Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 138, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

VARIATIONAL APPROACH TO FRACTIONAL BOUNDARY VALUE PROBLEMS WITH TWO CONTROL PARAMETERS

MASSIMILIANO FERRARA, ARMIN HADJIAN

Abstract. This article concerns the multiplicity of solutions for a fractional differential equation with Dirichlet boundary conditions and two control parameters. Using variational methods and three critical point theorems, we give some new criteria to guarantee that the fractional problem has at least three solutions.

1. Introduction

Fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of physic, chemistry, biology, engineering and economics. There has been significant development in fractional differential equations, one can see the monographs of Miller and Ross [19], Samko et al [26], Podlubny [21], Hilfer [12], Kilbas et al [14] and the papers [1, 3, 4, 6, 7, 15, 17, 28, 29, 31] and references therein.

Critical point theory has been very useful in determining the existence of solutions for integer order differential equations with some boundary conditions; see for instance, in the vast literature on the subject, the classical books [18, 24, 27, 30] and references therein. But until now, there are a few results for fractional boundary value problems (briefly BVP) which were established exploiting this approach, since it is often very difficult to establish a suitable space and variational functional for fractional problems.

The aim of this article is to study the nonlinear fractional boundary value problem

$$
\begin{gathered}
\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(t,u(t)) + \mu g(t,u(t)) = 0, \\
\text{a.e. } t\in[0,T], \\
u(0)=u(T)=0,
\end{gathered}
\tag{1.1}
$$

where $\alpha\in(1/2,1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order $0<\alpha\le 1$ respectively, λ and μ are positive real parameters, and $f,g:[0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

2010 *Mathematics Subject Classification.* 58E05, 26A33, 34A08, 34B15, 45J05, 91A80, 91B55.
Key words and phrases. Fractional differential equations; Caputo fractional derivatives; variational methods; multiple solutions.

Submitted February 18, 2015. Published May 20, 2015.

In this article, employing two three critical point theorems which we recall in the next section (Theorems 2.1 and 2.2), we establish the exact collections of the parameters λ and μ, for which the problem (1.1) admits at least three weak solutions; see Theorems 3.1 and 3.2.

For more information, we refer the reader to [5, 11, 20] where the existence and multiplicity of solutions for problem (1.1), with $\mu = 0$, using the critical point theory is proved; see also [2, 10] where analogous variational approaches have been developed on studying nonlinear perturbed differential equations. A special case of Theorem 3.1 is the following theorem.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(\xi) := \int_0^\xi f(x)dx$ for each $\xi \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d > 0$ and $F(\xi) \geq 0$ in $[0, d)$ and*

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^2} = \limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2} = 0.$$

Then, there is $\lambda^\star > 0$ such that for each $\lambda > \lambda^\star$ and for every continuous function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying the asymptotic condition

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} \int_0^\xi g(t, s)ds}{\xi^2} < +\infty,$$

there exists $\delta^\star_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta^\star_{\lambda,g}[$, the problem

$$\begin{gathered}\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) + \lambda f(u(t)) + \mu g(t, u(t)) = 0,\\ \text{a.e. } t \in [0, T],\\ u(0) = u(T) = 0,\end{gathered}$$

admits at least three solutions.

The following result is a consequence of Theorem 3.2.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that $\lim_{t\to 0} f(t)/t = 0$ and*

$$\int_0^{10} f(s)ds < \frac{25}{12} \int_0^1 f(s)ds.$$

Then, for every

$$\lambda \in \Big]\frac{24}{\int_0^1 f(s)ds}, \frac{50}{\int_0^{10} f(s)ds}\Big[$$

and for every nonnegative continuous function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$, there exists $\delta^\star > 0$ such that, for each $\mu \in [0, \delta^\star[$, the problem

$$\begin{gathered}2u''(t) + \lambda f(u(t)) + \mu g(t, u(t)) = 0, \quad \text{a.e. } t \in [0, 1],\\ u(0) = u(1) = 0,\end{gathered}$$

admits at least three solutions.

2. Preliminaries

The original three critical point theorem is due to Pucci and Serrin [22, 23] and establishes that if X is a real Banach space and a function $f : X \to \mathbb{R}$ is of class C^1, satisfies the Palais-Smale condition, and has two local minima, then f has at least three distinct critical points. This result has been extended in the framework of problems depending on a real parameter by Ricceri [25], who also established a precise range of the parameter that guarantees the existence of at least three critical points.

Our main tools are critical point theorems that we recall here in a convenient form. The first result has been obtained in [9] and it is a more precise version of [8, Theorem 3.2]. The second one has been established in [8].

Theorem 2.1 ([9, Theorem 3.6]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that*

(A1) $\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(A2) *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}[$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional I_λ has at least three distinct critical points in X.

Theorem 2.2 ([8, Theorem 3.3]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{x} \in X$, with $2r_1 < \Phi(\overline{x}) < r_2/2$, such that

(A3) $\frac{\sup_{\Phi(x)<r_1}\Psi(x)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(A4) $\frac{\sup_{\Phi(x)<r_2}\Psi(x)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(A5) *for each* λ *in*

$$\Lambda' := \Big]\frac{3}{2}\frac{\Phi(\overline{x})}{\Psi(\overline{x})},\ \min\Big\{\frac{r_1}{\sup_{\Phi(x)<r_1}\Psi(x)},\ \frac{\frac{r_2}{2}}{\sup_{\Phi(x)<r_2}\Psi(x)}\Big\}\Big[$$

and for every $x_1, x_2 \in X$, which are local minima for the functional $I_\lambda := \Phi - \lambda\Psi$, and such that $\Psi(x_1) \ge 0$ and $\Psi(x_2) \ge 0$ one has $\inf_{s\in[0,1]}\Psi(sx_1 + (1-s)x_2) \ge 0$.

