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HIGH ENERGY SOLUTIONS TO $p(x)$-LAPLACIAN EQUATIONS OF SCHRÖDINGER TYPE

XIAOYAN WANG, JINGHUA YAO, DUCHAO LIU

Abstract. In this article, we study nonlinear Schrödinger type equations in $\mathbb{R}^N$ under the framework of variable exponent spaces. We proposed new assumptions on the nonlinear term to yield bounded Palais-Smale sequences and then prove that the special sequences we found converge to critical points respectively. The main arguments are based on the geometry supplied by Fountain Theorem. Consequently, we showed that the equation under investigation admits a sequence of weak solutions with high energies.

1. Introduction

In recent years, there has been increasing interests in nonlinear partial differential equations with nonstandard variable growth. In this article, inspired by Fan [15, 16] and Jeanjean [29], we study the following nonlinear Schrödinger type equation on the whole space $\mathbb{R}^N$:

$$\begin{gathered} -\operatorname{div}(|Du|^{p(x)-2}Du) + V(x)|u|^{p(x)-2}u = f(x,u), x \in \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $\operatorname{div}(|Du|^{p(x)-2}Du)$ is called the $p(x)$-Laplacian and $V(x)$ satisfies the following condition.

- (V1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$ where V_0 is a constant, and for every constant $M > 0$, the Lebesgue measure of the set $\{x \in \mathbb{R}^N; V(x) \leq M\}$ is finite.

The equations involving the $p(x)$-Laplacian (also called $p(x)$-Laplacian equations) arise in the modeling of electrorheological fluids (see [2, 7, 40] and [36]) and image restorations among many other problems in physics and engineering. A number of classical equations, for example the classical fluid equations, are also studied in this general framework (see the new monograph [9] and the references therein). Different from the Laplacian $\Delta := \sum_j \partial_j^2$ (linear and homogeneous) and the p-Laplacian $\Delta_p u(x) := \operatorname{div}(|Du|^{p-2}Du)$ (nonlinear but homonegeous) where $0 < p < \infty$ is a positive number, the $p(x)$-Laplacian is nonlinear and nonhomogeneous. Consequently, the problems involving $p(x)$-Laplacian are usually much

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harder than those involving Laplacian or p-Laplacian from this point of view. Besides the applications we mentioned at the beginning of this paragraph, the $p(x)$-Laplacian equations can be regarded as a nonlinear and nonhomogeneous mathematical generalization of the stationary Schrödinger equation $\mathcal{H}u(x) = 0$ where the Hamiltonian is usually given by $\mathcal{H} := -\frac{\hbar^2}{2m}\Delta + V(x)$. For these connections and potential further generalizations, see [4, 6, 41].

To proceed, we recall the definitions of variable exponent spaces in order to describe our problem precisely.

Let Ω be an open domain in $\mathbb{R}^N$ and denote:

$$C_+(\Omega) := \{p(x) \in C(\Omega) : 1 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \sup_{x\in\Omega} p(x) < \infty\}.$$

For $p(x) \in C_+(\Omega)$, we consider the set:

$$L^{p(x)}(\Omega) = \{u : u \text{ is real-valued measurable function}, \int_\Omega |u|^{p(x)}\,dx < \infty\}.$$

We introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x),\Omega} := \inf\{k > 0 : \int_\Omega |\frac{u}{k}|^{p(x)}\,dx \le 1\},$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x),\Omega})$ is a Banach Space and we call it a variable exponent Lebesgue space.

Consequently, $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}; |Du| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{p(x),\Omega} = \inf\{k > 0; \int_\Omega |\frac{Du}{k}|^{p(x)} + |\frac{u}{k}|^{p(x)}\,dx \le 1\}.$$

Then $(W^{1,p(x)}\Omega, \|\cdot\|_{p(x),\Omega})$ also becomes a Banach space and we call it a variable exponent Sobolev space.

For any function $V(x)$ satisfying condition (V1), let

$$E := \{u \in W^{1,p(x)}(\mathbb{R}^N); \int_{\mathbb{R}^N} V(x)|u|^{p(x)}\,dx < \infty\}.$$

Then E is a Banach space with the following norm

$$\|u\| = \inf\{k > 0; \int_{\mathbb{R}^N} |\frac{Du}{k}|^{p(x)} + V(x)|\frac{u}{k}|^{p(x)}\,dx \le 1\}.$$

Of course, our working space is E. Under proper assumptions, we shall show that (1.1) has a sequence of high energy solutions $\{u_n\}$ in E in this paper (Theorem 2.2).

In the previous two decades, there have been many studies on variable exponent spaces; ssee [1, 2, 7, 10, 11], [12]-[23], [30], [40], [48]-[50]). These kinds of spaces are extensions of the usual Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{m,p}(\Omega)$ where $1 \le p < \infty$ is a constant. They are special Orlicz spaces (see [26]). A lot of mathematical work has been done under the framework of the variable exponent spaces (see [1, 5, 14, 36, 38, 45]). Meanwhile, a number of typical and interesting problems have come into light (see [5, 8, 13, 18, 23, 27, 28, 37, 38, 42]). For example, local conditions on the exponent $p(x)$ can assure the multiplicity of solutions to $p(x)$-Laplacian equation; see [45].

There is no doubt that there are mainly two characteristics when we work with variable exponent spaces. On the one hand, these spaces are more complicated than the usual spaces [3, 11, 20, 30]. As a result, the related problems are more difficult. On the other hand, we will obtain more general results if we work under the framework of the variable exponent spaces because there spaces are natural generalizations of the usual Sobolev and Lebesgue spaces.

Fan [15] considered a constrained minimization problem involving $p(x)$-Laplacian in $\mathbb{R}^N$. Under periodic assumptions, the author could elaborately deal with this unbounded problem by concentration-compactness principle of Lions [31, 32, 33, 34]. In a following paper, Fan [16] considered $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and non-periodic perturbations. Under proper conditions, the author was able to show the existence of solutions and gave a concise description of the ground sate solutions. It is worth noting that the periodicity assumptions are essential for the validity of concentration-compactness principle under the framework of variable exponent spaces (see the recent paper of Bonder and coworkers [24, 25] for the concentration-compactness theory in the variable exponent space framework involving critical exponents). In our paper, we also consider an unbounded problem. However, under condition (V1), we could get some compact embedding theorems. In fact, other tricks can be used to recover some kinds of compactness. For example, weight function method was used in [12]. In [46], we considered a combined effect of the symmetry of the space and the coerciveness of potential $V(x)$.

