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GLOBAL DISSIPATIVE SOLUTIONS FOR THE TWO-COMPONENT CAMASSA-HOLM SHALLOW WATER SYSTEM

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Abstract. This article presents a continuous semigroup of globally defined weak dissipative solutions for the two-component Camassa-Holm system. Such solutions are established by using a new approach based on characteristics a set of new variables overcoming the difficulties inherent in multi-component systems.

1. Introduction

We consider the two-component Camassa-Holm shallow water system (see [5, 10, 13])

$$\begin{gathered} m_t + um_x + 2u_x m - Au_x + \rho\rho_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ m = u - u_{xx}, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, \quad t > 0,\ x \in \mathbb{R}. \end{gathered} \tag{1.1}$$

Here $A > 0$ characterizes a linear underlying shear flow so that (1.1) models wave-current interactions. The variable $u(x,t)$ represents the horizontal velocity of the fluid, and $\rho(x,t)$ is the scalar density. This system appears in [19]; it was also derived by Constantin and Ivanov in [10] in the context of shallow water theory. It is an extension of the Camassa–Holm (CH), is formally integrable [5, 10, 13], and also has a bi-Hamiltonian structure with Hamiltonians

$$H_1 = \frac{1}{2}\int \big(um + (\rho-1)^2\big)dx$$

and

$$H_2 = \frac{1}{2}\int \Big(u(\rho-1)^2 + 2u(\rho-1) + u^3 + uu_x^2 - Au^2\Big)dx.$$

For $\rho \equiv 0$, one obtains the classical CH equation, which models the unidirectional propagation of shallow water waves over a flat bottom. It has a bi-Hamiltonian structure [6] and is completely integrable [4, 7]. The CH equation has attracted a lot of attention just because it has peaked solitons [4, 11] and models wave breaking [4, 9]. The presence of breaking waves means the solution remains bounded while its slope becomes unbounded in finite time [8, 9]. After wave breaking the solutions

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of the CH equation can be continued uniquely as either globally conservative [2] or globally dissipative solutions[3].

System (1.1) is an integrable multi-component generalization of the CH equation. System (1.1) has a physical interpretation[10], just like the CH equation, has an integrable structure [13], and can be expressed as a flow [17]. It has been shown that the two-component CH system is locally well-posed with initial data $(u_0,\ \rho_0) \in H^s \times H^{s-1}$, $s > 3/2$ [15]. The system also has global strong solutions which blow up in finite time [12]. More interestingly, it possesses a global continuous semigroup of weak conservative solutions [21, 22]. The goal of the present paper is to construct a global continuous semigroup of weak dissipative solutions for the two-component Camassa-Holm system (1.1). It should be stressed that system (1.1) is a multiple (rather than single) component system in which the mutual effect between the components u and ρ exits, making it quite challenging to address properties of the solutions associated with the system.

To circumvent the difficulties coming from the two-component coupling effect, we introduce a suitable characteristic and a new set of independent and dependent variables to transfer the system (1.1) into a semilinear hyperbolic system. By solving the corresponding semilinear system which contains a discontinuous non-local source term but has bounded directional variation, a global dissipative solution is derived. Then, by mapping the solution of the semilinear system into the solution of original system (1.1), the problem is solved. Furthermore, it is proved that the solutions actually construct a semigroup.

The remainder of this article is organized as follows. Section 2 is the introduction of the original system. In Section 3, a transformation from the original system to an equivalent semilinear system is conducted by applying a new set of variables. The unique global solution of the equivalent semilinear system is derived in Section 4 and then it is reversed to the weak dissipative solution of the original system in Section 5, which constructs a global continuous semigroup.

