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EXISTENCE OF PERIODIC SOLUTIONS FOR SUB-LINEAR FIRST-ORDER HAMILTONIAN SYSTEMS

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ABSTRACT. We prove the existence solutions for the sub-linear first-order Hamiltonian system $J\dot{u}(t) + Au(t) + \nabla H(t, u(t)) = h(t)$ by using the least action principle and a version of the Saddle Point Theorem.

1. INTRODUCTION

In this article, we consider the first-order Hamiltonian system

$$J\dot{u}(t) + Au(t) + \nabla H(t, u(t)) = h(t) \tag{1.1}$$

where A is a $(2N \times 2N)$ symmetric matrix, $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is T-periodic in the first variable $(T > 0)$ and $h \in C(\mathbb{R}, \mathbb{R}^{2N})$ is T-periodic.

When $A = 0$ and $h = 0$, it has been proved that system (1.1) has at least one T-periodic solution by the use of critical point theory and minimax methods [1, 2, 3, 4, 5, 6, 7, 13, 15, 16]. Many solvability conditions are given, such as the convex condition (see [3,5]), the super-quadratic condition (see [1, 4, 6, 7, 9, 12, 13, 16]), the sub-linear condition (see [2, 15]). When A is not identically null, the existence of periodic solutions for (1.1) has been studied in [7, 14]. In all these last papers, the Hamiltonian is assumed to be super-quadratic. As far as the general case (A not identically null) is concerned, to our best knowledge, there is no research about the existence of periodic solutions for (1.1) when H is sub-linear. In [2], the authors considered the special case $A = 0$ and $h = 0$ and obtain the existence of subharmonic solutions for (1.1) under the following assumptions:

(A1) There exist constants $a, b, c > 0$, $\alpha \in [0, 1[$, functions $p \in L^{\frac{2}{1-\alpha}}(0, T; \mathbb{R}^+)$, $q \in L^2(0, T; \mathbb{R}^+)$ and a nondecreasing function $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the following properties:

- (i) $\gamma(s + t) \le c(\gamma(s) + \gamma(t))$ for all $s, t \in \mathbb{R}^+$,
- (ii) $\gamma(t) \le at^\alpha + b$ for all $t \in \mathbb{R}^+$,
- (iii) $\gamma(t) \to +\infty$ *as* $t \to +\infty$, such that

$$|\nabla H(t, x)| \le p(t)\gamma(|x|) + q(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0, T];$$

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$$\lim_{|x|\to\infty} \frac{1}{\gamma^2(|x|)} \int_0^T H(t,x)dt = \pm\infty.$$

Similarly, in [15] the author considered the case $A = 0$ and $h = 0$ and obtained the existence os subharmonic solutions for (1.1) under the following assumptions:

(A2) There exist a positive constant a, $g \in L^2(0,T;\mathbb{R})$ and a non-increasing function $\omega \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the properties:

$$\liminf_{s\to\infty} \frac{\omega(s)}{\omega(\sqrt{s})} > 0,$$

$$\omega(s) \to 0, \quad \omega(s)s \to \infty \quad \text{as } s \to \infty,$$

such that

$$|\nabla H(t,x)| \le a\omega(|x|)|x| + g(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T];$$

$$\frac{1}{[\omega(|x|)|x|]^2} \int_0^T H(t,x)dt \to +\infty \quad \text{as } |x| \to \infty.$$

In Sections 4,5, we will use the Least Action Principle and a version of the Saddle Point Theorem to study the existence of periodic solutions for (1.1), when A and h are not necessary null and H satisfies some more general variants conditions replacing conditions (A1), (A2).

2. Preliminaries

Let $T > 0$ and A be a $(2N \times 2N)$ symmetric matrix. Consider the Hilbert space $H^{1/2}(S^1, \mathbb{R}^{2N})$ where $S^1 = \mathbb{R}/(T\mathbb{Z})$ and the continuous quadratic form Q defined on E by

$$Q(u) = \frac{1}{2}\int_0^T (J\dot{u}(t)\cdot u(t) + Au(t)\cdot u(t))dt$$

where $x \cdot y$ is the inner product of $x, y \in \mathbb{R}^{2N}$. Let us denote by E^0, E^-, E^+ respectively the subspaces of E on which Q is null, negative definite and positive definite. It is well known that these subspaces are mutually orthogonal in $L^2(S^1, \mathbb{R}^{2N})$ and in E with respect to the bilinear form

$$B(u,v) = \frac{1}{2}\int_0^T (J\dot{u}(t)\cdot v(t) + Au(t)\cdot v(t))dt, \ u, v \in E$$

associated with Q. If $u \in E^+$ and $v \in E^-$, then $B(u,v) = 0$ and $Q(u+v) = Q(u) + Q(v)$.

For $u = u^- + u^0 + u^+ \in E$, the expression $\|u\| = [Q(u^+) - Q(u^-) + |u^0|^2]^{1/2}$ is an equivalent norm in E. It is well known that the space E is compactly embedded in $L^s(S^1, \mathbb{R}^{2N})$ for all $s \in [1,\infty[$. In particular, for all $s \in [1,\infty[$, there exists $\lambda_s > 0$ such that for all $u \in E$,

$$\|u\|_{L^s} \le \lambda_s \|u\|. \tag{2.1}$$

Next, we have a version of the Saddle Point Theorem [11].

Lemma 2.1. *Let $E = E^1 \oplus E^2$ be a real Hilbert space with $E^2 = (E^1)^\perp$. Suppose that $f \in C^1(E,\mathbb{R})$ satisfies*

(a) *$f(u) = \frac{1}{2}\langle Lu, u\rangle + g(u)$ and $Lu = L_1P_1u + L_2P_2u$ with $L_i : E^i \to E^i$ bounded and self-adjoint, $i = 1, 2$;*
(b) *g' is compact;*
(c) *There exists $\beta \in \mathbb{R}$ such that $f(u) \le \beta$ for all $u \in E^1$;*

(d) *There exists $\gamma \in \mathbb{R}$ such that $f(u) \geq \gamma$ for all $u \in E^2$.*

Furthermore, if f satisfies the Palais-Smale condition $(PS)_c$ for all $c \geq \gamma$, then f possesses a critical value $c \in [\gamma, \beta]$.