Then, for each $\lambda \in \Lambda'$ the functional I_λ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Now, we introduce some necessary definitions and properties of the fractional calculus which are used in this article.

Definition 2.3. Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha>0$ for a function u are defined by

$$
\begin{aligned}
{}_aD_t^{-\alpha}u(t) &:= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds,\\
{}_tD_b^{-\alpha}u(t) &:= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)ds,
\end{aligned}
$$

for every $t\in[a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the standard gamma function given by

$$
\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1}e^{-z}dz.
$$

Set $AC^n([a,b],\mathbb{R})$ as the space of functions $u:[a,b]\to\mathbb{R}$ such that u belongs to $C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)}$ belongs to $AC([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings having $(n-1)$ times continuously differentiable on $[a,b]$. In particular we denote $AC([a,b],\mathbb{R}) := AC^1([a,b],\mathbb{R})$.

Definition 2.4. Let $\gamma\ge 0$ and $n\in\mathbb{N}$.

(i) If $\gamma\in(n-1,n)$ and $u\in AC^n([a,b],\mathbb{R})$, then the left and right Caputo fractional derivatives of order γ for function u denoted by ${}_a^cD_t^\gamma u(t)$ and ${}_t^cD_b^\gamma u(t)$, respectively, exist almost everywhere on $[a,b]$, ${}_a^cD_t^\gamma u(t)$ and ${}_t^cD_b^\gamma u(t)$ are represented by

$$
\begin{aligned}
{}_a^cD_t^\gamma u(t) &= \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-\gamma-1}u^{(n)}(s)ds,\\
{}_t^cD_b^\gamma u(t) &= \frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b (s-t)^{n-\gamma-1}u^{(n)}(s)ds,
\end{aligned}
$$

for every $t\in[a,b]$, respectively.

(ii) If $\gamma=n-1$ and $u\in AC^{n-1}([a,b],\mathbb{R})$, then ${}_a^cD_t^{n-1}u(t)$ and ${}_t^cD_b^{n-1}u(t)$ are represented by

$$
{}_a^cD_t^{n-1}u(t) = u^{(n-1)}(t), \quad \text{and} \quad {}_t^cD_b^{n-1}u(t) = (-1)^{(n-1)}u^{(n-1)}(t),
$$

for every $t\in[a,b]$.

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [14] and [26].

Proposition 2.5. *We have the following property of fractional integration*

$$
\int_a^b [{}_aD_t^{-\gamma}u(t)]v(t)dt = \int_a^b [{}_tD_b^{-\gamma}v(t)]u(t)dt, \quad \gamma>0, \tag{2.1}
$$

provided that $u\in L^p([a,b],\mathbb{R})$, $v\in L^q([a,b],\mathbb{R})$ *and* $p\ge 1$, $q\ge 1$, $1/p+1/q\le 1+\gamma$ *or* $p\ne 1$, $q\ne 1$, $1/p+1/q=1+\gamma$.

Proposition 2.6. *Let* $n\in\mathbb{N}$ *and* $n-1<\gamma\le n$. *If* $u\in AC^n([a,b],\mathbb{R})$ *or* $u\in C^n([a,b],\mathbb{R})$, *then*

$$
{}_aD_t^{-\gamma}({}_a^cD_t^\gamma u(t)) = u(t) - \sum_{j=0}^{n-1}\frac{u^{(j)}(a)}{j!}(t-a)^j,
$$

$$ {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}u(t)) = u(t) - \sum_{j=0}^{n-1}\frac{(-1)^j u^{(j)}(b)}{j!}(b-t)^j, $$

for every $t \in [a,b]$. In particular, if $0 < \gamma \leq 1$ and $u \in AC([a,b],\mathbb{R})$ or $u \in C^1([a,b],\mathbb{R})$, then

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}f(t)) = f(t) - f(a), \quad \text{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}f(t)) = f(t) - f(b). \tag{2.2} $$

Remark 2.7. By (2.1) and Definition 2.4, it is obvious that $u \in AC([0,T],\mathbb{R})$ is a solution of problem (1.1) if and only if u is a solution of the boundary value problem

$$ \begin{gathered} \frac{d}{dt}\Big({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\Big) + \lambda f(t,u(t)) = 0 \quad \text{a.e. } t \in [0,T] \\ u(0) = u(T) = 0, \end{gathered} \tag{2.3} $$

where $\beta := 2(1-\alpha) \in [0,1)$. Recall that a function $u \in AC([0,T],\mathbb{R})$ is called a solution of BVP (2.3) if:

(i) the map $t \mapsto {}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))$ is differentiable for almost every $t \in [0,T]$, and
(ii) the function u satisfies (2.3).

To establish a variational structure for the main problem, it is necessary to construct appropriate function spaces. Following [13], we denote by $C_0^{\infty}([0,T],\mathbb{R})$ the set of all functions $g \in C^{\infty}([0,T],\mathbb{R})$ with $g(0) = g(T) = 0$.

Definition 2.8. Let $0 < \alpha \leq 1$. The fractional derivative space E_0^{α} is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R})$ with respect to the norm

$$ \|u\| := \Big(\int_0^T |{}_0^cD_t^{\alpha}u(t)|^2 dt + \int_0^T |u(t)|^2 dt\Big)^{1/2}, $$

for every $u \in E_0^{\alpha}$.

Remark 2.9. It is obvious that the fractional derivative space E_0^{α} is the space of functions $u \in L^2([0,T],\mathbb{R})$ having an α-order Caputo fractional derivative ${}_0^cD_t^{\alpha}u \in L^2([0,T],\mathbb{R})$ and $u(0) = u(T) = 0$.

