m} = -\alpha_0.$$

By using Lemma 2.2, we obtain $e^{P^*} \equiv 0$, which is impossible.

$(**)$ Suppose now that there exists at most $j \in \{1, 2, \dots, n\}$ such that $d = j a_m$. Without loss of generality, we assume that $j = n$. Then we rewrite (3.14) as

$$\alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \dots + (\alpha_n - e^{P^*}) e^{n a_m z^m} = -\alpha_0.$$

By using Lemma 2.2, we have $\alpha_0 \equiv 0$, so $\Delta_c^n e^{-P} = 0$. Thus

$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-P_{ic}} \equiv 0. \tag{3.15}$$

Suppose that $\deg P = \deg h = m > 1$ and

$$P(z) = b_m z^m + b_{m-1} z^{m-1} + \dots + b_0, \quad (b_m \ne 0).$$

Note that for $j = 0, 1, \dots, n$, we have

$$P(z + jc) = b_m z^m + (b_{m-1} + m b_m jc) z^{m-1} + \beta_j(z),$$

where $\beta_j(z)$ are polynomials with degree less than $m - 1$. Rewrite (3.15) as

$$\begin{aligned}& e^{-\beta_n(z)} e^{-b_m z^m - (b_{m-1} + m b_m nc) z^{m-1}} \\ & - n e^{-\beta_{n-1}(z)} e^{-b_m z^m - (b_{m-1} + m b_m (n-1)c) z^{m-1}} + \dots \\ & + (-1)^n e^{-\beta_0(z)} e^{-b_m z^m - b_{m-1} z^{m-1}} \equiv 0.\end{aligned} \tag{3.16}$$

For any $0 \le l < k \le n$, we have

$$\begin{aligned}& \rho(e^{-b_m z^m - (b_{m-1} + m b_m lc) z^{m-1} - (-b_m z^m - (b_{m-1} + m b_m kc) z^{m-1})}) \\ & = \rho(e^{-m b_m (l-k) c z^{m-1}}) = m - 1,\end{aligned}$$

and for $j = 0, 1, \dots, n$, we see that

$$\rho(e^{\beta_j}) \le m - 2.$$

By this, together with (3.16) and Lemma 2.2, we obtain $e^{-\beta_n(z)} \equiv 0$, which is impossible. Suppose now that $P(z) = \mu z + \eta$ ($\mu \neq 0$) and $Q(z) = \alpha z + \beta$ because if $\deg Q > 1$, then we go back to case (ii). It easy to see that

$$\begin{aligned}
\Delta_c^n e^{-P} &= \sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-\mu(z+ic)-\eta} \\
&= e^{-P} \sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-\mu ic} \\
&= e^{-P}(e^{-\mu c} - 1)^n.
\end{aligned}$$

This together with $\Delta_c^n e^{-P} \equiv 0$ gives $(e^{-\mu c} - 1)^n \equiv 0$, which yields $e^{\mu c} \equiv 1$. Therefore, for any $j \in \mathbb{Z}$,

$$e^{P(z+jc)} = e^{\mu z + \mu jc + \eta} = (e^{\mu c})^j e^{P(z)} = e^{P(z)}.$$

To prove that $e^{Q(z)}$ is also periodic entire function with period c, we suppose the contrary, which means that $e^{\alpha c} \neq 1$. Since $e^{P(z)}$ is of period c, then by (3.14), we obtain

$$\alpha_1 e^{(\alpha-\mu)z} + \alpha_2 e^{2(\alpha-\mu)z} + \cdots + \alpha_n e^{n(\alpha-\mu)z} = e^{\mu z+\eta}, \tag{3.17}$$

where α_i ($i = 1, \dots, n$) are constants. In particular,

$$\alpha_n = e^{n(\beta-\eta)+\alpha c \frac{n(n-1)}{2}}$$

and

$$\begin{aligned}
\alpha_1 &= \Big[\sum_{i=1}^{n} C_n^i (-1)^{n-i} + \sum_{i=2}^{n} C_n^i (-1)^{n-i} e^{\alpha c} \\
&\quad + \sum_{i=3}^{n} C_n^i (-1)^{n-i} e^{2\alpha c} + \cdots + e^{(n-1)\alpha c} \Big] e^{(\beta-\eta)} \\
&= \Big[C_n^1 (-1)^{n-1} + C_n^2 (-1)^{n-2} (1 + e^{\alpha c}) + C_n^3 (-1)^{n-3} (1 + e^{\alpha c} + e^{2\alpha c}) \\
&\quad + \cdots + C_n^n (-1)^{n-n} (1 + e^{\alpha c} + \cdots + e^{(n-1)\alpha c}) \Big] e^{(\beta-\eta)} \\
&= \Big[C_n^1 (-1)^{n-1} \frac{e^{\alpha c} - 1}{e^{\alpha c} - 1} + C_n^2 (-1)^{n-2} \frac{e^{2\alpha c} - 1}{e^{\alpha c} - 1} + C_n^3 (-1)^{n-3} \frac{e^{3\alpha c} - 1}{e^{\alpha c} - 1} \\
&\quad + \cdots + C_n^n (-1)^{n-n} \frac{e^{n\alpha c} - 1}{e^{\alpha c} - 1} \Big] e^{(\beta-\eta)} \\
&= \Big[C_n^1 (-1)^{n-1} (e^{\alpha c} - 1) + C_n^2 (-1)^{n-2} (e^{2\alpha c} - 1) + C_n^3 (-1)^{n-3} (e^{3\alpha c} - 1) \\
&\quad + \cdots + C_n^n (-1)^{n-n} (e^{n\alpha c} - 1) \Big] \frac{e^{(\beta-\eta)}}{e^{\alpha c} - 1} \\
&= \Big[\sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{i\alpha c} - (-1)^n - \sum_{i=1}^{n} C_n^i (-1)^{n-i} \Big] \frac{e^{(\beta-\eta)}}{e^{\alpha c} - 1} \\
&= (e^{\alpha c} - 1)^{n-1} e^{(\beta-\eta)}.
\end{aligned}$$