3. Linear Hamiltonian systems

Let A be a $(2N \times 2N)$ symmetric matrix, we consider the linear Hamiltonian system

$$\dot{x} = JAx. \tag{3.1}$$

Let $\lambda_1, \dots, \lambda_s$ be all the distinct eigenvalues of $B = JA$ and $F_1, \dots, F_s$ be the corresponding root subspaces. The dimension of the root subspace F_σ is equal to the multiplicity m_σ of the corresponding root λ_σ of the characteristic equation $det(B - \lambda I_{2N}) = 0$ $(m_1 + \dots + m_s = 2N)$. The space $\mathbb{R}^{2N}$ splits into a direct sum of the B-invariant subspaces F_σ:

$$\mathbb{R}^{2N} = F_1 \oplus \dots \oplus F_s. \tag{3.2}$$

Each subspace F_σ possesses a basis $(a_1^\sigma, \dots, a_{m_\sigma}^\sigma)$ satisfying

$$Ba_1^\sigma = \lambda_\sigma a_1^\sigma,\ Ba_2^\sigma = \lambda_\sigma a_2^\sigma + a_1^\sigma, \dots, Ba_{m_\sigma}^\sigma = \lambda_\sigma a_{m_\sigma}^\sigma + a_{m_\sigma - 1}^\sigma.$$

The $(m_\sigma \times m_\sigma)$ matrix

$$Q_\sigma(\lambda_\sigma) = \begin{pmatrix} \lambda_\sigma & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_\sigma & 1 & 0 & \dots & 0 \\ . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \lambda_\sigma & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_\sigma \end{pmatrix}$$

is called an elementary Jordan matrix. We have $B = SQS^{-1}$ where Q is a direct sum of elementary Jordan matrices

$$Q = \begin{pmatrix} Q_1(\lambda_1) & 0 & 0 & \dots & 0 \\ 0 & Q_2(\lambda_2) & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & 0 & 0 & \dots & Q_s(\lambda_s) \end{pmatrix} = Q_1(\lambda_1) \oplus \dots \oplus Q_s(\lambda_s)$$

the columns of the matrix S,

$$a_1^1, \dots, a_{m_1}^1;\ a_1^2, \dots, a_{m_2}^2; \dots;\ a_1^s, \dots, a_{m_s}^s$$

form a basis for $\mathbb{R}^{2N}$ and so $det(S) \neq 0$.

The matrizant of equation (3.1) is given by

$$R(t) = e^{tB} = S[\exp(tQ_1(\lambda_1)) \oplus \dots \oplus \exp(tQ_s(\lambda_s))]S^{-1} = Se^{tQ}S^{-1}.$$

then the solution of equation (3.1) with initial condition $x(0)$ is

$$x(t) = e^{tB}x(0).$$

Therefore to each eigenvalue λ_σ corresponds a group of m_σ-linearly independent solutions:

$$\begin{gathered} x_1^\sigma(t) = e^{\lambda_\sigma t} a_1^\sigma \\ x_2^\sigma(t) = e^{\lambda_\sigma t}(t a_1^\sigma + a_2^\sigma) \\ \dots \\ x_{m_\sigma}^\sigma(t) = e^{\lambda_\sigma t}\Big(\frac{1}{(m_\sigma - 1)!} t^{m_\sigma - 1} a_1^\sigma + \dots + a_{m_\sigma}^\sigma\Big). \end{gathered} \tag{3.3}$$

Moreover, combining the solutions of all the groups (3.3) (there are obviously $2N$ in all, since $m_1+\dots+m_s=2N$), we obtain a complete system of linearly independent solutions of (3.1). Now, assume that $\lambda_1=0$ is an eigenvalue of $B=JA$ and let $1\le m\le m_1$ be the dimension of the corresponding eigenspace E_1. We can replace the basis $(a_1^1,\dots,a_{m_1}^1)$ of the root subspace F_1 by the basis $(b_1^1,\dots,b_{m_1}^1)$ where $(b_1^1,\dots,b_m^1)$ is a basis of E_1, $b_j^1=a_j^1$ for $m+1\le j\le m_1$ and such that $b_{m+1}^1=Bb_m^1$. To this basis corresponds the group of $2N$ linearly independent solutions:

$$\begin{gathered}
u_1^1(t)=b_1^1\\
\dots\\
u_m^1(t)=b_m^1\\
u_{m+1}^1(t)=b_m^1t+b_{m+1}^1\\
\dots\\
u_{m_1}^1(t)=\frac{1}{(m_1-m)!}b_m^1t^{m_1-m}+\dots+b_{m_1}^1\\
u_k^\sigma(t)=x_k^\sigma(t),\quad 2\le\sigma\le s,\ 1\le k\le m_\sigma.
\end{gathered}\tag{3.4}$$

A solution u of (3.1) may be written in the form

$$u(t)=\sum_{\sigma=1}^{s}\sum_{j=1}^{m_\sigma}\alpha_j^\sigma u_j^\sigma(t).$$

Let $T>0$ be such that $\lambda_\sigma T\notin 2i\pi\mathbb{Z}$ for all $1\le\sigma\le s$. If u is T-periodic, then for any $1\le\sigma\le s$, we have

$$\sum_{j=1}^{m_\sigma}\alpha_j^\sigma u_j^\sigma(kT)=\sum_{j=1}^{m_\sigma}\alpha_j^\sigma u_j^\sigma(0),\quad \forall k\in\mathbb{Z}.$$

It is easy to see that $\alpha_j^1=0$ for $m+1\le j\le m_1$ and $\alpha_j^\sigma=0$ for $2\le\sigma\le s$ and $1\le j\le m_{m_\sigma}$. Therefore, $u(t)=\sum_{j=1}^m\alpha_j^1b_j^1$. Hence the set of T-periodic solutions of (3.1) is equal to $N(A)$.

Example 3.1. Let

$$A=\begin{pmatrix}-12&6&5&1\\-2&1&0&1\\2&-1&0&-1\\2&-1&0&-1\end{pmatrix}$$

The characteristic equation corresponding to $B=JA$ is $\det(JA-XI_4)=X^3(X-5)=0$. To the eigenvalue $\lambda_1=0$ corresponds the eigenspace

$$E_1=\operatorname{span}\{e_1,e_2\}$$

and the root subspace

$$F_1=\operatorname{span}\{e_1,e_2,e_3\}$$

where $e_1=(1,2,0,0)$, $e_2=(1,1,1,1)$, $e_3=(0,0,0,1)$ with $Be_3=e_2$. To the eigenvalue $\lambda_2=5$ corresponds the root subspace

$$E_2=F_2=\operatorname{span}\{e_4\},$$

where $e_4 = (0, 0, 1, 0)$. Then we have $JA = SQS^{-1}$ with

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

The matrizant of the corresponding equation (3.1) is then

$$R(t) = SQS^{-1} = S \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{5t} \end{pmatrix} S^{-1}.$$

To the basis (e_1, e_2, e_3, e_4) corresponds the group of 4-linearly independent solutions

$$\begin{aligned} u_1(t) &= e_1 \\ u_2(t) &= e_2 \\ u_3(t) &= te_2 + e_3 \\ u_4(t) &= e^{5t} e_4. \end{aligned} \tag{3.5}$$

A solution of equation (3.1) takes the form

$$u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t) + \alpha_3 u_3(t) + \alpha_4 u_4(t)$$

and it is easy to verify that u is T-periodic for $T > 0$ if and only if $\alpha_3 = \alpha_4 = 0$, i.e. $u \in N(A)$.