We also want to mention the celebrated paper of Jeanjean [29]. In this paper, the author illustrated a completely new idea to guarantee bounded (PS) sequences for a given C^1 functional. Roughly speaking, we could consider a family of functionals which contains the original one we are interested in. When given additional structure assumptions, almost all the functional in the family have bounded (PS) sequences if the family of functionals enjoy specific geometry properties. In fact, the information of relevant functionals in the family can provide useful information for the original functional. Under our conditions (see Section 2), we could show that the functional we consider satisfies the fountain geometry. Then following Jeanjean's idea and [51, Theorem 3.6], we could show that equation (1.1) has a sequence of high energy solutions. We want to emphasize that our condition (C4) is somewhat mild and is first used in dealing with $p(x)$-Laplacian equations. In addition, we do not need the usual Ambrosetti-Rabinowits type condition here.

For the reader's convenience, we recall some basic properties of the variable exponent spaces and nonlinear functionals defined on these spaces in the following part of this section.

Proposition 1.1 ([20, 21]). *$L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ are both separable, reflexive and uniformly convex Banach Spaces.*

Proposition 1.2 ([20, 21]). *Let $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$ for $u \in L^{p(x)}(\Omega)$, then we have*

1. $|u|_{p(x),\Omega} = 1 \Leftrightarrow \rho(u) = 1$;
2. $|u|_{p(x),\Omega} \le 1 \Rightarrow |u|^{p^+}_{p(x),\Omega} \le \rho(u) \le |u|^{p^-}_{p(x),\Omega}$;
3. $|u|_{p(x),\Omega} \ge 1 \Rightarrow |u|^{p^-}_{p(x),\Omega} \le \rho(u) \le |u|^{p^+}_{p(x),\Omega}$;
4. *For* $u_n \in L^{p(x)}(\Omega), \rho(u_n) \to 0 \Leftrightarrow |u_n|_{p(x),\Omega} \to 0$ *as* $n \to \infty$;
5. *For* $u_n \in L^{p(x)}(\Omega), \rho(u_n) \to \infty \Leftrightarrow |u_n|_{p(x),\Omega} \to \infty$ *as* $n \to \infty$.

Proposition 1.3 ([20, 21, 39]). *Let $\rho(u) = \int_\Omega |Du(x)|^{p(x)} + |u(x)|^{p(x)}\,dx$ for $u \in W^{1,p(x)}(\Omega)$. Then we have*

1. $\|u\|_{p(x),\Omega} = 1 \Leftrightarrow \rho(u) = 1$;
2. $\|u\|_{p(x),\Omega} \le 1 \Rightarrow \|u\|^{p^+}_{p(x),\Omega} \le \rho(u) \le \|u\|^{p^-}_{p(x),\Omega}$;
3. $\|u\|_{p(x),\Omega} \ge 1 \Rightarrow \|u\|^{p^-}_{p(x),\Omega} \le \rho(u) \le \|u\|^{p^+}_{p(x),\Omega}$;
4. *For* $u_n \in W^{1,p(x)}(\Omega), \rho(u_n) \to 0 \Leftrightarrow \|u_n\|_{p(x),\Omega} \to 0$ *as* $n \to \infty$;
5. *For* $u_n \in W^{1,p(x)}(\Omega), \rho(u_n) \to \infty \Leftrightarrow \|u_n\|_{p(x),\Omega} \to \infty$ *as* $n \to \infty$.

The following property can be easily verified:

Proposition 1.4. *For $u \in E$, let $\rho(u) = \int_{\mathbb{R}^N} |Du(x)|^{p(x)} + V(x)|u(x)|^{p(x)}\,dx$. Then we have the following relations:*

1. $\|u\| = 1 \Leftrightarrow \rho(u) = 1$;
2. $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$;
3. $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$.

From the above-mentioned properties, we can see that the norm and the integral functionals (i.e., the $\rho(u)'s$) don't enjoy the equality relation, which is typical in variable exponent spaces and very different from the constant exponent case.

Notation. For $p(x) \in C_+(\Omega), p^*(x)$ refers to the critical exponent of $p(x)$ in the sense of Sobolev embedding, i.e., $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N; p^*(x) = \infty$, otherwise. For two continuous functions $a(x)$ and $b(x)$ in $C(\Omega)$, $a(x) \ll b(x)$ means that $\inf_{x\in\Omega}(b(x) - a(x)) > 0$. We will use the symbols "$\rightharpoonup$", "$\to$" to represent weak convergence and strong convergence in a Banach space respectively. And "$\hookrightarrow$", "$\hookrightarrow\hookrightarrow$" will be used to denote continuous embedding and compact embedding between spaces respectively. We use C to denote a generic positive constant which may be different from line to line.

Proposition 1.5 ([20, 21, 45]).
1. *Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that the boundary $\partial\Omega$ possesses cone property and $q(x) \in C(\overline{\Omega}, R)$ with $1 \le q(x) \ll p^*(x)$, then $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$*
2. *$W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ if $p^+ < N$ and $q(x) \in C_+(\mathbb{R}^N)$ satisfies $p(x) \le q(x) \ll p^*(x)$.*

Following the spirit of [21], we have the following proposition.

Proposition 1.6. *For $u \in E$, we define*

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)})\,dx,$$

then $I \in C^1(E, \mathbb{R})$ and the derivative operator L of I is

$$\langle L(u), v\rangle = \int_{\mathbb{R}^N} (|Du|^{p(x)-2}Du \cdot Dv + V(x)|u|^{p(x)-2}uv)\,dx, \quad \forall u, v \in E,$$

and we have:

1. *$L : E \to E^*$ (the dual space of E) is a continuous, bounded and strictly monotone operator;*
2. *L is a mapping of type (S_+), i.e. if $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty}\langle L(u_n) - L(u), u_n - u\rangle \le 0$, then $u_n \to u$ in E;*
3. *$L : E \to E^*$ is a homeomorphism.*

Proposition 1.7 ([20, 21, 45]). *Let Ω be a bounded domain in $\mathbb{R}^N$. If $f(x,t)$ is a Carathéodory function and satisfies*

$$|f(x,t)| \le a(x) + b|t|^{\frac{p_1(x)}{p_2(x)}}, quad \forall x \in \overline{\Omega}, t \in \mathbb{R}^1$$

where $p_1(x), p_2(x) \in C_+(\Omega), b \ge 0$ is a constant, $0 \le a(x) \in L^{p_2(x)}(\Omega)$, then the superposition operator S from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(Su)(x) = f(x,u(x))$ is a continuous and bounded operator. Moreover, if Ω is unbounded (e.g., $\Omega = \mathbb{R}^N$) and $a(x) \equiv 0$, the same conclusion is true.

In the variable Lebesgue space case, Hölder type inequality still holds.

Proposition 1.8 ([17]). *Let Ω be a domain in $\mathbb{R}^N$ (either bounded or unbounded) and $u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$ where $p'(x) := \frac{p(x)}{p(x)-1}$ is the conjugate exponent of $p(x) \in C_+(\Omega)$. Then the following Hölder type inequality holds*

$$\int_\Omega |uv|\, dx \le (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{p(x),\Omega}|v|_{p'(x),\Omega}.$$

We will use this inequality in the following sections .