2. The original system

Let $G(x) = \frac{1}{2}e^{-|x|}$ and $*$ denotes the spatial convolution such that $G * f = (1-\partial^2)^{-1} f$ for all $f \in L^2(\mathbb{R})$. System (1.1) can thus be rewritten as a form of quasi-linear evolution equation

$$\begin{gathered} u_t + uu_x + \partial_x G * (u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\eta^2 + \eta) = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \eta_t + u\eta_x + \eta u_x + u_x = 0, \quad t > 0,\ x \in \mathbb{R}, \end{gathered}$$

which can be further represented in the form

$$\begin{gathered} u_t + uu_x + P_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \eta_t + u\eta_x + \eta u_x + u_x = 0, \quad t > 0,\ x \in \mathbb{R}, \end{gathered} \tag{2.1}$$

where $\eta = \rho - 1$ and $P = G * (u^2 + u_x^2/2 - Au + \eta^2/2 + \eta)$, with the initial condition $(u_0, \eta_0) \in H^1 \times U$ with $U = L^2 \cap L^\infty$. For smooth solutions, the total energy

$$E(t) = \int_R u^2 + u_x^2 + \eta^2 \, dx \tag{2.2}$$

is constant in time. Indeed, by using the identity $\partial_x^2 G * f = G * f - f$ and differentiating the two equations in (2.1) with respect to x respectively, we have

$$\begin{gathered} u_{xt} + uu_{xx} + u_x^2 - (u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\eta^2 + \eta) + P = 0, \\ \eta_{xt} + 2u_x\eta_x + (u\eta_{xx} + \eta u_{xx} + u_{xx}) = 0. \end{gathered} \tag{2.3}$$

Multiplying the first equation in (2.1) by u and the second equation by η, and multiplying the first one in (2.3) by u_x, we obtain the following conservation laws

$$\Big(\frac{u^2}{2}\Big)_t + \Big(\frac{u^3}{3}\Big)_x + uP_x = 0, \tag{2.4}$$

$$\Big(\frac{u_x^2}{2}\Big)_t + (\frac{1}{2}uu_x^2 - \frac{1}{3}u^3 + \frac{1}{2}Au^2)_x - \frac{1}{2}\eta^2 u_x - \eta u_x + u_x P = 0, \tag{2.5}$$

$$\Big(\frac{\eta^2}{2}\Big)_t + \eta^2 u_x + \eta u_x + u\eta\eta_x = 0. \tag{2.6}$$

It then follows from (2.4)-(2.6) that

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_S (u^2 + u_x^2 + v^2)(t,x)dx = 0.$$

Thus (2.1) possesses the H^1-norm conservation law given by

$$\|z\|_{H^1} = \Big(\int_R [u^2 + u_x^2 + \eta^2]dx\Big)^{1/2},$$

where $z = (u, \eta)$. Since $z = (u, \eta) \in H^1 \times U$, Young's inequality ensures $P \in H^1$.

Definition 2.1. By a solution of the Cauchy problem (2.1) we mean a Hölder continuous function $z = z(t, x)$ defined on $[0, T] \times R$ with the following properties:

(i) $z(t,\ \cdot) \in H^1 \times [L^2 \cap L^\infty]$ for each fixed t.

(ii) The map $t \to z(t, \cdot)$ is Lipschitz continuous from $[0, T]$ to L^2, satisfying

$$\begin{gathered} z_t = -uz_x - f(z), \\ z(0, x) = \bar{z}(x), \end{gathered} \tag{2.7}$$

where $z = (u, \eta)$, $z_x = (u_x, \eta_x)$ and $f(z) = (P_x,\ (\eta + 1)u_x)$.

Definition 2.2. We call a solution of the Cauchy problem (2.1) a dissipative solution if it satisfies the Oleinik type inequality

$$u_x(t, x), \eta_x(t, x) \le C(1 + t^{-1}), \quad t > 0$$

for some constant C depending only on the norm of the initial data $\|\bar{z}\|_{H^1}$ and its energy $E(t)$ in (2.2) is a non-increasing function of time.

3. The equivalent semilinear system

In this section, a transformation is conducted by introducing a characteristic and a new set of Lagrangian variables, with which the original system is transformed into an equivalent semilinear hyperbolic system.