Proposition 2.10. *Let $\alpha \in (0,1]$. The fractional derivative space E_0^{α} is reflexive and separable Banach space.*

For $u \in E_0^{\alpha}$, set

$$ \begin{gathered} \|u\|_{L^s} := \Big(\int_0^T |u(t)|^s dt\Big)^{1/s}, \quad (s \geq 1), \\ \|u\|_{\infty} := \max_{t \in [0,T]} |u(t)|. \end{gathered} $$

One has the following two Lemmas.

Lemma 2.11. *Let $\alpha \in (1/2,1]$. For all $u \in E_0^{\alpha}$, we have*

$$ \|u\|_{L^2} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|{}_0^cD_t^{\alpha}u\|_{L^2}, \tag{2.4} $$

$$ \|u\|_{\infty} \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|{}_0^cD_t^{\alpha}u\|_{L^2}. \tag{2.5} $$

Hence, we can consider E_0^α with respect to the (equivalent) norm

$$\|u\|_\alpha := \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt \Big)^{1/2} = \|{}_0^cD_t^\alpha u\|_{L^2}, \quad \forall u \in E_0^\alpha \tag{2.6}$$

in the following analysis.

Lemma 2.12 ([13]). *Let $\alpha \in (1/2, 1]$, then for every $u \in E_0^\alpha$, we have*

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \le -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t) dt \le \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2. \tag{2.7}$$

In the rest of this article, $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\lambda, \mu > 0$ are real parameters. Put

$$F(t,\xi) := \int_0^\xi f(t,s)\, ds, \quad G(t,\xi) := \int_0^\xi g(t,s)\, ds,$$

for all $(t,\xi) \in [0,T] \times \mathbb{R}$. Set $G^c := \int_0^T \max_{|\xi| \le c} G(t,\xi)\, dt$ for all $c > 0$ and $G_d := \inf_{[0,T]\times[0,d]} G$ for all $d > 0$. Clearly, $G^c \ge 0$ and $G_d \le 0$.

We consider the functional $I_\lambda : E_0^\alpha \to \mathbb{R}$, defined by

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \quad u \in E_0^\alpha, \tag{2.8}$$

where

$$\Phi(u) := -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t)\, dt, \tag{2.9}$$

$$\Psi(u) := \int_0^T [F(t,u(t)) + \frac{\mu}{\lambda} G(t,u(t))] dt. \tag{2.10}$$

Clearly, Φ and Ψ are Gâteaux differentiable functionals whose derivatives at the point $u \in E_0^\alpha$ are

$$\begin{aligned}
\Phi'(u)(v) &= -\int_0^T ({}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha v(t) + {}_t^cD_T^\alpha u(t) \cdot {}_0^cD_t^\alpha v(t))\, dt, \\
\Psi'(u)(v) &= \int_0^T \big[f(t,u(t)) + \frac{\mu}{\lambda} g(t,u(t))\big] v(t)\, dt \\
&= -\int_0^T \int_0^t \big[f(s,u(s)) + \frac{\mu}{\lambda} g(s,u(s))\big]\, ds \cdot v'(t)\, dt,
\end{aligned}$$

for every $v \in E_0^\alpha$. By Definition 2.4 and (2.2), we have

$$\Phi'(u)(v) = \int_0^T ({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))) \cdot v'(t)\, dt.$$

Hence, $I_\lambda = \Phi - \lambda\Psi \in C^1(E_0^\alpha, \mathbb{R})$. Moreover, a critical point of the functional I_λ is a solution of (1.1). Indeed, if $u_\star \in E_0^\alpha$ is a critical point of I_λ, then

$$\begin{aligned}
0 = I_\lambda'(u_\star)(v) = \int_0^T \Big(&{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u_\star(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u_\star(t)) \\
&+ \lambda \int_0^t f(s,u_\star(s))\, ds + \mu \int_0^t g(s,u_\star(s))\, ds \Big) \cdot v'(t)\, dt,
\end{aligned} \tag{2.11}$$

for every $v \in E_0^\alpha$. We can choose $v \in E_0^\alpha$ such that

$$v(t) = \sin\frac{2k\pi t}{T} \quad \text{or} \quad v(t) = 1 - \cos\frac{2k\pi t}{T}, \quad (k = 1, 2, \dots).$$

The theory of Fourier series and (2.11) imply

$$\begin{aligned} &{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u_\star(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u_\star(t)) \\ &+ \lambda\int_0^t f(s,u_\star(s))\,ds + \mu\int_0^t g(s,u_\star(s))\,ds = \kappa \end{aligned} \tag{2.12}$$

a.e. on $[0,T]$ for some $\kappa\in\mathbb{R}$. By (2.12), it is easy to show that $u_\star\in E_0^\alpha$ is a solution of (1.1).

To conclude this section, we cite a recent monograph by Kristály, Rădulescu and Varga [16] as a general reference on variational methods adopted here.