This article is divided into three sections. For the readers' convenience, we have recalled some basic properties of the variable exponent spaces $W^{1,p(x)}(\Omega), L^{p(x)}(\Omega)$ in this section. In Section 2, we will state our assumptions on the nonlinear term and our main result. Meanwhile, we shall prove some useful auxiliary results in this section. In our opinion, these results are interesting and important when we study variable exponent problems. In Sections 3, we are devoted to proving the main result.

2. Main result

In this section, we first specify our assumptions on the nonlinear term f. Then some comments about these assumptions will be given. Finally, we state the main result.

We use the following assumptions:

(C1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies

$$|f(x,t)| \le C(|t|^{p(x)-1} + |t|^{q(x)-1}), \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N,$$
$$f(x,t)t \ge 0, \quad \text{for } t \ge 0, x \in \mathbb{R}^N,$$
$$p(x) \le q(x) \ll p^*(x), \quad \forall x \in \mathbb{R}^N.$$

(C2) There exists a constant $\mu > p^+$ such that

$$\liminf_{|t|\to\infty} \frac{f(x,t)t}{|t|^\mu} \ge C_0 \quad \text{uniformly for } x \in \mathbb{R}^N.$$

where C_0 is a positive constant.

(C3) $\limsup_{|t|\to 0} \frac{f(x,t)t}{|t|^{p^+}} = 0$, uniformly for $x \in \mathbb{R}^N$.

(C4) Let $F(x,t) = \int_0^t f(x,s)ds$ and G, F be defined as

$$G(x,t) := f(x,t)t - p^- F(x,t), \quad H(x,t) := f(x,t)t - p^+ F(x,t).$$

We assume G and H satisfy the monotonicity condition: there exist two positive constants D_1 and D_2 such that

$$G(x,t) \le D_1 G(x,s) \le D_2 H(x,s), \quad \text{for } 0 \le t \le s.$$

(C5) $f(x,-t) = -f(x,t), \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^N$.

Definition 2.1. We say $u \in E$ is a weak solution to the equation (1.1) if for any $v \in E$,

$$\int_{\mathbb{R}^N} |Du|^{p(x)-2} DuDv + V(x)|u|^{p(x)-2}uv\,dx = \int_{\mathbb{R}^N} f(x,u)v\,dx.$$

Define a functional Φ from E to $\mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

Under our assumptions, we know that the functional is C^1 (Proposition 1.6, Lemma 2.7 below) and for $v \in E$,

$$\Phi'(u)v = \int_{\mathbb{R}^N} |Du|^{p(x)-2} DuDv + V(x)|u|^{p(x)-2}uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx.$$

So the critical points of the functional Φ are corresponding to the weak solutions of the equation (1.1).

Now we are in a position to comment and analyze the assumptions proposed above.

1. Conditions (C1)-(C4) are compatible. We shall give two examples to demonstrate this claim. Let $f(x,t) = |t|^{q(x)-2}t$ with $q(x) \in C_+(\mathbb{R}^N)$ satisfying $q(x) \ll p^*(x), q_- > p^+$. Obviously, (C1), (C2), (C3), (C5) hold. In order to verify (C4), we know that $F(x,t) = \frac{|t|^{q(x)}}{q(x)}$, $f(x,t)t = |t|^{q(x)}$. Consequently, $G(x,t) = (1 - \frac{p^-}{q(x)})|t|^{q(x)}$, $H(x,t) = (1 - \frac{p^+}{q(x)})|t|^{q(x)}$. It is easy to verify that $G(x,t)$ is nondecreasing in $t \geq 0$. Therefore, $G(x,t) \leq G(x,s)$ if $0 \leq t \leq s$. In view of $G, H \geq 0$, we know that

$$\frac{G(x,s)}{H(x,s)} = \frac{q(x) - p^-}{q(x) - p^+} \leq \frac{q^+ - p^-}{q^- - p^+}.$$

Choosing $D_2 = \frac{q^+ - p^-}{q^- - p^+}$, we obtain $G(x,s) \leq D_2 H(x,s)$ when $s \geq 0$. Therefore, (C4) holds.

Next, we illustrate another example. Let $f(x,t) = |t|^{q(x)-2}t \ln^a(|t|+1)$ where $q(x)$ satisfies $q(x) \ll p^*(x)$, $q^- > p^+$ and $\epsilon > a > 0$ is a real number. In view of the following two relations:

$$\lim_{|t| \to \infty} \frac{\ln^a(|t|+1)}{|t|^\epsilon} = 0 \quad \forall a \geq 0, \epsilon > 0;$$
$$\lim_{|t| \to 0} \frac{\ln^a(|t|+1)}{|t|^\epsilon} = \infty \quad \forall a \geq 0, \epsilon > 0.$$

we can verify (C4) similarly. Obviously, (C1), (C2), (C3), (C5) hold.

From the two examples we gave, we know that there are many functions which satisfy our assumptions. As a result, our main result is quite general.

2. Condition (C1) means that $f(x,t)$ is subcritical in the variable sense. Different from things in constant case (i.e. $p^+ = p^-$), here we need $q(x) \ll p^*(x)$.

3. Condition (C4) is crucial for our proof. It is because of this condition that we could obtain bounded Palais-Smale sequence (bounded (PS) sequences for short). We impose this condition on f other than the famous Ambrosetti-Rabinowitz type condition. However, we could still get bounded (PS) sequences via an indirect method. Lots of authors have tried to weaken the Ambrosetti-Rabinowits type

condition and they can only get weak type (PS) sequences (usually the Cerami Condition). It is known that (C5) is much weaker than the Ambrosetti-Rabinowitz type condition in the constant exponent case ($p^+ = p^-$) (see [26]).

4. Condition $(C5)$ assures that the functional Φ we defined before is an even functional. So the condition is necessary for us to take advantage of the fountain geometry.

In this article, we always assume condition (V1) holds and $p^+ < N$. Hence, we know $E \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$. Consequently, $E \hookrightarrow L^{p(x)}(\mathbb{R}^N), E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ if $q(x) \in C_+(\mathbb{R}^N)$ satisfies $p(x) \leq q(x) \ll p^*(x)$.

Now we can state our main result clearly.

Theorem 2.2. *Under conditions* (V1), (C1)–(C5)*, equation* (1.1) *has a sequence of solutions $\{u_n\}$. Moreover, these solutions have high energies; i.e., $\Phi(u_n) \to \infty$ as $n \to \infty$.*

To make the exposition more concise, we give some auxiliary results some of which are very useful.