For given initial data $\bar{z} = (\bar{u}, \bar{\eta}) \in H^1 \times U$, we consider the following initial problem,

$$\begin{gathered} \frac{\partial}{\partial t}q(t, \xi) = u(t, q(t, \xi)), \quad t \in [0, T], \\ q(0, \xi) = \bar{q}(\xi), \quad x \in \mathbb{R}, \end{gathered} \tag{3.1}$$

where the solution $z = (u, \eta)$ to (2.1) remains Lipschitz continuous for $t \in [0, T]$, and the non-decreasing maps $\xi \mapsto \bar{q}(\xi)$ is defined as

$$\int_0^{\bar{q}(\xi)} \bar{u}_x^2 dx = \xi. \tag{3.2}$$

The following notation is used:

$$\begin{gathered} u(t,\ \xi) = u(t,\ q(t,\xi)), \quad \eta(t,\ \xi) = \eta(t,\ q(t,\xi)), \quad P(t,\ \xi) = P(t,\ q(t,\xi)), \\ u_x(t,\ \xi) = u_x(t,\ q(t,\xi)), \quad \eta_x(t,\ \xi) = \eta_x(t,\ q(t,\xi)), \quad P_x(t,\ \xi) = P_x(t,\ q(t,\xi)). \end{gathered}$$

Define the variables $\theta = \theta(t, \xi)$ and $w = w(t, \xi)$ as

$$\theta = 2 \operatorname{arcsec} u_x, \quad w = u_x^2 \cdot \frac{\partial q}{\partial \xi}. \tag{3.3}$$

(θ in $[0, \pi) \cup (\pi, 2\pi]$).

We remark that the transformed variable θ used in this paper is of the form $\theta = 2 \operatorname{arcsec} u_x$, which makes the calculation much simple and convenient to set up the dissipative solution in contrast to the applied variable $v = 2 \arctan u_x$ in [2, 3], which further overcomes the difficulties existing in the multi-component system.

The following useful identities are prepared for later use from (3.1)-(3.3),

$$w(0, \xi) \equiv 1, \tag{3.4}$$

$$u_x = \sec \frac{\theta}{2}, \quad \frac{1}{u_x^2} = \cos^2 \frac{\theta}{2}, \tag{3.5}$$

$$\frac{\partial q}{\partial \xi} = \frac{w}{u_x^2} = \cos^2 \frac{\theta}{2} \cdot w. \tag{3.6}$$

According to (3.6), we obtain

$$q(t, \xi') - q(t, \xi) = \int_\xi^{\xi'} \cos^2 \frac{\theta}{2}(t, s) \cdot w(t, s) ds. \tag{3.7}$$

By using the new variable ξ, we represent P and P_x as follows,

$$\begin{aligned} P(\xi) = \frac{1}{2} \int_{-\infty}^{+\infty} & \exp\Big\{ -\Big| \int_\xi^{\xi'} \cos^2 \frac{\theta(s)}{2} \cdot w(s) ds \Big| \Big\} \\ & \times \big[\big(u^2 - Au + \frac{1}{2}\eta^2 + \eta \big) \cos^2 \frac{\theta}{2} + \frac{1}{2} \big] w(\xi') d\xi', \end{aligned}$$

$$\begin{aligned} P_x(\xi) = \frac{1}{2} \Big(\int_\xi^{+\infty} - \int_{-\infty}^{\xi} \Big) & \exp\Big\{ -\Big| \int_\xi^{\xi'} \cos^2 \frac{\theta(s)}{2} w(s) ds \Big| \Big\} \\ & \times \big[\big(u^2 - Au + \frac{1}{2}\eta^2 + \eta \big) \cos^2 \frac{\theta}{2} + \frac{1}{2} \big] w(\xi') d\xi', \end{aligned} \tag{3.8}$$

System (2.1) can be further rewritten with the new variables (t, ξ) as

$$\begin{gathered} \frac{\partial}{\partial t} u(t, \xi) = u_t + u u_x = -P_x(t, \xi), \\ \frac{\partial}{\partial t} \eta(t, \xi) = \eta_t + u \eta_x = -(\eta + 1) u_x(t, \xi) \end{gathered} \tag{3.9}$$