4. First class of sub-linear Hamiltonian systems

Consider the first-order Hamiltonian system

$$J\dot{u}(t) + Au(t) + \nabla H(t, u(t)) = h(t) \tag{4.1}$$

where A is a $(2N \times 2N)$ symmetric matrix, $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is a continuous function, T-periodic in the first variable ($T > 0$) and differentiable with respect to the second variable with continuous derivative $\nabla H(t, x) = \frac{\partial H}{\partial x}(t, x)$, $h \in C(\mathbb{R}, \mathbb{R}^{2N})$ is T-periodic and J is the standard symplectic matrix $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Let $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing continuous function satisfying the properties:

- (i) $\gamma(s + t) \le c(\gamma(s) + \gamma(t))$ for all $s, t \in \mathbb{R}^+$,
- (ii) $\gamma(t) \le at^\alpha + b$ for all $t \in \mathbb{R}^+$,
- (iii) $\gamma(t) \to +\infty$ as $t \to +\infty$,

where a, b, c are positive constants and $\alpha \in [0, 1[$. Consider the following assumptions

- (C1) $\dim(N(A)) = m \ge 1$ and A has no eigenvalue of the form $ki\frac{2\pi}{T}$ ($k \in \mathbb{N}^*$);
- (H1) There exist two functions $p \in L^{\frac{2}{1-\alpha}}(0, T; \mathbb{R}^+)$ and $q \in L^2(0, T; \mathbb{R}^+)$ such that

$$|\nabla H(t, x)| \le p(t)\gamma(|x|) + q(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0, T].$$

Our main results in this section are the following theorems.

Theorem 4.1. *Assume* (C1) *and* (H1) *hold and*

(H2) *H satisfies*

$$\limsup_{|x|\to\infty, x\in N(A)} \frac{|x|}{\gamma^2(|x|)} < +\infty, \quad \lim_{|x|\to\infty, x\in N(A)} \frac{1}{\gamma^2(|x|)} \int_0^T H(t,x)dt = +\infty.$$

Then (4.1) *possesses at least one T-periodic solution.*

Example 4.2. Let A be the matrix defined in Example 3.1 and let

$$H(t,x) = (\frac{3}{4}T - t)|x|^{8/5}, \quad \forall x \in \mathbb{R}^{2N},\ \forall t \in [0,T].$$

Then

$$|\nabla H(t,x)| = \frac{8}{5}|\frac{3}{4}T - t||x|^{3/5}.$$

Let $\gamma(t) = t^{3/5}, t \geq 0$. It is clear that properties (i), (ii), (iii) are satisfied. Moreover, we have

$$\limsup_{|x|\to\infty, x\in N(A)} \frac{|x|}{\gamma^2(|x|)} = \limsup_{|x|\to\infty, x\in N(A)} \frac{|x|}{|x|^{\frac{6}{5}}} = 0 < +\infty,$$

$$\lim_{|x|\to\infty, x\in N(A)} \frac{1}{\gamma^2(|x|)} \int_0^T H(t,x)dt = \lim_{|x|\to\infty, x\in N(A)} \frac{\frac{1}{4}T^2|x|^{8/5}}{|x|^{\frac{6}{5}}} = +\infty$$

Hence, by Theorem 4.1, the corresponding system (4.1) possesses at least one T-periodic solution.

Theorem 4.3. *Assume* (C1) *and* (H1) *hold and*

(H3) *H satisfies*

$$\limsup_{|x|\to\infty, x\in N(A)} \frac{\gamma^2(|x|)}{|x|} < \infty, \quad \lim_{|x|\to\infty} \frac{1}{|x|} \int_0^T H(t,x)dt = +\infty.$$

Then (4.1) *possesses at least one T-periodic solution.*

Theorem 4.4. *Assume* (C1) *and* (H1) *hold and*

(H4) *H satisfies*

$$\limsup_{|x|\to\infty, x\in N(A)} \frac{\gamma^2(|x|)}{|x|} = 0, \quad \lim_{|x|\to\infty} \frac{1}{|x|} \int_0^T H(t,x)dt > \int_0^T |h(t)|dt.$$

Then (4.1) *possesses at least one T-periodic solution.*

Example 4.5. Let A be the matrix defined in Example 3.1 and let

$$H(t,x) = (\frac{1}{2}T - t)ln^{\frac{3}{2}}(1+|x|^2) + \frac{l(t)|x|^3}{1+|x|^2}, \quad \forall x \in \mathbb{R}^{2N},\ \forall t \in [0,T],$$

where $l \in C([0,T], \mathbb{R}^+)$ with $\int_0^T l(t)dt > \int_0^T |h(t)|dt$. Then

$$\begin{aligned}
|\nabla H(t,x)| &\leq \frac{3}{2}|\frac{1}{2}T - t|\big(\ln(1+|x|^2)\big)^{1/2} \frac{|x|}{1+|x|^2} + \frac{l(t)(5|x|^4) + 3|x|^2)}{1+2|x|^2+|x|^4} \\
&\leq \frac{3}{2}|\frac{1}{2}T - t|\big(\ln(1+|x|^2)\big)^{1/2} \frac{|x|}{1+|x|^2} + c_1
\end{aligned}$$

where c_1 is a positive constant. Let $\gamma(t) = \big(\ln(1+|t|^2)\big)^{1/2}$, $t \geq 0$. It is clear that conditions (i), (ii), (iii) are satisfied. Moreover,

$$\limsup_{|x|\to\infty, x\in N(A)} \frac{\gamma^2(|x|)}{|x|} = \limsup_{|x|\to\infty, x\in N(A)} \frac{ln(1+|x|^2)}{|x|} = 0 < +\infty,$$

$$\lim_{|x|\to\infty, x\in N(A)} \frac{1}{|x|}\int_0^T H(t,x)dt = \int_0^T l(t)dt > \int_0^T |h(t)|dt.$$

Hence, by Theorem 4.4, the corresponding system (4.1) possesses at least one T-periodic solution.