Lemma 2.3. *Let Ω be a nonempty domain in $\mathbb{R}^N$ which can be bounded or unbounded. We also allow $\Omega = \mathbb{R}^N$. Then*

$$L^{p(x)}(\Omega) \cap L^{q(x)}(\Omega) \subset L^{a(x)}(\Omega)$$

if $p(x), q(x), a(x) \in C_+(\Omega)$ and $p(x) \leq a(x) \leq q(x)$. Moreover, if $p(x) \ll a(x) \ll q(x)$, the following interpolation inequality holds for $u \in L^{p(x)}(\Omega) \cap L^{q(x)}(\Omega)$:

$$\int_\Omega |u|^{a(x)}\,dx \leq 2||u|^{a_1(x)}|_{m(x),\Omega}||u|^{a_2(x)}|_{m'(x),\Omega}, \tag{2.1}$$

where

$$a_1(x) = \frac{p(x)(q(x)-a(x))}{q(x)-p(x)}, \quad a_2(x) = \frac{q(x)(a(x)-p(x))}{q(x)-p(x)};$$
$$m(x) = \frac{q(x)-p(x)}{q(x)-a(x)}, \quad m'(x) = \frac{q(x)-p(x)}{a(x)-p(x)}.$$

Sketch of the proof. For $L^{p(x)}(\Omega) \cap L^{q(x)}(\Omega)$, we have

$$\int_\Omega |u|^{p(x)}\,dx < \infty, \quad \int_\Omega |u|^{q(x)}\,dx < \infty.$$

Obviously, $|u(x)|^{a(x)} \leq |u(x)|^{p(x)} + |u(x)|^{q(x)}$ for $x \in \Omega$. Hence, $\int_\Omega |u|^{a(x)} \leq \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |u|^{q(x)}\,dx < \infty$, which means $u \in L^{a(x)}(\Omega)$. For the interpolation inequality, the readers can see [20]. □

Lemma 2.4. *Under condition* (V1)*, $E \hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$.*

Proof. We know that $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$. Next, we assume $u_n \rightharpoonup 0$ in E. We need to show $u_n \to 0$ in $L^{p(x)}(\mathbb{R}^N)$ to complete the proof. By Proposition 1.2, it suffices to verify that $\int_{\mathbb{R}^N} |u_n|^{p(x)}\,dx \to 0$ as $n \to \infty$. For any given $R > 0$, we write

$$\begin{aligned} I(n) &:= \int_{\mathbb{R}^N} |u_n|^{p(x)}\,dx \\ &= \int_{B(0,R)} |u_n|^{p(x)}\,dx + \int_{\mathbb{R}^N \setminus B(0,R)} |u_n|^{p(x)}\,dx := I_1(n) + I_2(n). \end{aligned}$$

Since $E \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(B(0,R)) \hookrightarrow\hookrightarrow L^{p(x)}(B(0,R))$, it follows that $I_1(n) \to 0$ as $n \to \infty$.

For any constant $M > 0$, Let $A = \{x \in R^N \backslash B(0,R); V(x) > M\}$ and $B = \{x \in \mathbb{R}^N \backslash B(0,R); V(x) \leq M\}$. Then we have

$$\int_A |u_n|^{p(x)}\,dx \leq \int_A \frac{V(x)}{M}|u_n|^{p(x)}\,dx \leq \frac{1}{M}\int_{\mathbb{R}^N} V(x)|u_n|^{p(x)}\,dx \leq \frac{C}{M}.$$

Since for the constant $M > 0, mes\{x \in \mathbb{R}^N; V(x) \leq M\}$ is finite, we can choose $R > 0$ large enough such that meas$\{x \in \mathbb{R}^N \backslash B(0,R); V(x) \leq M\} \to 0$. Consequently, $\int_B |u_n|^{p(x)} \to 0$.

Now Let $M \to \infty$ and $R \to \infty$, we have $I(n) \to 0$ as $n \to \infty$. □

Lemma 2.5. *Under condition* (V1), $E \hookrightarrow\hookrightarrow L^{a(x)}(\mathbb{R}^N)$ *if* $a(x) \in C_+(\mathbb{R}^N)$ *and* $p(x) \leq a(x) \ll p^*(x)$.

Proof. Let $u_n \rightharpoonup 0$ in E. We need to show $u_n \to 0$ in $L^{a(x)}(\mathbb{R}^N)$ to complete the proof.

First, we assume that $p(x) \ll a(x) \ll p^*(x)$. We can choose $q(x) \in C_+(\mathbb{R}^N)$ such that $a(x) \ll q(x) \ll p^*(x)$. It is obvious that $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$. In view of $p(x) \ll a(x) \ll q(x)$, we use Lemma 2.3 with $\Omega = \mathbb{R}^N$ and obtain

$$\int_\Omega |u_n|^{a(x)}\,dx \leq 2||u_n|^{a_1(x)}|_{m(x),\Omega}||u_n|^{a_2(x)}|_{m'(x),\Omega}, \tag{2.2}$$

where the symbols are the same as those of Lemma 2.3.

Let $\lambda_n := ||u_n|^{a_1(x)}|_{m(x),\Omega}$ and $\mu_n := ||u_n|^{a_2(x)}|_{m'(x),\Omega}$. By Proposition 1.2, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\frac{|u_n|^{a_1(x)}}{\lambda_n}|^{m(x)}\,dx &= \int_{\mathbb{R}^N} \frac{|u_n|^{p(x)}}{\lambda_n^{m(x)}}\,dx = 1;\\ \int_{\mathbb{R}^N} |\frac{|u_n|^{a_2(x)}}{\mu_n}|^{m'(x)}\,dx &= \int_{\mathbb{R}^N} \frac{|u_n|^{q(x)}}{\mu_n^{m'(x)}}\,dx = 1.\end{aligned}$$

From the two equalities above and Lemma 2.4, we know

$$\begin{gathered}\min\{\lambda_n^{m^+}, \lambda_n^{m^-}\} \leq \int_{\mathbb{R}^N} |u_n|^{p(x)}\,dx \to 0,\\ \min\{\mu_n^{m'^+}, \mu_n^{m'^-}\} \leq \int_{\mathbb{R}^N} |u_n|^{q(x)}\,dx \leq C.\end{gathered}$$

We have $\lambda_n \to 0$ as $n \to \infty$ and $0 \leq \mu_n \leq C$. So (2.2) yields $\int_{\mathbb{R}^N} |u_n|^{a(x)}\,dx \to 0$ as $n \to \infty$.

Next, we assume $p(x) \leq a(x) \ll p^*(x)$. We can choose $q(x) \in C_+(\mathbb{R}^N)$ such that $a(x) \ll q(x) \ll p^*(x)$. By the arguments above, we have

$$\int_{\mathbb{R}^N} |u_n|^{q(x)}\,dx \to 0.$$

By Lemma 2.3 and Lemma 2.4, we have

$$\int_{\mathbb{R}^N} |u_n|^{a(x)}\,dx \leq \int_{\mathbb{R}^N} |u_n|^{p(x)}\,dx + \int_{\mathbb{R}^N} |u_n|^{q(x)}\,dx \to 0.$$

□

The following lemma can be considered as an extension of the result in [44, Appendix A].