From (3.1), (3.3) and (2.3), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\theta(t,\xi) &= \frac{2}{u_x\sqrt{u_x^2-1}}(u_{xt}+uu_{xx})\\ &= -\csc\frac{\theta}{2}+(2u^2-2Au+\eta^2+2\eta-2P)\cos\frac{\theta}{2}\cdot\cot\frac{\theta}{2}.\end{aligned} \tag{3.10}$$

Furthermore, it follows from (3.1), (3.3) and (2.5) that

$$\frac{\partial}{\partial t}w(t,\xi) = (u_x^2)_t+(uu_x^2)_x = (2u^2-2Au+\eta^2+2\eta-2P)\cos\frac{\theta}{2}\cdot w. \tag{3.11}$$

The functions P and P_x used in the above (3.9)-(3.11) are given in (3.8).

Now the corresponding Cauchy problems (3.9)-(3.11) for the variables (u,η,θ,w) becomes the semilinear system

$$\begin{gathered}\frac{\partial u}{\partial t} = -P_x,\\ \frac{\partial \eta}{\partial t} = -(\eta+1)\sec\frac{\theta}{2},\\ \frac{\partial \theta}{\partial t} = -\csc\frac{\theta}{2}+(2u^2-2Au+\eta^2+2\eta-2P)\cos\frac{\theta}{2}\cdot\cot\frac{\theta}{2},\\ \frac{\partial w}{\partial t} = (2u^2-2Au+\eta^2+2\eta-2P)\cos\frac{\theta}{2}\cdot w,\end{gathered} \tag{3.12}$$

with the initial condition

$$\begin{gathered}u(0,\xi)=\bar u(\bar q(\xi)),\\ \eta(0,\xi)=\bar\eta(\bar q(\xi)),\\ \theta(0,\xi)=2\operatorname{arcsec}\bar u_x(\bar q(\xi)),\\ w(0,\xi)=1,\end{gathered} \tag{3.13}$$

which can be regarded as an ordinary differential equation (ODE) in the Banach space

$$X = H^1\times[L^2\cap L^\infty]\times[L^2\cap L^\infty]\times L^\infty,$$

endowed with the norm

$$\|(u,\eta,\theta,w)\|_X = \|u\|_{H^1}+\|\eta\|_{L^2}+\|\eta\|_{L^\infty}+\|\theta\|_{L^2}+\|\theta\|_{L^\infty}+\|w\|_{L^\infty}.$$

In the dissipative case, we need modify the system (3.12) suitably. Suppose that, along a given characteristic $t\to q(t,\xi)$, the wave breaks at a first time $t=\tau(\xi)$. As $t\uparrow\tau(\xi)$, the variable $\theta=2\operatorname{arcsec}u_x$ implies that $u_x(t,\xi)\to-\infty$. For all $t\ge\tau$, we set $\theta(t,\xi)\equiv\pi$. Then the P and P_x in (3.8) are replaced by

$$\begin{aligned}P(\xi) = \frac12\int_{\{\theta(\xi')\neq\pi\}} &\exp\Big\{-\Big|\int_{\{s\in[\xi,\xi'],\theta(s)\neq\pi\}}\cos^2\frac{\theta(s)}{2}\cdot w(s)ds\Big|\Big\}\\ &\times(u^2+\frac12u_x^2-Au+\frac12\eta^2+\eta)\cos^2\frac{\theta}{2}\cdot w(\xi')d\xi'\end{aligned} \tag{3.14}$$