3. Main results

In this section we establish our main abstract results. We put

$$\begin{gathered} \Omega := \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}, \\ \begin{aligned} C(T,\alpha) := \int_0^{T/4} t^{2-2\alpha}\,dt &+ \int_{T/4}^{3T/4}\big[t^{1-\alpha}-\big(t-\frac{T}{4}\big)^{1-\alpha}\big]^2\,dt \\ &+ \int_{3T/4}^{T}\big[t^{1-\alpha}-\big(t-\frac{T}{4}\big)^{1-\alpha}+\big(t-\frac{3T}{4}\big)^{1-\alpha}\big]^2\,dt, \end{aligned} \\ \omega_{\alpha,d} := \frac{16d^2}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|}C(T,\alpha). \end{gathered}$$

Fixing $c,d>0$ such that

$$\frac{\omega_{\alpha,d}}{\int_{T/4}^{3T/4}F(t,d)\,dt} < \frac{c^2|\cos(\pi\alpha)|}{\Omega^2\int_0^T\max_{|\xi|\le c}F(t,\xi)\,dt}$$

and selecting

$$\lambda\in\Lambda := \Big]\frac{\omega_{\alpha,d}}{\int_{T/4}^{3T/4}F(t,d)\,dt}, \frac{c^2|\cos(\pi\alpha)|}{\Omega^2\int_0^T\max_{|\xi|\le c}F(t,\xi)\,dt}\Big[, \tag{3.1}$$

put

$$\delta := \min\Big\{\frac{c^2|\cos(\pi\alpha)| - \lambda\Omega^2\int_0^T\max_{|\xi|\le c}F(t,\xi)\,dt}{\Omega^2G^c}, \frac{\omega_{\alpha,d}-\lambda\int_{T/4}^{3T/4}F(t,d)\,dt}{TG_d}\Big\} \tag{3.2}$$

and

$$\overline{\delta} := \min\Big\{\delta, \frac{1}{\max\big\{0, \frac{2T\Omega^2}{|\cos(\pi\alpha)|}\limsup_{|\xi|\to+\infty}\frac{\sup_{t\in[0,T]}G(t,\xi)}{\xi^2}\big\}}\Big\}, \tag{3.3}$$

where we read $\frac{r}{0}=+\infty$ whenever this case occurs. With the above notation we are able to prove the following result.

Theorem 3.1. *Assume that there exist positive constants c,d, with*

$$c < \Big(\frac{4\Omega d}{T\Gamma(2-\alpha)}\Big)\sqrt{C(T,\alpha)}, \tag{3.4}$$

such that

(A6) $F(t,\xi)\ge 0$, *for each* $(t,\xi)\in([0,\frac{T}{4}]\cup[\frac{3T}{4},T])\times[0,d]$;

(A7)
$$\frac{\int_0^T \max_{|\xi|\le c} F(t,\xi)\,dt}{c^2} < \frac{|\cos(\pi\alpha)|}{\Omega^2}\frac{\int_{T/4}^{3T/4} F(t,d)\,dt}{\omega_{\alpha,d}};$$

(A8)
$$\limsup_{|\xi|\to+\infty}\frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} < \frac{\int_0^T \max_{|\xi|\le c} F(t,\xi)\,dt}{2c^2T}.$$

Then, for every $\lambda\in\Lambda$, where Λ is given by (3.1)*, and for every continuous function $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ such that*
$$\limsup_{|\xi|\to+\infty}\frac{\sup_{t\in[0,T]} G(t,\xi)}{\xi^2} < +\infty,$$
there exists $\overline{\delta}>0$ given by (3.3) *such that, for each $\mu\in[0,\overline{\delta}[$, problem* (1.1) *admits at least three solutions.*

Proof. Fix λ, g and μ as in the conclusion. It suffices to show the functional I_λ defined in (2.8) has at least three critical points in E_0^α. We prove this by verifying the conditions given in Theorem 2.1. Note that Φ defined in (2.9) is a nonnegative Gâteaux differentiable and sequentially weakly lower semicontinuous functional, and its Gâteaux derivative admits a continuous inverse on $(E_0^\alpha)^*$. Further, from Lemma 2.12, the functional Φ is coercive. Indeed, one has
$$\Phi(u)\ge|\cos(\pi\alpha)|\|u\|_\alpha^2\to+\infty,$$
as $\|u\|_\alpha\to+\infty$. Moreover, Ψ defined in (2.10) is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. We will verify (A1) and (A2) of Theorem 2.1.

Let w be the function defined by
$$w(t):=\begin{cases}\frac{4d}{T}t, & t\in[0,T/4),\\ d, & t\in[T/4,3T/4],\\ \frac{4d}{T}(T-t), & t\in(3T/4,T],\end{cases} \tag{3.5}$$
and put
$$r:=\frac{|\cos(\pi\alpha)|}{\Omega^2}c^2.$$
It is easy to check that $w(0)=w(T)=0$ and $w\in L^2([0,T])$. Moreover, w is Lipschitz continuous on $[0,T]$, and hence w is absolutely continuous on $[0,T]$. By calculations, we have
$${}_0^cD_t^\alpha w(t)=\begin{cases}\frac{4d}{T\Gamma(2-\alpha)}t^{1-\alpha}, & t\in[0,T/4),\\ \frac{4d}{T\Gamma(2-\alpha)}\big[t^{1-\alpha}-\big(t-\frac{T}{4}\big)^{1-\alpha}\big], & t\in[T/4,3T/4],\\ \frac{4d}{T\Gamma(2-\alpha)}\big[t^{1-\alpha}-\big(t-\frac{T}{4}\big)^{1-\alpha}+\big(t-\frac{3T}{4}\big)^{1-\alpha}\big], & t\in(3T/4,T].\end{cases}$$
Obviously, ${}_0^cD_t^\alpha w$ is continuous on $[0,T]$ and
$$\begin{aligned}&\int_0^T|{}_0^cD_t^\alpha w(t)|^2\,dt\\ &=\frac{16d^2}{T^2\Gamma^2(2-\alpha)}\Big\{\int_0^{T/4}t^{2-2\alpha}\,dt+\int_{T/4}^{3T/4}\big[t^{1-\alpha}-\big(t-\frac{T}{4}\big)^{1-\alpha}\big]^2\,dt\end{aligned}$$

$$
\begin{aligned}
&+\int_{3T/4}^{T}\big[t^{1-\alpha}-\big(t-\frac{T}{4}\big)^{1-\alpha}+\big(t-\frac{3T}{4}\big)^{1-\alpha}\big]^2\,dt\Big\}\\
&:=\frac{16d^2}{T^2\Gamma^2(2-\alpha)}C(T,\alpha).
\end{aligned}
$$