Lemma 2.6. *Assume* $1 \le p_1(x),\ p_2(x), q_1(x), q_2(x) \in C(\Omega)$. *Let* $f(x,t)$ *be a Carathéodory function on* $\Omega\times\mathbb{R}$ *and satisfy*

$$|f(x,t)| \le a|t|^{\frac{p_1(x)}{q_1(x)}} + b|t|^{\frac{p_2(x)}{q_2(x)}}, \quad (x,t)\in\Omega\times\mathbb{R},$$

where $a,b>0$ *and* Ω *is either bounded or unbounded. Define a Carathéodory operator by*

$$Bu := f(x,u(x)), \quad u\in\mathscr{H} := L^{p_1(x)}(\Omega)\cap L^{p_2(x)}(\Omega)$$

Define the space $\mathscr{E} := L^{q_1(x)}(\Omega) + L^{q_2(x)}(\Omega)$ *with the norm*

$$\|u\|_{\mathscr{E}} = \inf\{|v|_{q_1(x),\Omega} + |w|_{q_2(x),\Omega} : u = v+w, v\in L^{q_1(x)}(\Omega), w\in L^{q_2(x)}(\Omega)\}.$$

If $\frac{p_1(x)}{q_1(x)} \le \frac{p_2(x)}{q_2(x)}$ *for* $x\in\Omega$, *then* $B = B_1 + B_2$, *where* B_i *is a bounded and continuous mapping from* $L^{p_i(x)}(\Omega)$ *to* $L^{q_i(x)}(\Omega)$, $i=1,2$. *In particular,* B *is a bounded continuous mapping from* $\mathcal{H}$ *to* $\mathscr{E}$.

Proof. Let $\psi:\mathbb{R}\to[0,1]$ be a smooth function such that $\psi(t)=1$ for $t\in(-1,1)$; $\psi(t)=0$ for $t\notin(-2,2)$. Let

$$g(x,t) = \psi(t)f(x,t), h(x,t) = (1-\psi(t))f(x,t).$$

Because $\frac{p_1(x)}{q_1(x)} \le \frac{p_2(x)}{q_2(x)}$ for $x\in\Omega$, there are two constants $d>0, m>0$ such that

$$|g(x,t)| \le d|t|^{\frac{p_1(x)}{q_1(x)}}, |h(x,t)| \le m|t|^{\frac{p_2(x)}{q_2(x)}}.$$

Define

$$B_1u = g(x,u), u\in L^{p_1(x)}(\Omega), B_2u = h(x,u), u\in L^{p_2(x)}(\Omega).$$

Then by Proposition 1.7, B_i is a bounded and continuous mapping from $L^{p_i(x)}(\Omega)$ to $L^{q_i(x)}(\Omega)$, $i=1,2$. It is readily to see that $B := B_1+B_2$ is a bounded continuous mapping from $\mathcal{H}$ to $\mathscr{E}$. ☐

From Lemmas 2.4 and 2.5, we know that the condition (V1) plays an important role. It enables E to be compactly embedded into $L^{p(x)}(\mathbb{R}^N)$ type spaces. Using Lemmas 2.5 and 2.6, we can prove the following result.

Lemma 2.7. *Under assumptions* (V1), (C1), *the functional* $J(u) = \int_{\mathbb{R}^N} F(x,u)\,dx$ *on* E *is a* C^1 *functional. Moreover,* J' *is compact.*

Proof. The verification that J is a C^1 functional is routine and we omit it here. We only show that J' is compact. Because $E \hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$ (Lemma 2.4) and $E \hookrightarrow\hookrightarrow L^{q(x)}(\mathbb{R}^N)$ (Lemma 2.5), any bounded sequence $\{u_k\}$ in E has a renamed subsequence still denoted by $\{u_k\}$ which converges to u_0 in $L^{p(x)}(\mathbb{R}^N)$ and $L^{q(x)}(\mathbb{R}^N)$. Using Lemma 2.6 with $p_1(x)=p(x)$, $q_1(x)=\frac{p(x)}{p(x)-1}$, $p_2(x)=q(x)$, $q_2(x)=\frac{q(x)}{q(x)-1}$ and $\Omega=\mathbb{R}^N$, we have $J'(u)v = \int_{R^N}(B_1u+B_2u)v\,dx$ for $v\in E$. Hence, $B_1(u_k)\to B_1(u_0)$ in $L^{q_1(x)}(\Omega)$ and $B_2(u_k)\to B_2(u_0)$ in $L^{q_2(x)}(\Omega)$. Then Hölder type inequality (Proposition 1.8) and Sobolev embedding (Lemma 2.5) assure $J'(u_k)\to J'(u_0)$ in E^*, i.e., J' is compact. ☐

For convenience, we give the definition of $(PS)_c$ sequence for $c\in\mathbb{R}$.

Definition 2.8. Let Π be a C^1 functional defined on a real Banach space X. Any sequence $\{u_n\}$ satisfying $\Pi(u_n) \to c$ and $\Pi'(u_n) \to 0$ is called a $(PS)_c$ sequence. In addition, we call c here a prospective critical level of Π.

Remark 2.9 (See [17])**.** Under the assumptions of Theorem 2.2, we have the following comments. $\Phi(u) = I(u) + J(u)$ and $\Phi'(u) = I'(u) + J'(u)$ for $u \in E$. Since I' is of type (S_+) (Proposition 1.6) and J' is a compact (Lemma 2.7), we can easily derive that Φ' is of type (S_+). It is well-known that any bounded $(PS)_c$ sequence of a functional whose Fréchet derivative is of type (S_+) in a reflexive Banach space has a convergent subsequence and so does Φ here.

3. Proof of Theorem 2.2

We state the Fountain Theorem, before presenting the proof of the main result. Let X be a Banach space with the norm $\|\cdot\|$ and let $\{X_j\}$ be a sequence of subspaces of X with $\dim X_j < \infty$ for each $j \in \mathbb{N}$. Further, $X = \overline{\oplus_{j=1}^{\infty} X_j}, W_k := \oplus_{j=1}^{k} X_j, Z_k := \overline{\oplus_{j=k}^{\infty} X_j}$. Moreover, for $k \in \mathbb{N}$ and $\rho_k > r_k > 0$, we denote:

$$B_k = \{u \in W_k : \|u\| \le \rho_k\}; \quad S_k = \{u \in Z_k : \|u\| = r_k\};$$
$$c_k := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi(\gamma(u)), \quad \text{where}$$
$$\Gamma_k := \{\gamma \in C(B_k, X) : \gamma \text{ is odd and } \gamma|_{\partial B_k} = id\}.$$

Theorem 3.1 (Fountain Theorem, Bartsch 1992 [34])**.** *Under the aforementioned assumptions, let $\Phi \in C^1(X, R)$ be an even functional. If for $k > 0$ large enough, there exists $\rho_k > r_k > 0$ such that*

$$a_k := \max\{\Phi(u) : u \in W_k, \|u\| = \rho_k\} \le 0, \tag{3.1}$$

$$b_k := \inf\{\Phi(u) : u \in Z_k, \|u\| = r_k\} \to \infty \quad \text{as } k \to \infty. \tag{3.2}$$

then Φ has a $(PS)_{c_k}$ sequence for each prospective critical value c_k and $c_k \to \infty$ as $k \to \infty$.