and

$$
\begin{aligned}
P_x(\xi) = \frac{1}{2}\int_{\{\xi'>\xi,\theta(\xi')\neq\pi\}} & \exp\Big\{-\Big|\int_{\{s\in[\xi,\xi'],\theta(s)\neq\pi\}} \cos^2\frac{\theta(s)}{2}\cdot w(s)ds\Big|\Big\} \\
& \times (u^2+\frac{1}{2}u_x^2-Au+\frac{1}{2}\eta^2+\eta)\cdot\cos^2\frac{\theta}{2}\cdot w(\xi')d\xi' \\
& -\frac{1}{2}\int_{\{\xi'<\xi,\theta(\xi')\neq\pi\}} \exp\Big\{-\Big|\int_{\{s\in[\xi,\xi'],\theta(s)\neq\pi\}} \cos^2\frac{\theta(s)}{2}\cdot w(s)ds\Big|\Big\} \\
& \times (u^2+\frac{1}{2}u_x^2-Au+\frac{1}{2}\eta^2+\eta)\cdot\cos^2\frac{\theta}{2}\cdot w(\xi')d\xi',
\end{aligned}
\tag{3.15}
$$

System (3.12) can thus be rewritten in the form

$$
\begin{gathered}
\frac{\partial u}{\partial t} = -P_x, \\
\frac{\partial \eta}{\partial t} = \begin{cases} -(\eta+1)\sec\frac{\theta}{2} & \text{if } \theta\neq\pi \\ 0 & \text{if } \theta=\pi, \end{cases} \\
\frac{\partial \theta}{\partial t} = \begin{cases} -\csc\frac{\theta}{2} + (2u^2-2Au+\eta^2+2\eta-2P)\cos\frac{\theta}{2}\cdot\cot\frac{\theta}{2} & \text{if } \theta\neq\pi \\ 0 & \text{if } \theta=\pi, \end{cases} \\
\frac{\partial w}{\partial t} = \begin{cases} (2u^2-2Au+\eta^2+2\eta-2P)\cos\frac{\theta}{2}\cdot w & \text{if } \theta\neq\pi \\ 0 & \text{if } \theta=\pi. \end{cases}
\end{gathered}
\tag{3.16}
$$

where the right hand side is now discontinuous. The discontinuity occurs precisely when $\theta=\pi$.

4. Global solutions of the equivalent semilinear system

A unique local solution of the equivalent semilinear system defined on some time interval $[0,T]$ is first obtained, and then it is proved that this local solution can be globally extended for all times $t\geq 0$. Denote

$$
U=(u,\eta,\theta,w)\in\mathbb{R}^4,
$$

$$
F(U)=\begin{cases} \Big(0,\ -(\eta+1)\sec\frac{\theta}{2},\ -\csc\frac{\theta}{2}+(2u^2-2Au+\eta^2+2\eta)\cos\frac{\theta}{2} & \\ \times\cot\frac{\theta}{2}, (2u^2-2Au+\eta^2+2\eta)\cos\frac{\theta}{2}\cdot w\Big) & \text{if } \theta\neq\pi, \\ (0,0,0,0) & \text{if } \theta=\pi, \end{cases}
$$

$$
G(\xi,U(\cdot))=\begin{cases} \Big(-P_x, 0, -2P\cos\frac{\theta}{2}\cdot\cot\frac{\theta}{2}, -2P\cos\frac{\theta}{2}\cdot w\Big) & \text{if } \theta\neq\pi, \\ (-P_x,0,0,0) & \text{if } \theta=\pi. \end{cases}
$$

The Cauchy problem for (3.16) is rewritten in more compact form with this notation,

$$
\frac{\partial}{\partial t}U(t,\xi)=F(U(t,\xi))+G(\xi,U(t,\cdot)),\quad \xi\in\mathbb{R}
\tag{4.1}
$$

with the initial condition

$$
U(0,\xi)=\bar{U}(\xi).
$$

After a solution (u,η,θ,w) of (4.1) is obtained, a solution of (3.16) will be soon provided by the mapping $(t,\xi)\to(u,\eta,\theta,w)$. We regard (4.1) as an ODE on the space $L^\infty(\mathbb{R},\mathbb{R}^4)$. Observe that the vector field $F:R^4\to R^4$ is uniformly bounded and Lipschitz continuous as long as u, η remain in a bounded set. However, the nonlocal operator G is discontinuous.