Rewrite (3.17) as

$$\alpha_1 e^{(\alpha-2\mu)z} + \alpha_2 e^{(2\alpha-3\mu)z} + \cdots + \alpha_n e^{(n\alpha-(n+1)\mu)z} = e^{\eta}, \tag{3.18}$$

it is clear that for each $1 \le l < m \le n$, we have

$$\rho\big(e^{(m\alpha-(m+1)\mu-l\alpha+(l+1)\mu)z}\big) = \rho\big(e^{(m-l)(\alpha-\mu)z}\big) = 1.$$

We have the following two cases:

(i1) If $j\alpha-(j+1)\mu \ne 0$ for all $j \in \{1,2,\dots,n\}$, which means that

$$\rho\big(e^{(j\alpha-(j+1)\mu)z}\big) = 1, \quad 1 \le j \le n$$

then, by applying Lemma 2.2 we obtain $e^{\eta} \equiv 0$, which is a contradiction.

(i2) If there exists (at most one) an integer $j \in \{1,2,\dots,n\}$ such that $j\alpha-(j+1)\mu = 0$. Without loss of generality, assume that $e^{(n\alpha-(n+1)\mu)z} = 1$, the equation (3.18) will be

$$\alpha_1 e^{(\alpha-2\mu)z} + \alpha_2 e^{(2\alpha-3\mu)z} + \dots + \alpha_{n-1} e^{((n-1)\alpha-n\mu)z} = e^{\eta} - e^{n(\beta-\eta)+\alpha c\frac{n(n-1)}{2}}$$

and by applying Lemma 2.2, we obtain $\alpha_1 = (e^{\alpha c}-1)^{n-1}e^{(\beta-\eta)} \equiv 0$, which is impossible. So, by (i1) and (i2), we deduce that $e^{\alpha c} \equiv 1$. Therefore, for any $j \in \mathbb{Z}$ we have

$$e^{Q(z+jc)} = e^{\alpha z+\beta}(e^{\alpha c})^j = e^{Q(z)},$$

which implies that e^{Q} is periodic of period c. Since $e^{P(z)}$ is of period c, then by (3.1), we obtain

$$\Delta_c^{n+1} f(z) = e^{P}\Delta_c f(z), \tag{3.19}$$

then $\Delta_c^{n+1} f(z)$ and $\Delta_c f(z)$ share 0 CM. Substituting (3.19) into the second equation (3.2), we obtain

$$e^{P(z)}\Delta_c f(z) = e^{Q(z)}(f(z)-a(z)) + a(z). \tag{3.20}$$

Since $e^{P(z)}$ and $e^{Q(z)}$ are of period c, then by (3.20), we obtain

$$\Delta_c^{n+1} f(z) = e^{Q-P}\Delta_c^{n} f(z). \tag{3.21}$$

So, $\Delta^{n+1} f(z)$ and $\Delta^{n} f(z)$ share $0, a(z)$ CM, combining (3.1), (3.2) and (3.21), we deduce that

$$\frac{\Delta^{n+1} f(z)-a(z)}{\Delta^{n} f(z)-a(z)} = \frac{\Delta^{n+1} f(z)}{\Delta^{n} f(z)},$$

and we obtain

$$\Delta^{n+1} f(z) = \Delta^{n} f(z)$$

which is a contradiction. Suppose now that $P = c_1$ and $Q = c_2$ are constants ($e^{c_1} \ne e^{c_2}$). By (3.8) we have

$$g_c(z) = (e^{c_2-c_1}+1)g(z) + a(z)e^{-c_1}$$

by the same,

$$g_{2c}(z) = (e^{c_2-c_1}+1)^2 g(z) + a(z)e^{-c_1}((e^{c_2-c_1}+1)+1).$$

By induction, we obtain

$$\begin{aligned} g_{nc}(z) &= (e^{c_2-c_1}+1)^n g(z) + a(z)e^{-c_1}\sum_{i=0}^{n-1}(e^{c_2-c_1}+1)^i \\ &= (e^{c_2-c_1}+1)^n g(z) + a(z)e^{-c_2}((e^{c_2-c_1}+1)^n-1). \end{aligned}$$