Definition 3.2. Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The functional Φ satisfies the $(PS)_c$ condition if any sequence $\{u_k\} \subset X$ such that

$$\Phi(u_n) \to c, \quad \Phi'(u_n) \to 0 \tag{3.3}$$

has a convergent subsequence.

Remark 3.3. In fact, if the following condition holds

(C) Φ satisfies the $(PS)_c$ condition for every $c > 0$,

the sequence $\{c_k\}$ in Theorem 3.1 is a sequence of unbounded critical values of Φ. However, the condition (C) is not necessary to guarantee that c_k is a critical level. We just need $(PS)_{c_k}$ condition.

To use the decomposition technique, we need a theorem on the structure of a reflexive and separable Banach space.

Lemma 3.4 ([47, Section 17])**.** *Let X be a reflexive and separable Banach space, then there are $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that*

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \ne m \end{cases},$$
$$X = \overline{\operatorname{span}}\{e_n : n = 1, 2, \dots, \}, \quad X^* = \overline{\operatorname{span}}^{W^*}\{f_n : n = 1, 2, \dots, \}.$$

For $k = 1, 2, \dots$, and $X = E$, we choose

$$X_j = \operatorname{span}\{e_j\}, W_k = \oplus_{j=1}^{k} X_j, Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

In the following, we shall identify the Banach space E and the functional Φ as those we consider. Next, we will prove the main result step by step. First, we give a useful lemma. For simplicity, we write $|u|_{p(x),\mathbb{R}^N}$ as $|u|_{p(x)}$ when $\Omega = \mathbb{R}^N$ for $p(x) \in C_+(\mathbb{R}^N)$.

Lemma 3.5. *Let $q(x) \in C_+(\mathbb{R}^N)$ with $p(x) \le q(x) \ll p^*(x)$ and denote*

$$\alpha_k = \sup\{|u|_{q(x)} : \|u\| = 1, u \in Z_k\}, \tag{3.4}$$

then $\alpha_k \to 0$ as $k \to \infty$.

Proof. Obviously, α_k is decreasing as $k \to \infty$. Noting that $\alpha_k \ge 0$, we may assume that $\alpha_k \to \alpha \ge 0$. For every $k > 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $|u_k|_{q(x)} > \frac{\alpha_k}{2}$. By definition of Z_k, $u_k \rightharpoonup 0$ in E. Then Lemma 2.5 implies that $u_k \to 0$ in $L^{q(x)}(\mathbb{R}^N)$. Thus we have proved that $\alpha = 0$. □

Using lemma 3.5, we can prove the following Lemma.

Lemma 3.6. *Under the assumptions of Theorem 3.1, the geometry conditions of the Fountain Theorem hold, i.e.* (3.1) *and* (3.2) *hold.*

Proof. By (C2) and (C3), for any $\epsilon > 0$, there exists a $C(\epsilon) > 0$ such that

$$f(x, u)u \ge C(\epsilon)|u|^{\mu} - \epsilon|u|^{p^+}.$$

In view of (C5), we have a constant, still denoted by $C(\epsilon)$, such that

$$F(x, u) \ge C(\epsilon)|u|^{\mu} - \epsilon|u|^{p^+}.$$

When $\|u\| > 1$, we have

$$\begin{aligned} \Phi(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)})\,dx - \int_{\mathbb{R}^N} F(x, u)\,dx \\ &\le \frac{1}{p^-}\|u\|^{p^+} - C(\epsilon)\int_{\mathbb{R}^N} |u|^{\mu}\,dx + \epsilon\int_{\mathbb{R}^N} |u|^{p^+}\,dx. \end{aligned} \tag{3.5}$$

Let $u \in W_k$, since $\dim(W_k) < \infty$. all norms on W_k are equivalent. Hence $\Phi(u) \le C\|u\|^{p^+} - C\|u\|^{\mu}$. Because $\mu > p^+$, we can choose $\rho_k > 0$ large enough such that $\Phi(u) \le 0$ when $\|u\| = \rho_k$. We have shown that (3.1) holds.

To verify (3.2), we can still let $\|u\| > 1$ without loss of generality. By (C1) and (C3), for any $\epsilon > 0$, there exists a $C = C(\epsilon) > 0$ such that

$$|F(x, u)| \le \epsilon|u|^{p^+} + C|u|^{q(x)},$$

So

$$\begin{aligned} \Phi(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)}\,dx) - \int_{R^N} F(x, u)\,dx \\ &\ge \frac{1}{p^+}\|u\|^{p^-} - \epsilon|u|_{p^+}^{p^+} - C\max\{|u|_{q(x)}^{q^-}, |u|_{q(x)}^{q^+}\}. \end{aligned} \tag{3.6}$$

Let $u \in Z_k$ with $\|u\| = r_k > 0$. We can choose uniformly an $\epsilon > 0$ small enough such that $\epsilon|u|_{p^+}^{p^+} \le \frac{1}{2p^+}\|u\|^{p^-}$. Hence

$$\Phi(u) \ge \frac{1}{2p^+}\|u\|^{p^-} - C\max\{|u|_{q(x)}^{q^-}, |u|_{q(x)}^{q^+}\}.$$

If $\max\{|u|_{q(x)}^{q^-}, |u|_{q(x)}^{q^+}\} = |u|_{q(x)}^{q^-}$, we choose $r_k = (2q^- C\alpha_k^{q^-})^{\frac{1}{p^- - q^-}}$ and get that

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2p^+}\|u\|^{p^-} - C|u|_{q(x)}^{q^-} \geq \frac{1}{2p^+} - C\alpha_k^{p^-}\|u\|^{q^-} \\ &\geq (\frac{1}{2p^+} - \frac{1}{2q^-})r_k^{p^-}.\end{aligned} \tag{3.7}$$

Since $q^- > p^+$ and $\alpha_k \to 0$, we obtain $b_k \to \infty$.