Theorem 4.6. *Assume* (C1) *and* (H1) *hold and*

(H5) *H satisfies*

$$\int_0^T h(t)dt \perp N(A), \quad \lim_{|x|\to\infty, x\in N(A)} \frac{1}{\gamma^2(|x|)}\int_0^T H(t,x)dt = +\infty.$$

Then (4.1) *possesses at least one T-periodic solution.*

Theorem 4.6 generalizes the result concerning the existence of periodic solutions for (4.1) in [2, Theorem 3.1].

Example 4.7. Let A be the matrix defined in Example 3.1 and let

$$H(t,x) = (\frac{3}{4}T - t)ln^{\frac{3}{2}}(1+|x|^2) + l(t)\big(\ln(1+|x|^2)\big)^{1/2}, \quad x\in\mathbb{R}^{2N},\ t\in[0,T],$$

where $l \in C([0,T],\mathbb{R}^+)$ and $h(t) = c(t)v_1 + d(t)v_2$, with $v_1 = (2,-1,0,-1), v_2 = (0,0,1,-1) \in (N(A))^{\perp}$, $c,d \in C(\mathbb{R},\mathbb{R})$. Then $\int_0^T h(t)dt \perp N(A)$ and

$$|\nabla H(t,x)| \leq \frac{3}{2}|\frac{3}{4}T - t|\big(\ln(1+|x|^2)\big)^{1/2} + l(t).$$

Let $\gamma(t) = \big(\ln(1+|x|^2)\big)^{1/2}$, $t \geq 0$. It is easy to verify that γ satisfies conditions (i), (ii), (iii). Moreover,

$$\lim_{|x|\to\infty, x\in N(A)} \frac{1}{\gamma^2(|x|)}\int_0^T H(t,x)dt = \lim_{|x|\to\infty, x\in N(A)} \frac{T^2}{4}\big(\ln(1+|x|^2)\big)^{1/2} = +\infty$$

Hence, by Theorem 4.6, the corresponding system (4.1) possesses at least one T-periodic solution.

Remark 4.8. Let $u(t)$ be a periodic solution of (4.1), then by replacing t by $-t$ in (4.1), we obtain

$$\dot{u}(-t) = JH'(-t, u(-t)).$$

So it is clear that the function $v(t) = u(-t)$ is a periodic solution of the system

$$\dot{v}(t) = -JH'(-t, v(t)).$$

Moreover, $-H(-t,x)$ satisfies (H2)–(H5) whenever $H(t,x)$ satisfies the following assumptions

(H2')

$$\limsup_{|x|\to\infty, x\in N(A)} \frac{|x|}{\gamma^2(|x|)} < +\infty, \quad \lim_{|x|\to\infty, x\in N(A} \frac{1}{\gamma^2(|x|)}\int_0^T H(t,x)dt = -\infty;$$

(H3')
$$\limsup_{|x|\to\infty, x\in N(A)} \frac{\gamma^2(|x|)}{|x|} < \infty, \quad \lim_{|x|\to\infty} \frac{1}{|x|}\int_0^T H(t,x)dt = -\infty;$$
(H4')
$$\limsup_{|x|\to\infty, x\in N(A)} \frac{\gamma^2(|x|)}{|x|} = 0, \quad \lim_{|x|\to\infty} \frac{1}{|x|}\int_0^T H(t,x)dt < -\int_0^T |h(t)|dt;$$
(H5')
$$\int_0^T h(t)dt \perp N(A), \quad \lim_{|x|\to\infty, x\in N(A)} \frac{1}{\gamma^2(|x|)}\int_0^T H(t,x)dt = -\infty.$$
Consequently, the previous Theorems remains true if we replace (H2)–(H5) by (H2')–(H5').

Proofs of Theorems. Consider the functional
$$\varphi(u) = \frac{1}{2}\int_0^T (J\dot{u}(t)\cdot u(t) + Au(t)\cdot u(t))dt + \int_0^T H(t,u(t))]\,dt - \int_0^T h(t)\cdot u(t))\,dt$$
Let E be the space introduced in Section 2. By assumption (H1) and the property (ii) of γ, [11, Proposition B37] implies that $\varphi \in C^1(E,\mathbb{R})$ and the critical points of φ on E correspond to the T-periodic solutions of (4.1), moreover
$$\varphi'(u)v = \int_0^T [J\dot{u}(t) + Au(t) + \nabla H(t,u(t))]\cdot v(t)\,dt - \int_0^T h(t)\cdot v(t)\,dt.$$

Lemma 4.9. *Assume* (H1) *holds. Then for any* (PS) *sequence* $(u_n) \subset E$ *of the functional* φ*, there exists a constant* $c_0 > 0$ *such that*
$$\|\tilde{u}_n\| \le c_0(\gamma(\|u_n^0\|) + 1), \quad \forall n \in \mathbb{N} \tag{4.2}$$
where $\tilde{u}_n = u_n^+ + u_n^- = u_n - u_n^0$*, with* $u_n^0 \in E^0$*,* $u_n^- \in E^-$*,* $u_n^+ \in E^+$*.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a (PS) sequence, i.e. $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. We have
$$\varphi'(u_n)(u_n^+ - u_n^-) = 2\|\tilde{u}_n\|^2 + \int_0^T \nabla H(t,u_n)\cdot(u_n^+ - u_n^-)dt - \int_0^T h(t)\cdot(u_n^+ - u_n^-)dt.$$
Since $\varphi'(u_n) \to 0$ as $n \to \infty$, there exists a constant $c_2 > 0$ such that
$$\left|\varphi'(u_n)(u_n^+ - u_n^-)\right| \le c_2\|\tilde{u}_n\|, \ \forall n \in \mathbb{N}.$$
By Hölder's inequality and (H1), we have
$$\begin{aligned}
\Big|\int_0^T \nabla H(t,u_n)\cdot(u_n^+ - u_n^-)dt\Big| &\le \|\tilde{u}_n\|_{L^2}\Big(\int_0^T |\nabla H(t,u_n)|^2 dt\Big)^{1/2} \\
&\le \|\tilde{u}_n\|_{L^2}\Big(\int_0^T [p(t)\gamma(|u_n|) + q(t)]dt\Big)^{1/2} \\
&\le \|\tilde{u}_n\|_{L^2}\big[\Big(\int_0^T p^2(t)\gamma^2(|u_n|)dt\Big)^{1/2} + \|q\|_{L^2}\big].
\end{aligned} \tag{4.3}$$
Now, by nondecreasing condition and the properties (i) and (ii) of γ, we have
$$\Big(\int_0^T p^2(t)\gamma^2(|u_n|)dt\Big)^{1/2} \le \Big(\int_0^T p^2(t)\gamma^2(|\tilde{u}_n| + |u_n^0|)dt\Big)^{1/2}$$