Therefore, from inequality (3.4) one has

$$
\Phi(w)\geq|\cos(\pi\alpha)|\|w\|_\alpha^2=\frac{16d^2|\cos(\pi\alpha)|}{T^2\Gamma^2(2-\alpha)}C(T,\alpha)>r.
$$

Also, by using condition (A6), since $0\leq w(t)\leq d$ for each $t\in[0,T]$, we infer

$$
\begin{aligned}
\Psi(w)&=\int_0^T[F(t,w(t))+\frac{\mu}{\lambda}G(t,w(t))]dt\\
&\geq\int_{T/4}^{3T/4}F(t,d)\,dt+\frac{\mu}{\lambda}\int_0^T G(t,w(t))\,dt\\
&\geq\int_{T/4}^{3T/4}F(t,d)\,dt+\frac{\mu}{\lambda}TG_d.
\end{aligned}
$$

For all $u\in E_0^\alpha$ with $\Phi(u)\leq r$, by Lemma 2.12, we have

$$
|\cos(\pi\alpha)|\|u\|_\alpha^2\leq\Phi(u)\leq r,
$$

which implies

$$
\|u\|_\alpha^2\leq\frac{1}{|\cos(\pi\alpha)|}r.
$$

On the other hand, by Lemma 2.11, when $\alpha>1/2$, for each $u\in E_0^\alpha$ we have

$$
\|u\|_\infty\leq\Omega\Big(\int_0^T|{}_0^cD_t^\alpha u(t)|^2\,dt\Big)^{1/2}=\Omega\|u\|_\alpha. \tag{3.6}
$$

Thus, we obtain

$$
|u(t)|\leq\Omega\sqrt{\frac{r}{|\cos(\pi\alpha)|}}=c,\quad\forall t\in[0,T].
$$

Therefore,

$$
\begin{aligned}
\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r}&\leq\frac{\int_0^T\max_{|\xi|\leq c}F(t,\xi)\,dt+\frac{\mu}{\lambda}\int_0^T\max_{|\xi|\leq c}G(t,\xi)\,dt}{\frac{|\cos(\pi\alpha)|}{\Omega^2}c^2}\\
&=\frac{\Omega^2}{c^2|\cos(\pi\alpha)|}\Big(\int_0^T\max_{|\xi|\leq c}F(t,\xi)\,dt+\frac{\mu}{\lambda}G^c\Big).
\end{aligned}
$$

From this, if $G^c=0$, it is clear that

$$
\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r}<\frac{1}{\lambda}, \tag{3.7}
$$

while, if $G^c>0$, it turns out to be true bearing in mind that

$$
\mu<\frac{c^2|\cos(\pi\alpha)|-\lambda\Omega^2\int_0^T\max_{|\xi|\leq c}F(t,\xi)\,dt}{\Omega^2G^c}.
$$

On the other hand, taking into account that

$$
\Phi(w)\leq\frac{1}{|\cos(\pi\alpha)|}\|w\|_\alpha^2=\omega_{\alpha,d},
$$

we have

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_{T/4}^{3T/4} F(t,d)\,dt + \frac{\mu}{\lambda} G_d}{\omega_{\alpha,d}}.$$

Hence, if $G_d = 0$, we find

$$\frac{\Psi(w)}{\Phi(w)} > \frac{1}{\lambda}, \tag{3.8}$$

while, if $G_d < 0$, the same relation holds since

$$\mu < \frac{\omega_{\alpha,d} - \lambda \int_{T/4}^{3T/4} F(t,d)\,dt}{TG_d}.$$

Therefore, from (3.7) and (3.8), condition (A1) of Theorem 2.1 is verified.

Now, to prove the coercivity of the functional I_λ, first we assume that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} > 0.$$

Therefore, fixing

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} < \varepsilon < \frac{\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt}{2c^2T},$$

from (A8) there is a function $h_\varepsilon \in L^1([0,T])$ such that $F(t,\xi) \leq \varepsilon\xi^2 + h_\varepsilon(t)$, for each $t \in [0,T]$ and $\xi \in \mathbb{R}$. Taking (3.6) into account and since

$$\lambda < \frac{c^2|\cos(\pi\alpha)|}{\Omega^2 \int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt},$$

it follows that

$$\begin{aligned} \lambda \int_0^T F(t,u(t))\,dt &\leq \lambda\Big(\varepsilon \int_0^T (u(t))^2 dt + \int_0^T h_\varepsilon(t)\,dt\Big) \\ &< \frac{c^2|\cos(\pi\alpha)|}{\Omega^2 \int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt} \Big(\varepsilon\Omega^2 T\|u\|_\alpha^2 + \|h_\varepsilon\|_{L^1([0,T])}\Big), \end{aligned} \tag{3.9}$$

for each $u \in E_0^\alpha$. Since $\mu < \overline{\delta}$, we obtain

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} G(t,\xi)}{\xi^2} < \frac{|\cos(\pi\alpha)|}{2T\mu\Omega^2};$$

then, there is a function $h_\mu \in L^1([0,T])$ such that

$$G(t,\xi) \leq \frac{|\cos(\pi\alpha)|}{2T\mu\Omega^2}\xi^2 + h_\mu(t),$$

for each $t \in [0,T]$ and $\xi \in \mathbb{R}$. Thus, taking again (3.6) into account, it follows that