If $\max\{|u|_{q(x)}^{q^-}, |u|_{q(x)}^{q^+}\} = |u|_{q(x)}^{q^+}$, we can similarly derive that $b_k \to \infty$. Hence we have shown (3.2) holds. □

By far, we have shown that the geometry conditions of the Fountain Theorem hold. In fact, in order to use the Fountain Theorem to get our main result, we do not need to verify the functional Φ satisfies the $(PS)_c$ condition for every $c > 0$. It suffices if we could find a special (PS) sequence for each c_k and verify the sequence we find has a convergence subsequence. Of course, the first step is to show that the $(PS)_{c_k}$ sequence is bounded. Because there is no Ambrosetti-Rabinowits type condition, we couldn't give a direct proof. Following the ideas in Jeanjean [29] and Zou [51], we consider Φ as a member in a family of functional. We will show almost all the functional in the family have bounded (PS) sequences. The following result (Theorem 3.7) due to Zou and Schechter [51] is crucial for this purpose.

Let the notions be the same as in Theorem 3.1. Consider a family of real C^1 functional Φ_λ of the form: $\Phi_\lambda(u) := I(u) - \lambda J(u)$, where $\lambda \in \Lambda$ and Λ is a compact interval in $[0, \infty)$. We make the following assumptions:

- (A1) Φ_λ maps bounded sets into bounded sets uniformly for $\lambda \in \Lambda$. Moreover, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in \Lambda \times X$.
- (A2) $J(u) \geq 0$ for all $u \in E; I(u) \to \infty$ or $J(u) \to \infty$ as $\|u\| \to \infty$.

Let

$$a_k(\lambda) := \max\{\Phi_\lambda(u) : u \in W_k, \|u\| = \rho_k\}, \tag{3.8}$$

$$b_k(\lambda) := \inf\{\Phi_\lambda(u) : u \in Z_k, \|u\| = r_k\}. \tag{3.9}$$

Define

$$c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)),$$

$$\Gamma_k := \{\gamma \in C(B_k, X) : \gamma \text{ is odd and } \gamma|_{\partial B_k} = id\}.$$

Theorem 3.7. *Assume that* (A1) *and* (A2) *hold. If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in \Lambda$, then $c_k(\lambda) \geq b_k(\lambda)$ for all $\lambda \in \Lambda$. Moreover, for almost every $\lambda \in \Lambda$, there exists a sequence of $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that* $\sup_n \|u_n^k(\lambda)\| < \infty, \Phi'_\lambda(u_n^k(\lambda)) \to 0$ *and $\Phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda)$ as $n \to \infty$.*

Next, we let $I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)})\,dx, J(u) = \int_{\mathbb{R}^N} F(x,u)\,dx$ for $u \in E$ and $\Lambda = [1, 2]$. Under these terminologies, $\Phi(u) = \Phi_1(u)$. Under the assumptions of Theorem 3.1. It is easy to see that (A1) and (A2) hold.

Lemma 3.8. *Under the assumptions of Theorem 3.1, $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1, 2]$ when k is large enough.*

Sketch of the proof. Let $\rho_k > r_k > 0$ large enough. Using same reasoning, we can show that $a_k(\lambda) \leq 0$ and $b_k(\lambda) \to \infty$ uniformly for $\lambda \in [1, 2]$ as $k \to \infty$. Hence, we have shown the Lemma. Moreover, $c_k(\lambda) \le \sup_{u\in B_k} \Phi_\lambda(u) \le \sup_{u\in B_k} \Phi(u) = \max_{u\in B_k} \Phi_1(u) = \max_{u\in B_k} \Phi(u) := \overline{c_k} < \infty$. $\square$

Remark 3.9. Since $\Phi'_\lambda(u)$ is of type (S_+) (Remark 2.9), we know that any bounded $(PS)_{c(\lambda)}$ sequence of Φ_λ has a convergent subsequence which converges to a critical point of Φ_λ with critical level $c(\lambda)$.

Now, applying Theorem 3.7, we obtain that for almost every $\lambda \in [1, 2]$, there exists a sequence of $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that $\sup_n \|u_n^k(\lambda)\| < \infty, \Phi'_\lambda(u_n^k(\lambda)) \to 0$ and $\Phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda)$ as $n \to \infty$. Denote the set of these λ by Λ_0. If $1 \in \Lambda_0$, we have found bounded $(PS)_{c_k}$ sequence for the functional Φ.

If $1 \notin \Lambda_0$, we can choose a sequence $\{\lambda_n\} \subset \Lambda_0$ such that $\lambda_n \to 1$ decreasingly. In view of Note 3.9, for each $\lambda \in \Lambda_0$, the bounded $(PS)_{c_k(\lambda)}$ sequence has a convergent subsequence. We denote the limit by $u^k(\lambda)$. Accordingly, $u^k(\lambda)$ is the critical point of the functional Φ_λ with critical level $c_k(\lambda)$. Next, we are going to show the sequence $\{u^k(\lambda_n)\}_{n=1}^\infty$ is a bounded $(PS)_{c_k}$ sequence of Φ. For simplicity, we write $\{u^k(\lambda_n)\}$ as $\{u(\lambda_n)\}$.

In fact, we only need to show $\{u(\lambda_n)\}$ is bounded. Indeed, if $\{u(\lambda)\}$ is bounded, we have

$$\begin{aligned}
\Phi(u(\lambda_n)) &= \Phi_{\lambda_n}(u(\lambda_n)) + (1-\lambda_n)J(u(\lambda_n)) \to c_k, \\
\Phi'(u(\lambda_n)) &= \Phi'_{\lambda_n}(u(\lambda_n)) + (1-\lambda_n)J'(u(\lambda_n)) \to 0.
\end{aligned}$$

We have used the fact that Φ_λ, J map bounded sets into bounded sets under the assumptions of Theorem 2.2.

Lemma 3.10. *Under the assumption of Theorem 2.2, the sequence $\{u(\lambda_n)\}$ is bounded.*

Proof. By contradiction. We assume $\|u(\lambda_n)\| \to \infty$ and consider $w_n = \frac{u(\lambda_n)}{\|u(\lambda_n)\|}$. Then up to a subsequence, we get that $w_n \rightharpoonup w$ in $E, w_n \to w$ in $L^{q(x)}(\mathbb{R}^N)$ for $p(x) \le q(x) \ll p^*(x), w_n \to w$ a.e. in $\mathbb{R}^N$.