$$
\begin{aligned}
&\le c\Big(\int_0^T [p^2(t)[\gamma(|\tilde{u}_n|)+\gamma(|u_n^0|)]^2 dt\Big)^{1/2}\\
&\le c\Big[\Big(\int_0^T p^2(t)\gamma^2(|\tilde{u}_n|)dt\Big)^{1/2}+\|p\|_{L^2}\gamma(|u_n^0|)\Big)\Big]\\
&\le c\Big[\Big(\int_0^T p^2(t)(a|\tilde{u}_n|^\alpha+b)^2 dt\Big)^{1/2}+\|p\|_{L^2}\gamma(|u_n^0|)\Big]\\
&\le c\Big[a\Big(\int_0^T p^2(t)|\tilde{u}_n|^{2\alpha}dt\Big)^{1/2}+b\|p\|_{L^2}+\|p\|_{L^2}\gamma(|u_n^0|)\Big]\\
&\le c\big[a\|p\|_{L^{\frac{2}{1-\alpha}}}\|\tilde{u}_n\|_{L^2}^\alpha+b\|p\|_{L^2}+\|p\|_{L^2}\gamma(|u_n^0|)\big].
\end{aligned}
$$

On the other hand, by (2.1) we have

$$
\begin{aligned}
2\|\tilde{u}_n\|^2 &\le |\varphi'(u_n)(u_n^+-u_n^-)|+|\int_0^T \nabla H(t,u_n)\cdot(u_n^+-u_n^-)dt|\\
&\quad +|\int_0^T h(t)\cdot(u_n^+-u_n^-)|\le c_2\|\tilde{u}_n\|+\|\tilde{u}_n\|_{L^2}c\big[a\|p\|_{L^{\frac{2}{1-\alpha}}}\|\tilde{u}_n\|_{L^2}^\alpha\\
&\quad +b\|p\|_{L^2}+\|q\|_{L^2}+\|p\|_{L^2}\gamma(|u_n^0|)\big]+\|h\|_{L^2}\|\tilde{u}_n\|_{L^2}\\
&\le ac\|p\|_{L^{\frac{2}{1-\alpha}}}\lambda_2^{\alpha+1}\|\tilde{u}_n\|^{\alpha+1}+[c_1+cb\|p\|_{L^2}\\
&\quad +\|q\|_{L^2}+\|h\|_{L^2}]\lambda_2\|\tilde{u}_n\|+c\lambda_2\|p\|_{L^2}\gamma(|u_n^0|)\|\tilde{u}_n\|.
\end{aligned}
$$

Since $0\le\alpha<1$, we deduce that there exists a constant $c_0>0$ satisfying (4.2). □

We will apply Lemma 2.1 to the functional φ to obtain critical points.

Lemma 4.10. *If* (H1) *holds and H satisfies one of the assumptions* (H2)–(H5), *then φ satisfies the $(PS)_c$ condition for all $c\in\mathbb{R}$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a $(PS)_c$ sequence, that is $\varphi(u_n)\to c$ and $\varphi'(u_n)\to 0$ as $n\to\infty$. Then there exists a positive constant c_3 such that

$$
|\varphi(u_n)|\le c_3,\quad \|\varphi'(u_n)\|\le c_3,\quad \forall n\in\mathbb{N}.
$$

By the Mean Value Theorem and Hölder's inequality, we have

$$
\begin{aligned}
&|\int_0^T (H(t,u_n)-H(t,u_n^0))dt|\\
&=|\int_0^T\int_0^1 \nabla H(t,u_n^0+s\tilde{u}_n)\cdot\tilde{u}_n\,ds\,dt|\\
&\le\|\tilde{u}_n\|_{L^2}\int_0^1\Big(\int_0^T|\nabla H(t,u_n^0+s\tilde{u}_n)|^2dt\Big)^{1/2}ds.
\end{aligned}\tag{4.4}
$$

As in the proof of Lemma 4.9, we have

$$
\begin{aligned}
&\Big(\int_0^T|\nabla H(t,u_n^0+s\tilde{u}_n)|^2dt\Big)^{1/2}\\
&\le ac\|p\|_{L^{\frac{2}{1-\alpha}}}\|\tilde{u}_n\|_{L^2}^\alpha+cb\|p\|_{L^2}+\|q\|_{L^2}+c\|p\|_{L^2}\gamma(|u_n^0|)\|q\|_{L^2}\big].
\end{aligned}\tag{4.5}
$$

Therefore, by properties (2.1), (4.2), (4.4), (4.5) and since $0 \leq \alpha < 1$, there exists a positive constant c_4 such that

$$
\begin{aligned}
\Big|\int_0^T (H(t,u_n) - H(t,u_n^0))dt\Big| &\leq c_0(\gamma(|u_n^0|)+1)[ac\|p\|_{L^{\frac{2}{1-\alpha}}} c_0^\alpha(\gamma(|u_n^0|)+1)^\alpha \\
&\quad + c\|p\|_{L^2}\gamma(|u_n^0|) + cb\|p\|_{L^2} + \|q\|_{L^2}] \\
&\leq c_4(\gamma^2(|u_n^0|)+1).
\end{aligned} \tag{4.6}
$$

Combining (2.1), (4.2), (4.3) and (4.6) yields

$$
\begin{aligned}
c_3 &\geq \varphi(u_n) \\
&\geq -\|\tilde{u}_n\|^2 + \int_0^T (H(t,u_n) - H(t,u_n^0))dt + \int_0^T H(t,u_n^0)dt \\
&\quad - \int_0^T h(t)(\tilde{u}_n + u_n^0)dt \\
&\geq -c_0^2(\gamma(|u_n^0)|+1)^2 - c_4(\gamma^2(|u_n^0)|)+1) - c_0\|h\|_{L^2}(\gamma(|u_n^0)|+1) \\
&\quad - \|h\|_{L^1}|u_n^0| + \int_0^T H(t,u_n^0)dt \\
&\geq -c_5(\gamma^2(|u_n^0)|)+1) - \|h\|_{L^1}|u_n^0| + \int_0^T H(t,u_n^0)dt,
\end{aligned} \tag{4.7}
$$

where c_5 is a positive constant.