$$\begin{aligned} \int_0^T G(t,u(t))\,dt &\leq \frac{|\cos(\pi\alpha)|}{2T\mu\Omega^2} \int_0^T (u(t))^2 dt + \int_0^T h_\mu(t)\,dt \\ &\leq \frac{|\cos(\pi\alpha)|}{2\mu}\|u\|_\alpha^2 + \|h_\mu\|_{L^1([0,T])}, \end{aligned} \tag{3.10}$$

for each $u \in E_0^\alpha$. Finally, putting together (3.9) and (3.10), we have

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

$$
\begin{aligned}
&\geq |\cos(\pi\alpha)|\|u\|_\alpha^2 - \frac{c^2|\cos(\pi\alpha)|}{\Omega^2\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt}\Big(\varepsilon\Omega^2 T\|u\|_\alpha^2 + \|h_\varepsilon\|_{L^1([0,T])}\Big)\\
&\quad - \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \mu\|h_\mu\|_{L^1([0,T])}\\
&= |\cos(\pi\alpha)|\Big(\frac{1}{2} - \frac{c^2T}{\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt}\varepsilon\Big)\|u\|_\alpha^2\\
&\quad - \frac{c^2|\cos(\pi\alpha)|\|h_\varepsilon\|_{L^1([0,T])}}{\Omega^2\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt} - \mu\|h_\mu\|_{L^1([0,T])}.
\end{aligned}
$$

On the other hand, if

$$
\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{\xi^2} \leq 0,
$$

then there exists a function $h_\varepsilon \in L^1([0,T])$ such that $F(t,\xi) \leq h_\varepsilon(t)$ for each $t\in[0,T]$ and $\xi\in\mathbb{R}$. Arguing as before we obtain

$$
I_\lambda(u) \geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{c^2|\cos(\pi\alpha)|\|h_\varepsilon\|_{L^1([0,T])}}{\Omega^2\int_0^T \max_{|\xi|\leq c} F(t,\xi)\,dt} - \mu\|h_\mu\|_{L^1([0,T])}\,.
$$

Both cases lead to the coercivity of I_λ and condition (A2) of Theorem 2.1 is verified.

Since, from (3.7) and (3.8),

$$
\lambda \in \Lambda \subseteq \Big]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\Big[,
$$

Theorem 2.1 ensures the existence of at least three critical points for the functional I_λ and the proof is complete. □

Now, we state a variant of Theorem 3.1 in which no asymptotic condition on g is requested. In such a case, the functions f and g are supposed to be nonnegative.

Fixing positive constants c_1, c_2 and d such that

$$
\frac{3}{2}\frac{\omega_{\alpha,d}}{\int_{T/4}^{3T/4} F(t,d)\,dt} < \frac{|\cos(\pi\alpha)|}{\Omega^2}\min\Big\{\frac{c_1^2}{\int_0^T F(t,c_1)\,dt}, \frac{c_2^2}{2\int_0^T F(t,c_2)\,dt}\Big\},
$$

and selecting

$$
\lambda \in \Lambda' := \Big]\frac{3}{2}\frac{\omega_{\alpha,d}}{\int_{T/4}^{3T/4} F(t,d)\,dt}, \frac{|\cos(\pi\alpha)|}{\Omega^2}\min\Big\{\frac{c_1^2}{\int_0^T F(t,c_1)\,dt}, \frac{c_2^2}{2\int_0^T F(t,c_2)\,dt}\Big\}\Big[, \tag{3.11}
$$

we put

$$
\delta^\star := \min\Big\{\frac{c_1^2|\cos(\pi\alpha)| - \lambda\Omega^2\int_0^T F(t,c_1)\,dt}{\Omega^2 G^{c_1}}, \frac{c_2^2|\cos(\pi\alpha)| - 2\lambda\Omega^2\int_0^T F(t,c_2)\,dt}{2\Omega^2 G^{c_2}}\Big\}. \tag{3.12}
$$

With the above notation we have the following multiplicity result.

Theorem 3.2. *Assume that there exist three positive constants c_1, c_2 and d with*

$$
c_1 < \frac{2d\Omega}{T\Gamma(2-\alpha)}\sqrt{2C(T,\alpha)} < \frac{c_2|\cos(\pi\alpha)|}{2}, \tag{3.13}
$$

such that

(A9) *$f(t,\xi)\geq 0$ for all $(t,\xi)\in[0,T]\times[0,c_2]$;*

(A10)

$$\max\Big\{\frac{\int_0^T F(t,c_1)\,dt}{c_1^2}, \frac{2\int_0^T F(t,c_2)\,dt}{c_2^2}\Big\} < \frac{2}{3}\frac{|\cos(\pi\alpha)|\int_{T/4}^{3T/4} F(t,d)\,dt}{\omega_{\alpha,d}\Omega^2}.$$

Then, for each $\lambda \in \Lambda'$, where Λ' is given by (3.11), and for every nonnegative continuous function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, there exists $\delta^\star > 0$ given by (3.12) such that, for each $\mu \in [0, \delta^\star[$, problem (1.1) admits at least three distinct solutions u_i, $i = 1,2,3$, such that

$$0 \le u_i(t) < c_2, \quad \forall t \in [0,T],\ i = 1,2,3.$$

Proof. Without loss of generality, we can assume $f(t,\xi) \ge 0$ for all $(t,\xi) \in [0,T]\times\mathbb{R}$. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Then, our aim is to verify (A3) and (A4).