Rewrite the equation (3.6) as

$$\Delta_c^n g(z) = \sum_{i=0}^{n} C_n^i(-1)^{n-i}[(e^{c_2-c_1}+1)^i g(z) + a(z)e^{-c_2}\big((e^{c_2-c_1}+1)^i-1\big)]$$

$$= e^{c_1} g(z) + a(z).$$

Since $A(z) \equiv 0$, we have

$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} (e^{c_2 - c_1} + 1)^i = e^{c_1},$$
$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} ((e^{c_2 - c_1} + 1)^i - 1) = e^{c_2}$$

which are equivalent to

$$e^{n(c_2 - c_1)} = e^{c_1},$$
$$e^{n(c_2 - c_1)} = e^{c_2}$$

which is a contradiction.

Part (2). $h(z)$ is a constant. We show first that $P(z)$ is a constant. If $\deg P > 0$, from the equation (3.12), we see

$$\deg P \le \deg P - 1,$$

which is a contradiction. Then $P(z)$ must be a constant and since $h(z) = Q(z) - P(z)$ is a constant, we deduce that both of $P(z)$ and $Q(z)$ is constant. This case is impossible too (the last case in Part (1)), and we deduced that $h(z)$ can not be a constant. Thus, the proof complete. □

Proof of the Theorem 1.10. Setting $g(z) = f(z) + b(z) - a(z)$, we can remark that

$$g(z) - b(z) = f(z) - a(z),$$
$$\Delta_c^n g(z) - b(z) = \Delta_c^n f(z) - b(z),$$
$$\Delta_c^{n+1} g(z) - b(z) = \Delta_c^n f(z) - b(z), \ n \ge 2.$$

Since $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, it follows that $g(z)$, $\Delta_c^n g(z)$ and $\Delta_c^{n+1} g(z)$ share $b(z)$ CM. By using Theorem 1.7, we deduce that $\Delta_c^{n+1} g(z) \equiv \Delta_c^n g(z)$, which leads to $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$ and the proof complete. □

Proof of the Theorem 1.11. Note that $f(z)$ is a nonconstant entire function of finite order. Since $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, it follows that

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{P(z)}, \tag{3.22}$$
$$\frac{\Delta_c^{n+1} f(z)}{f(z)} = e^{Q(z)}, \tag{3.23}$$

where P and Q are polynomials. If $Q - P$ is a constant, then we can get easily from (3.22) and (3.23)

$$\Delta_c^{n+1} f(z) = e^{Q(z) - P(z)} \Delta_c^n f(z) :\equiv C \Delta_c^n f(z).$$

This completes the proof. If $Q - P$ is a not constant, with a similar arguing as in the proof of Theorem 1.7, we can deduce that the case $\deg P = \deg(Q - P) > 1$ is impossible. For the case $\deg P = \deg(Q - P) = 1$, we can obtain that $e^{P(z)}$ is periodic entire function with period c. This together with (3.22) yields

$$\Delta_c^{n+1} f(z) = e^{P(z)} \Delta_c f(z) \tag{3.24}$$

which means that $f(z)$, $\Delta_c f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM. Thus, by Theorem 1.6, we obtain

$$\Delta_c^{n+1} f(z) \equiv C \Delta_c f(z)$$

which is a contradiction to (3.22) and $\deg P = 1$. Theorem 1.11 is thus proved. □

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References

[1] W. Bergweiler, J. K. Langley; *Zeros of differences of meromorphic functions*, Math. Proc. Cambridge Philos. Soc. 142 (2007), no. 1, 133–147.

[2] B. Chen, Z. X. Chen, S. Li; *Uniqueness theorems on entire functions and their difference operators or shifts*, Abstr. Appl. Anal. 2012, Art. ID 906893, 8 pp.

[3] B. Chen, S. Li; *Uniquness problems on entire functions that share a small function with their difference operators*, Adv. Difference Equ. 2014, 2014:311, 11 pp.

[4] Y. M. Chiang, S. J. Feng; *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. 16 (2008), no. 1, 105-129.

[5] R. G. Halburd, R. J. Korhonen; *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. 314 (2006), no. 2, 477-487.

[6] R. G. Halburd, R. J. Korhonen; *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. 31 (2006), no. 2, 463–478.

[7] W. K. Hayman; *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford 1964.

[8] G. Jank. E. Mues, L. Volkmann; *Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endlichen Wert teilen*, Complex Variables Theory Appl. 6 (1986), no. 1, 51–71.

[9] I. Laine; *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.

[10] Z. Latreuch, B. Belaïdi; *Estimations about the order of growth and the type of meromorphic functions in the complex plane*, An. Univ. Oradea, Fasc. Matematica, Tom XX (2013), Issue No. 1, 179-186.

[11] P. Li, C. C. Yang; *Uniqueness theorems on entire functions and their derivatives*, J. Math. Anal. Appl. 253 (2001), no. 1, 50–57.

[12] C. C. Yang, H. X. Yi; *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.

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