Case 1: H satisfies (H2). By (4.7), we have

$$
c_3 \geq \gamma^2(|u_n^0)|)[-c_5 - \|h\|_{L^1}\frac{|u_n^0|}{\gamma^2(|u_n^0)|)} + \frac{1}{\gamma^2(|u_n^0)|)}\int_0^T H(t,u_n^0)dt] - c_5.
$$

It follows from (H2) that (u_n^0) is bounded.

Case 2: H satisfies (H3) or (H4). Note that by (4.7)

$$
c_3 \geq |u_n^0|[-c_5\frac{\gamma^2(|u_n^0)|)}{|u_n^0)|} - \|h\|_{L^1} + \frac{1}{|u_n^0|}\int_0^T H(t,u_n^0)dt] - c_5.
$$

Hence (H3) or (H4) implies that (u_n^0) is bounded.

Case 2: H satisfies (H5). Since $\int_0^T h(t)dt \perp N(A)$, we get as in (4.7)

$$
\begin{aligned}
c_3 &\geq \varphi(u_n) \\
&\geq -\|\tilde{u}_n\|^2 + \int_0^T (H(t,u_n) - H(t,u_n^0))dt + \int_0^T H(t,u_n^0)dt \\
&\quad - \int_0^T h(t)\cdot \tilde{u}_n dt \\
&\geq -c_5(\gamma^2(|u_n^0)|)+1) + \int_0^T H(t,u_n^0)dt, \\
&\geq \gamma^2(|u_n^0)|)[-c_5 + \frac{1}{\gamma^2(|u_n^0)|)}\int_0^T H(t,u_n^0)dt] - c_5.
\end{aligned} \tag{4.8}
$$

Hence (H5) implies that (u_n^0) is bounded.

In all the above cases, (u_n^0) is bounded. We deduce from Lemma 4.9 that (u_n) is also bounded in E. By a standard argument, we conclude that (u_n) possesses a convergent subsequence. The proof of Lemma 4.10 is complete. $\square$

Now, decompose $E = E^- \oplus (E^0 \oplus E^+)$ and let $E^1 = E^-$ and $E^2 = E^0 \oplus E^+$. Remark that by Section 3, we have $E^0 = N(A)$. We will verify that φ satisfies condition c) of Lemma 2.1. For $u \in E^1$, we have

$$\varphi(u) = -\|u\|^2 + \int_0^T (H(t,u) - H(t,0))dt + \int_0^T H(t,0)dt - \int_0^T h(t)\cdot u dt.$$

As in the proof of Lemma 4.10,

$$\Big|\int_0^T (H(t,u) - H(t,0))dt\Big| \le [ac\|p\|_{L^{\frac{2}{1-\alpha}}}\|u\|_{L^2}^{\alpha} + cb\|p\|_{L^2} + \|q\|_{L^2}]\|u\|_{L^2}.$$

Hence by (2.1), we deduce

$$\begin{aligned} \varphi(u) \le -\|u\|^2 &+ ac\|p\|_{L^{\frac{2}{1-\alpha}}}\lambda_2^{\alpha+1}\|u\|^{\alpha+1} + (cb\|p\|_{L^2} \\ &+ \|q\|_{L^2} + \|h\|_{L^2})\lambda_2\|u\| + \int_0^T H(t,0)dt. \end{aligned} \tag{4.9}$$

Since $0 \le \alpha < 1$, (4.9) implies that $\varphi(u) \to -\infty$ as $\|u\| \to \infty$. Hence there exists $\beta \in \mathbb{R}$ such that $f(u) \le \beta$ for all $u \in E^1$. Condition (c) of Lemma 2.1 is then proved.

Let us verify that φ satisfies condition (d) of Lemma 2.1. In fact, for $u \in E^2 = E^0 \oplus E^+$, as in the proof of Lemma 4.10, we have

$$\begin{aligned} &\Big|\int_0^T (H(t,u) - H(t,u^0))dt\Big| \\ &\le \big[ac\|p\|_{L^{\frac{2}{1-\alpha}}}\|u^+\|_{L^2}^{\alpha} + cb\|p\|_{L^2} + c\|p\|_{L^2}\gamma(|u^0|) + \|q\|_{L^2}\big]\|u^+\|_{L^2}^{\alpha}. \end{aligned} \tag{4.10}$$

From (2.1) and (4.10), we deduce that

$$\begin{aligned} \varphi(u) \ge \|u^+\|^2 &- ac\|p\|_{L^{\frac{2}{1-\alpha}}}\lambda_2^{\alpha+1}\|u^+\|^{\alpha+1} - c\|p\|_{L^2}\lambda_2\|u^+\|\gamma(|u^0|) \\ &- (cb\|p\|_{L^2} + \|q\|_{L^2} + \|h\|_{L^2})\lambda_2\|u^+\| - \int_0^T |h|dt|u^0| + \int_0^T H(t,u^0)dt. \end{aligned} \tag{4.11}$$

For $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that

$$c\|p\|_{L^2}\lambda_2\|u^+\|\gamma(|u^0|) \le \epsilon\|u^+\|^2 + C(\epsilon)\gamma^2(|u^0|).$$

Taking $\epsilon = 1/2$, it follows from (4.11) that

$$\begin{aligned} \varphi(u) \ge \frac{1}{2}\|u^+\|^2 &- ac\|p\|_{L^{\frac{2}{1-\alpha}}}\lambda_2^{\alpha+1}\|u^+\|^{\alpha+1} - \lambda_2(cb\|p\|_{L^2} + \|q\|_{L^2} \\ &+ \|h\|_{L^2})\|u^+\| - C(\frac{1}{2})\gamma^2(|u^0|) - \int_0^T |h|dt|u^0| + \int_0^T H(t,u^0)dt. \end{aligned} \tag{4.12}$$

Since $0 \le \alpha < 1$, the term

$$\frac{1}{2}\|u^+\|^2 - ac\|p\|_{L^{\frac{2}{1-\alpha}}}\lambda_2^{\alpha+1}\|u^+\|^{\alpha+1} - \lambda_2(cb\|p\|_{L^2} + \|q\|_{L^2} + \|h\|_{L^2})\|u^+\|$$

approaches $+\infty$ as $\|u^+\| \to \infty$. It remains to study the following member of (4.12)

$$\psi(u^0) = -C(\frac{1}{2})\gamma^2(|u^0|) - \int_0^T |h|dt|u^0| + \int_0^T H(t,u^0)dt.$$

Case 1: (H2) holds. We have

$$\psi(u^0) \geq \gamma^2(|u^0|)\big(-C(\frac{1}{2}) - \int_0^T |h|dt\frac{|u^0|}{\gamma^2(|u^0|)} + \frac{1}{\gamma^2(|u^0|)}\int_0^T H(t,u^0)dt\big).$$

It follows from (H2) that $\psi(u^0) \to +\infty$ as $|u^0| \to \infty$.