To this end, put w as given in (3.5), and

$$r_1 := \frac{|\cos(\pi\alpha)|}{\Omega^2}c_1^2, \quad r_2 := \frac{|\cos(\pi\alpha)|}{\Omega^2}c_2^2.$$

By using the condition (3.13), we get $2r_1 < \Phi(w) < \frac{r_2}{2}$. Since $\mu < \delta^\star$ and $G_d = 0$, one has

$$\begin{aligned}
\frac{\sup_{\Phi(u)<r_1}\Psi(u)}{r_1} &= \frac{\sup_{\Phi(u)<r_1}\big[\int_0^T F(t,u(t))\,dt + \frac{\mu}{\lambda}\int_0^T G(t,u(t))\,dt\big]}{r_1} \\
&\le \frac{\int_0^T F(t,c_1)\,dt + \frac{\mu}{\lambda}G^{c_1}}{\frac{|\cos(\pi\alpha)|}{\Omega^2}c_1^2} \\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_{T/4}^{3T/4} F(t,d)\,dt + \frac{\mu}{\lambda}TG_d}{\omega_{\alpha,d}} \\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{2\sup_{\Phi(u)<r_2}\Psi(u)}{r_2} &= \frac{2\sup_{\Phi(u)<r_2}\big[\int_0^T F(t,u(t))\,dt + \frac{\mu}{\lambda}\int_0^T G(t,u(t))\,dt\big]}{r_2} \\
&\le \frac{2\int_0^T F(t,c_2)\,dt + 2\frac{\mu}{\lambda}G^{c_2}}{\frac{|\cos(\pi\alpha)|}{\Omega^2}c_2^2} \\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_{T/4}^{3T/4} F(t,d)\,dt + \frac{\mu}{\lambda}TG_d}{\omega_{\alpha,d}} \\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},
\end{aligned}$$

Therefore, (A3) and (A4) of Theorem 2.2 are satisfied.

Finally, we verify that I_λ satisfies the assumption (A5) of Theorem 2.2. Let u_1 and u_2 be two local minima for I_λ. Then u_1 and u_2 are critical points for I_λ, and so, they are solutions for problem (1.1). We claim that the solutions obtained are nonnegative. Indeed, if $\bar u$ is a solution of problem (1.1), then one has

$$\begin{aligned}
&-\int_0^T ({}_0^cD_t^\alpha \bar u(t)\cdot {}_t^cD_T^\alpha v(t) + {}_t^cD_T^\alpha \bar u(t)\cdot {}_0^cD_t^\alpha v(t))\,dt\\
&= \lambda\int_0^T f(t,\bar u(t))v(t)\,dt + \mu\int_0^T g(t,\bar u(t))v(t)\,dt
\end{aligned}$$

for all $v\in E_0^\alpha$. Arguing by a contradiction, assume that the set $A := \{t\in[0,T] : \bar u(t) < 0\}$ is non-empty and of positive measure. Put $\bar v := \min\{\bar u, 0\}$. Clearly, $\bar v\in E_0^\alpha$. So, taking into account that $\bar u$ is a solution and by choosing $v=\bar v$, from our sign assumptions on the data, one has

$$\begin{aligned}
&-\int_A ({}_0^cD_t^\alpha \bar u(t)\cdot {}_t^cD_T^\alpha \bar u(t) + {}_t^cD_T^\alpha \bar u(t)\cdot {}_0^cD_t^\alpha \bar u(t))\,dt\\
&= \lambda\int_A f(t,\bar u(t))\bar u(t)\,dt + \mu\int_A g(t,\bar u(t))\bar u(t)\,dt \le 0.
\end{aligned}$$

On the other hand, by Lemma 2.12, we have

$$2|\cos(\pi\alpha)|\|\bar u\|^2_{E_0^\alpha(A)} \le -\int_A ({}_0^cD_t^\alpha \bar u(t)\cdot {}_t^cD_T^\alpha \bar u(t) + {}_t^cD_T^\alpha \bar u(t)\cdot {}_0^cD_t^\alpha \bar u(t))\,dt$$

Hence, $\bar u\equiv 0$ on A which is absurd. Then, we deduce $u_1(t)\ge 0$ and $u_2(t)\ge 0$ for every $t\in[0,T]$. Thus, it follows that $su_1 + (1-s)u_2\ge 0$ for all $s\in[0,1]$, and that

$$(\lambda f + \mu g)(t, su_1 + (1-s)u_2) \ge 0,$$

and consequently, $\Psi(su_1 + (1-s)u_2)\ge 0$, for every $s\in[0,1]$. So, also (A5) holds.

From Theorem 2.2, for every

$$\lambda \in \Big]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\Big\{\frac{r_1}{\sup_{\Phi(u)<r_1}\Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{\Phi(u)<r_2}\Psi(u)}\Big\}\Big[,$$

the functional I_λ has at least three distinct critical points which are the solutions of problem (1.1) and the conclusion is achieved. □

Proof of Theorem 1.1. Fix $\lambda > \lambda^\star := \frac{2\omega_{\alpha,d}}{TF(d)}$ for some $d>0$ such that $F(d)>0$. Recalling that

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{c_n\}\subset]0,+\infty[$ such that $\lim_{n\to+\infty} c_n = 0$ and

$$\lim_{n\to+\infty}\frac{\max_{|\xi|\le c_n} F(\xi)}{c_n^2} = 0.$$

Indeed, one has

$$\lim_{n\to+\infty}\frac{\max_{|\xi|\le c_n} F(\xi)}{c_n^2} = \lim_{n\to+\infty}\frac{F(\xi_{c_n})}{\xi_{c_n}^2}\frac{\xi_{c_n}^2}{c_n^2} = 0,$$

where $F(\xi_{c_n}) = \max_{|\xi|\le c_n} F(\xi)$. Therefore, there exists $\bar c>0$ such that

$$\frac{\max_{|\xi|\le\bar c} F(\xi)}{\bar c^2} < \frac{|\cos(\pi\alpha)|}{\Omega^2}\min\Big\{\frac{F(d)}{2\omega_{\alpha,d}}, \frac{1}{T\lambda}\Big\}$$

and $\bar c < \frac{4\Omega d}{T\Gamma(2-\alpha)}\sqrt{C(T,\alpha)}$. Hence, the conclusion follows from Theorem 3.1 □