We first consider the case $w \ne 0$ in E. Since $\Phi'_{\lambda_n}(u(\lambda_n)) = 0$, we have

$$\int_{\mathbb{R}^N} |Du(\lambda_n)|^{p(x)} + V(x)|u(\lambda_n)|^{p(x)}\,dx = \lambda_n \int_{\mathbb{R}^N} f(x, u(\lambda_n))u(\lambda_n)\,dx.$$

Assume $\|u(\lambda_n)\| > 1$. Dividing both sides by $\|u(\lambda_n)\|^{p^+}$, we get

$$\int_{\mathbb{R}^N} \frac{f(x, u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^+}}\,dx \le \frac{1}{\lambda_n} \le 1.$$

Further, by Fatou's Lemma and $(C2)$, we have

$$\int_{\mathbb{R}^N} \frac{f(x, u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^+}}\,dx = \int_{\mathbb{R}^N} \frac{f(x, u(\lambda_n))u(\lambda_n)|w_n(x)|^{p^+}}{\|u_n(x)\|^{p^+}}\,dx \to \infty,$$

a contradiction.

For the case $w = 0$ in E, we define $\Phi_{\lambda_n}(t_n u(\lambda_n)) = \max_{t\in[0,1]} \Phi_{\lambda_n}(tu(\lambda_n))$. Then for any $C > 1, \overline{w_n} := \frac{Cu(\lambda_n)}{\|u(\lambda_n)\|}$ and n large enough, we have

$$\begin{aligned}
&\Phi_{\lambda_n}(t_n u(\lambda_n)) \\
&\ge \Phi_{\lambda_n}(\overline{w_n})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|CDw_n|^{p(x)} + V(x)|Cw_n|^{p(x)})\,dx - \lambda_n \int_{R^N} F(x, Cw_n)\,dx \\
&\geq \frac{1}{p^+}C^{p^-} - \lambda_n \int_{\mathbb{R}^N} F(x, Cw_n)\,dx.
\end{aligned}
$$

Since $w_n \to 0$ a.e. in $\mathbb{R}^N$ and $\lambda_n \in [1, 2]$, we have $\lambda_n \int_{\mathbb{R}^N} F(x, Cw_n)\,dx \to 0$ as $n \to \infty$. Since C is arbitrary, we have $\Phi_{\lambda_n}(t_n u(\lambda_n)) \to \infty$ as $n \to \infty$. Consequently, we know $t_n \in (0, 1)$ when n is large enough, which implies $\Phi'_{\lambda_n}(t_n u(\lambda_n))t_n u(\lambda_n) = 0$. Thus,

$$
\Phi_{\lambda_n}(t_n u(\lambda_n)) - \frac{1}{p^-}\Phi'_{\lambda_n}(t_n u(\lambda_n))t_n u(\lambda_n) \to \infty,
$$

which implies

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Big(\frac{1}{p(x)} - \frac{1}{p^-}\Big)(|t_n Du(\lambda_n)|^{p(x)} + V(x)|t_n u(\lambda_n)|^{p(x)})\,dx \\
&+ \lambda_n \int_{\mathbb{R}^N} \frac{1}{p^-} f(x, t_n u(\lambda_n))t_n u(\lambda_n) - F(x, t_n u(\lambda_n))\,dx \to \infty.
\end{aligned}
$$

So

$$
\int_{\mathbb{R}^N} \frac{1}{p^-} f(x, t_n u(\lambda_n))t_n u(\lambda_n) - F(x, t_n u(\lambda_n))\,dx \to \infty.
$$

However,

$$
\begin{aligned}
\Phi_{\lambda_n}(u(\lambda_n)) &= \Phi_{\lambda_n}(u(\lambda_n)) - \frac{1}{p^+}\Phi'_{\lambda_n}(u(\lambda_n))u(\lambda_n) \\
&= \int_{\mathbb{R}^N} \Big(\frac{1}{p(x)} - \frac{1}{p^+}\Big)(|Du(\lambda_n)|^{p(x)} + V(x)|u(\lambda_n)|^{p(x)})\,dx \\
&\quad + \lambda_n \int_{\mathbb{R}^N} \frac{1}{p^+} f(x, u(\lambda_n))u(\lambda_n) - F(x, u(\lambda_n))\,dx \\
&\geq \lambda_n \int_{\mathbb{R}^N} \frac{1}{p^+} f(x, u(\lambda_n))u(\lambda_n) - F(x, u(\lambda_n))\,dx.
\end{aligned}
$$

In view of (C4), there exist two positive constants C_1 and C_2 such that

$$
\begin{aligned}
\Phi_{\lambda_n}(u(\lambda_n)) &\geq \lambda_n \int_{\mathbb{R}^N} \frac{1}{p^+} f(x, u(\lambda_n))u(\lambda_n) - F(x, u(\lambda_n))\,dx \\
&\geq \lambda_n C_1 \int_{\mathbb{R}^N} \frac{1}{p^-} f(x, u(\lambda_n))u(\lambda_n) - F(x, u(\lambda_n))\,dx \\
&\geq \lambda_n C_1 C_2 \int_{\mathbb{R}^N} \frac{1}{p^-} f(x, t_n u(\lambda_n))t_n u(\lambda_n) - F(x, t_n u(\lambda_n))\,dx \\
&\geq C \int_{\mathbb{R}^N} \frac{1}{p^-} f(x, t_n u(\lambda_n))t_n u(\lambda_n) - F(x, t_n u(\lambda_n))\,dx \to \infty.
\end{aligned}
$$

However, for each k large enough, $\Phi_{\lambda_n}(u(\lambda_n)) = c_k(\lambda_n) \leq \overline{c_k} < \infty$ (See Lemma 3.8), a contradiction. □

Proof of Theorem 2.2. Whether $1 \in \Lambda_0$ or not, we have found a special bounded $(PS)_{c_k}$ sequence $\{u^k(\lambda_n)\}_{n=1}^{\infty}$ for each c_k in the Fountain Theorem when k is large enough. In view of Remark 3.9, we know $\{u^k(\lambda_n)\}_{n=1}^{\infty}$ has a convergent subsequence and c_k is indeed an critical level of Φ and Theorem 2.2 follows. □

We end this paper with the following brief comments on our argument structure. We prove Theorem 2.2 in such a way to emphasize the procedure of finding critical points. First, we consider the original functional and verify the functional satisfies some geometry properties (e.g. Mountain Pass Geometry in [29], Fountain geometry in this paper, general linking geometry, etc) to ensure prospective critical levels. Then, we consider our functional as a member in a family of functionals. Some given structure conditions on the family will yield bounded (PS) sequences for almost all the functionals. Using the information supplied by these functionals, we could find special bounded (PS) sequences for those prospective critical levels. At last, we prove that the special (PS) sequences we found converge to critical points respectively up to subsequences.

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Xiaoyan Wang
Department of Mathematics, Indiana University Bloomington, IN 47405, USA
E-mail address: wang264@indiana.edu

Jinghua Yao (corresponding author)
Department of Mathematics, The University of Iowa, Iowa City, IA 52246, USA
E-mail address: jinghua-yao@uiowa.edu

Duchao Liu
Department of Mathematics, Lanzhou University, Lanzhou 730000, China
E-mail address: liuduchao@gmail.com, liudch@lzu.edu.cn