Case 2: (H3) or (H4) holds. We have

$$\psi(u^0) \geq |u^0|\big(-C(\frac{1}{2})\frac{\gamma^2(|u^0|)}{|u^0|} - \int_0^T |h|dt + \frac{1}{|u^0|}\int_0^T H(t,u^0)dt\big).$$

It follows from (H3) or (H4) that $\psi(u^0) \to +\infty$ as $|u^0| \to \infty$.

Case 3: (H5) holds. Since $\int_0^T h(t)dt \perp N(A)$, we have

$$\psi(u^0) \geq \gamma^2(|u^0|)\big(-C(\frac{1}{2}) + \frac{1}{\gamma^2(|u^0|)}\int_0^T H(t,u^0)dt\big).$$

It follows from (H5) that $\psi(u^0) \to +\infty$ as $|u^0| \to \infty$.

Therefore, if one of assumptions (H2)–(H5) is satisfied, then $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. So there exists a constant ρ such that $\varphi(u) \geq \rho$ for all $u \in E^2$. Condition d) of Lemma 2.1 is satisfied. Moreover, it is well known that the derivative of the functional $d(u) = \int_0^T H(t,u)dt - \int_0^T hudt$ is compact. All the conditions of Lemma 2.1 are satisfied, so φ possesses a critical point u which is a T-periodic solution of system (4.1)

5. Second class of Hamiltonian systems

For A, H and h be defined as in Section 4, we have the following result.

Theorem 5.1. *Let $\omega \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non-increasing function with the following properties:*

(a) $\liminf_{s\to\infty} \frac{\omega(s)}{\omega(\sqrt{s})} > 0$,

(b) $\omega(s) \to 0$ *and* $\omega(s)s \to +\infty$ *as* $s \to \infty$.

Assume that A satisfies (C1)*, and H satisfies*

(H6) *There exist a positive constant a and a function $g \in L^2(0,T;\mathbb{R})$ such that*

$$|\nabla H(t,x)| \leq a\omega(|x|) + g(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T];$$

(H7)

$$\lim_{|x|\to\infty, x\in N(A)} \frac{1}{(\omega(|x|)|x|)^2}\int_0^T H(t,x)dt = +\infty;$$

(H8) *There exists $f \in L^1(0,T;\mathbb{R})$ such that*

$$H(t,x) \geq f(t), \quad \forall x \in \mathbb{R}^{2N}, \text{ a.e. } t \in [0,T].$$

Then system (4.1) *possesses at least one T-periodic solution.*

The above theorem generalizes [15, Theorem 1.1].

Example 5.2. Take $\omega(s) = \frac{1}{\ln(2+s^2)}$, $s \geq 0$,

$$H(t,x) = (\frac{1}{2} + \cos(\frac{2\pi}{T}t))\frac{|x|^2}{\ln(2+|x|^2)}, \quad \forall t \in [0,T], \ \forall x \in \mathbb{R}^{2N}$$

and let A be the matrix defined in Section 3, $h \in C([0,T],\mathbb{R})$. Then A, H, h satisfy assumptions of Theorem 5.1.

Proof of Theorem 5.1. As in Section 4, we will apply Lemma 2.1 to the functional φ defined on the space E introduced in section 2.

Lemma 5.3 ([15]). *Assume* (H6) *and* (H7) *hold, then there exists a non-increasing function* $\theta \in C(]0,+\infty[,\mathbb{R}^+)$ *and a positive constant* c_0 *such that*

(i) $\theta(s) \to 0$ *and* $\theta(s)s \to \infty$ *as* $s \to \infty$,
(ii) $\|\nabla H(t,u)\|_{L^2} \le c_0(\theta(\|u\|)\|u\| + 1)$ *for all* $u \in E$
(iii)
$$\frac{1}{(\theta(\|u^0\|)\|u^0\|)^2}\int_0^T H(t,u^0)dt \to +\infty \quad \text{as } \|u^0\| \to \infty.$$

Lemma 5.4. *Assume* (H6) *holds. Then for any* (PS) *sequence of the functional* φ, *there exists a constant* $c_1 > 0$ *such that*
$$\|\tilde{u}_n\| \le c_1\big(\theta(\|u_n^0\|)\|u_n^0\| + 1\big). \tag{5.1}$$

Proof. Let (u_n) be a Palais-Smale sequence, that is $(\varphi(u_n))$ is bonded and $\varphi'(u_n) \to 0, as\ n \to \infty$. We have
$$\varphi'(u_n)(u_n^+ - u_n^-) = 2\|\tilde{u}_n\|^2 + \int_0^T \nabla H(t,u_n(t))\cdot(u_n^+ - u_n^-)dt - \int_0^T h(t)\cdot(u_n^+ - u_n^-)dt.$$
Since θ is non-increasing and $\|u\| \ge max(\|\tilde{u}\|, \|u^0\|)$, we have
$$\theta(\|u\|) \le \min(\theta(\|\tilde{u}\|), \theta(\|u^0\|)). \tag{5.2}$$
By Hölder's inequality, inequalities (2.1), (5.1), (5.2) and Lemma 5.3, we have
$$\begin{aligned}
&|\int_0^T \nabla H(t,u_n(t))\cdot(u_n^+ - u_n^-)dt| \\
&\le \|u_n^+ - u_n^-\|_{L^2}(\int_0^T |\nabla H(t,u_n)|^2 dt)^{1/2} \\
&\le c_2\|\tilde{u}_n\|(\theta(\|u_n\|)\|u_n\| + 1) \\
&\le c_2\|\tilde{u}_n\|\big(\theta(\|\tilde{u}_n\|)\|\tilde{u}_n\| + \theta(\|u_n^0\|)\|u_n^0\| + 1\big).
\end{aligned}$$
Thus there exists positive constants c_3, c_4 such that
$$\begin{aligned}
c_3\|\tilde{u}_n\| &\ge \varphi'(u_n)(u_n^+ - u_n^-) \\
&\ge 2\|\tilde{u}_n\|^2 - c_2\|\tilde{u}_n\|\big(\theta(\|\tilde{u}_n\|)\|\tilde{u}_n\| + \theta(\|u_n^0\|)\|u_n^0\| + 1\big) - c_4\|\tilde{u}_n\|.
\end{aligned}$$
Hence
$$c_2\theta(\|u_n^0\|)\|u_n^0\| \ge \|\tilde{u}_n\|[2 - c_2\|\tilde{u}_n\|] - c_3 - c_4.$$
Since $\theta(s) \to 0$ as $s \to \infty$, this implies the existence of a constant c_1 satisfying (5.1). □

Lemma 5.5. φ *satisfies the* $(PS)_c$ *condition for all real* c.