Proof of Theorem 1.2. Our aim is to apply Theorem 3.2 by choosing $c_2 = 10$ and $d = 1$. Therefore, taking into account that $\alpha = T = 1$, one has

$$\frac{3}{2}\frac{\omega_{\alpha,d}}{\int_{T/4}^{3T/4} F(t,d)dt} = \frac{24}{\int_0^1 f(s)ds},$$

$$\frac{|\cos(\pi\alpha)|}{\Omega^2}\frac{c_2^2}{2\int_0^T F(t,c_2)dt} = \frac{50}{\int_0^{10} f(s)ds}.$$

Since $\lim_{t\to 0} f(t)/t = 0$, one has

$$\lim_{t\to 0}\frac{\int_0^t f(s)ds}{t^2} = 0.$$

Then, there exists a positive constant $c_1 < 2$ such that

$$\frac{\int_0^{c_1} f(s)ds}{c_1^2} < \frac{1}{24}\int_0^1 f(s)ds,$$

$$\frac{c_1^2}{\int_0^{c_1} f(s)ds} > \frac{50}{\int_0^{10} f(s)ds}.$$

Hence, a simple computation shows that all assumptions of Theorem 3.2 are satisfied, and the conclusion follows. □

References

[1] G. A. Afrouzi, A. Hadjian, G. Molica Bisci; Some results for one dimensional fractional problems, submitted.

[2] G. A. Afrouzi, A. Hadjian, V. Rădulescu; Variational approach to fourth-order impulsive differential equations with two control parameters, *Results Math.*, **65** (2014), 371-384.

[3] C. Bai; Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative, *J. Math. Anal. Appl.*, **384** (2011), 211-231.

[4] C. Bai; Solvability of multi-point boundary value problem of nonlinear impulsive fractional differential equation at resonance, *Electron. J. Qual. Theory Differ. Equ.*, **89** (2011), 1-19.

[5] C. Bai; Existence of solutions for a nonlinear fractional boundary value problem via a local minimum theorem, *Electron. J. Differential Equations*, **176** (2012), 1-9.

[6] Z. Bai and H. Lu; Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311** (2005), 495-505.

[7] M. Benchohra, S. Hamani, S. K. Ntouyas; Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.*, **71** (2009), 2391-2396.

[8] G. Bonanno, P. Candito; Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, *J. Differential Equations*, **244** (2008), 3031-3059.

[9] G. Bonanno, S. A. Marano; On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, **89** (2010), 1-10.

[10] G. D'Aguì, S. Heidarkhani, G. Molica Bisci; Multiple solutions for a perturbed mixed boundary value problem involving the one-dimensional p-Laplacian, *Electron. J. Qual. Theory Differ. Equ.*, (2013), No. 24, 14 pp.

[11] M. Galewski, G. Molica Bisci; Existence results for one-dimensional fractional equations, preprint.

[12] R. Hilfer; *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[13] F. Jiao and Y. Zhou; Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.*, **62** (2011), 1181-1199.

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[15] N. Kosmatov; Integral equations and initial value problems for nonlinear differential equations of fractional order, *Nonlinear Anal.*, **70** (2009), 2521-2529.

[16] A. Kristály, V. Rădulescu, Cs. Varga; *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, no. 136, Cambridge University Press, Cambridge, 2010.
[17] V. Lakshmikantham, A.S. Vatsala; Basic theory of fractional differential equations, *Nonlinear Anal.*, **69** (2008), 2677-2682.
[18] J. Mawhin, M. Willem; *Critical Point Theorey and Hamiltonian Systems*, Springer, New York, 1989.
[19] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[20] N. Nyamoradi; Infinitely many solutions for a class of fractional boundary value problems with Dirichlet boundary conditions, *Mediterr. J. Math.*, **11** (2014), 75-87.
[21] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[22] P. Pucci, J. Serrin; Extensions of the mountain pass theorem, *J. Funct. Anal.*, **59** (1984), 185-210.
[23] P. Pucci, J. Serrin; A mountain pass theorem, *J. Differential Equations*, **60** (1985), 142-149.
[24] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, in: CBMS, vol. 65, Amer. Math. Soc., 1986.
[25] B. Ricceri; On a three critical points theorem, *Arch. Math. (Basel)*, **75** (2000), 220-226.
[26] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach, Longhorne, PA, 1993.
[27] M. Struwe; *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Springer-Verlag, Berlin-Heidelberg, 1990.
[28] J. Wang, Y. Zhou; A class of fractional evolution equations and optimal controls, *Nonlinear Anal. Real World Appl.* **12** (2011), 262-272.
[29] Z. Wei, W. Dong, J. Che; Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative, *Nonlinear Anal.*, **73** (2010), 3232-3238.
[30] M. Willem; *Minimax Theorems*, Birkhäuser, 1996.
[31] S. Zhang; Positive solutions to singular boundary value problem for nonlinear fractional differential equation, *Comput. Math. Appl.*, **59** (2010), 1300-1309.

Massimiliano Ferrara
Department of Law and Economics, University Mediterranea of Reggio Calabria, Via dei Bianchi, 2-89127 Reggio Calabria, Italy
E-mail address: `massimiliano.ferrara@unirc.it`

Armin Hadjian
Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran
E-mail address: `a.hadjian@ub.ac.ir`