Proof. Let (u_n) be a $(PS)_c$-sequence. Assume that (u_n^0) is unbounded. Going to a subsequence if necessary, we can assume that $\|u_n^0\| \to \infty$ as $n \to \infty$. By the

Mean Value Theorem, Hölder's inequality, inequality (2.1) and Lemma 5.3 (ii), there exists a positive constant c_5 such that

$$\begin{aligned}
&\Big|\int_0^T (H(t,u_n)-H(t,u_n^0))dt\Big| \\
&= \Big|\int_0^T\int_0^1 \nabla H(t,u_n^0+s\tilde{u}_n)\cdot \tilde{u}_n\,ds\,dt\Big| \\
&\le \|\tilde{u}_n\|_{L^2}\int_0^1 \Big(\int_0^T |\nabla H(t,u_n^0+s\tilde{u}_n)dt|\Big)^{1/2} ds \\
&\le c_5\|\tilde{u}_n\|\big[\theta(\|u_n^0\|)\|u_n^0\|+\theta(\|u_n^0\|)\|\tilde{u}_n\|+1\big].
\end{aligned} \tag{5.3}$$

Hence by Lemma 5.4, there exists a positive constant c_6 such that

$$\Big|\int_0^T (H(t,u_n)-H(t,u_n^0))dt\Big| \le c_6\big([\theta(\|u_n^0\|)\|\tilde{u}_n^0\|]^2+1\big). \tag{5.4}$$

Combining (2.1), (5.1) and (5.4) yields

$$\varphi(u_n)\ge -c_7\big([\theta(\|u_n^0\|)\|\tilde{u}_n^0\|]^2+1\big)-\frac{1}{T}\int_0^T |h(t)|dt\|u_n^0\|+\int_0^T H(t,u_n^0)dt$$

where c_7 is a positive constant.

On the other hand, it is easy to see that $\lim inf_{s\to\infty}\frac{\theta(s)}{\theta(\sqrt{s})}>0$. So there exists a positive constant c_8 such that for s large enough $\theta(s)\ge c_8\theta(\sqrt{s})$. Hence for n large enough

$$\frac{\|u_n^0\|}{[\theta(\|u_n^0\|)\|u_n^0\|]^2}\ge \frac{1}{c_8^2[\theta(\|u_n^0\|^{1/2})\|u_n^0\|^{1/2}]^2}\to 0 \quad \text{as } n\to\infty\,.$$

Therefore,

$$\begin{aligned}
\varphi(u_n)\ge [\theta(\|u_n^0\|)\|u_n^0\|]^2[-c_7-\frac{1}{T}\int_0^T |h(t)|dt\frac{\|u_n^0\|}{[\theta(\|u_n^0\|)\|u_n^0\|]^2} \\
+\frac{1}{[\theta(\|u_n^0\|)\|u_n^0\|]^2}\int_0^T H(t,u_n^0)dt]-c_7\to+\infty
\end{aligned}$$

as $n\to\infty$, which contradicts the boundedness of $(\varphi(u_n))$. Hence $(\|u_n^0\|)$ is bounded, and by Lemma 5.4, (u_n) is also bounded. By a standard argument, we conclude that (u_n) possesses a convergent subsequence. The proof is complete. □

Now, for $u=u^0+u^+\in E^2=E^0\oplus E^+$, we have as in (5.3),

$$\Big|\int_0^T (H(t,u)-H(t,u^0))dt\Big|\le c_5\|u^+\|\big[\theta(\|u^0\|)\|u^0\|+\theta(\|u^0\|)\|u^+\|+1\big].$$

Since $c_5\theta(\|u^0\|)\|u^0\|\|u^+\|\le \frac{1}{2}\|u^+\|^2+2c_5^2[\theta(\|u^0\|)\|u^0\|]^2$, we obtain

$$\begin{aligned}
\varphi(u)&\ge (\frac{1}{2}-c_5\theta(\|u^0\|))\|u^+\|^2-c_5\|u^+\| \\
&\quad+\big[\theta(\|u^0\|)\|u^0\|\big]^2\Big(-2c_5^2-\frac{1}{T}\int_0^T |h(t)|dt\frac{\|u^0\|}{[\theta(\|u^0\|)\|u^0\|]^2} \\
&\quad+\frac{1}{[\theta(\|u^0\|)\|u^0\|]^2}\int_0^T H(t,u^0)dt\Big).
\end{aligned}$$

Since $\theta(s) \to 0$ as $s \to \infty$, there exists $r > 0$ such that $c_5\theta(s) \leq \frac{1}{4}$ for $s \geq r$. Then, if $\|u^0\| \geq r$, we have

$$\begin{aligned}\varphi(u) \geq \frac{1}{4}\|u^+\|^2 - c_5\|u^+\| + [\theta(\|u^0\|)\|u^0\|]^2(-2c_5^2 \\ - \frac{1}{T}\int_0^T |h(t)|dt \frac{\|u^0\|}{[\theta(\|u^0\|)\|u^0\|]^2} + \frac{1}{[\theta(\|u^0\|)\|u^0\|]^2}\int_0^T H(t, u^0)dt).\end{aligned}$$

then $\varphi(u) \to +\infty$ as $\|u^0 + u^+\| \to \infty$, $\|u^0\| \geq r$.

If $\|u^0\| \leq r$, we have by (H8) and (2.1)

$$\varphi(u) \geq \|u^+\|^2 + \int_0^T f(t)dt - \frac{r}{T}\int_0^T |h(t)|dt - \lambda_2\|h\|_{L^2}\|u^+\|$$

then $\varphi(u) \to +\infty$ as $\|u^0 + u^+\| \to \infty$, $\|u^0\| \leq r$. Therefore $\varphi(u) \to +\infty$ as $\|u\| \to \infty$, $u \in E^2$.

In E^1, as in [15], we obtain $\varphi(u) \to -\infty$ as $\|u\| \to \infty$. Hence, by Lemma 2.1, φ possesses at least a critical point u which is a T-periodic solution of (4.1).

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