To prove the unique local solution of the system (4.1), we begin with some assumptions.

Assumption 1. Suppose F and G are given in (4.1), there exists some constant $C > 0$ and constant $\kappa^* > 0$ depending only on C such that, for $U = (u, v, \theta, w) \in L^\infty(\mathbb{R}, \mathbb{R}^4)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{w}) \in L^\infty(\mathbb{R}, \mathbb{R}^4)$, the following inequalities hold:

$$\|u\|_{L^\infty}, \|\eta\|_{L^\infty}, \|\tilde{u}\|_{L^\infty}, \|\tilde{\eta}\|_{L^\infty} \le C, \quad \frac{1}{C} \le w(\xi), \quad \tilde{w}(\xi) \le C, \tag{4.2}$$

$$\operatorname{meas}(\{\xi; \theta(\xi) \ne \pi, |\theta(\xi) - \pi| \ge \frac{\pi}{2}\}) \le C, \tag{4.3}$$

$$\operatorname{meas}(\{\xi; \tilde{\theta}(\xi) \ne \pi, |\tilde{\theta}(\xi) - \pi| \ge \frac{\pi}{2}\}) \le C, \tag{4.4}$$

$$\|P\|_{L^\infty} + \|P_x\|_{L^\infty} \le \kappa^*, \tag{4.5}$$

$$\|P\|_{L^1} + \|P_x\|_{L^1} \le \kappa^*(1 + \|u\|_{L^1} + \|\eta\|_{L^1} + \|\theta\|_{L^1}), \tag{4.6}$$

$$\|F(U)\|_{L^\infty}, \|G(U)\|_{L^\infty} \le \kappa^*, \tag{4.7}$$

$$\|F(U) - F(\tilde{U})\|_{L^\infty} \le \kappa \|U - \tilde{U}\|_{L^\infty}, \tag{4.8}$$

$$\begin{aligned} &\|G(U) - G(\tilde{U})\|_{L^\infty} \\ &\le \kappa\big[\|U - \tilde{U}\|_{L^\infty} + \operatorname{meas}(\{\xi; \theta \ne \pi, \tilde{\theta} = \pi\}) + \operatorname{meas}(\{\xi; \tilde{\theta} \ne \pi, \theta = \pi\})\big], \end{aligned} \tag{4.9}$$

where κ is a Lipschitz constant.

Assumption 2. Given initial data $\bar{z} = (\bar{u}, \bar{\eta}) \in H^1 \times L^2$, there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty}, \|\eta\|_{L^\infty} \le \frac{C}{2}, \quad \operatorname{meas}(\{\xi; \theta(\xi) \ne \pi, |\theta(\xi) - \pi| \ge \frac{\pi}{4}\}) \le \frac{C}{2}.$$

Define the set $\Omega^\delta = \{\xi \in \mathbb{R}; \bar{\theta}(\xi) \in (\pi, \pi + \delta]\ \}$, where $\delta > 0$ is a constant small enough. By possibly reducing the size of $\delta > 0$, thus we can assume that $\operatorname{meas}(\Omega^\delta) \le 1/(8\kappa)$.

Given $T > 0$, let D be the set of all continuous mappings $t \to U(t) : [0, T] \to L^\infty(R, R^4)$, with the following properties:

$$\begin{gathered} U(0) = \bar{U}, \\ \|U(t) - U(s)\|_{L^\infty} \le 2k^*|t - s|, \\ \theta(t, \xi) - \theta(t, \xi) \le -\frac{t - s}{2}, \quad \xi \in \Omega^\delta, \ 0 \le s < t \le T. \end{gathered}$$

Let $\Pi : D \to D$ be defined by

$$(\Pi(U))(t, \xi) = \bar{U} + \int_0^t [F(U(\tau, \xi)) + G(\xi, U(\tau, \cdot))] d\tau, \tag{4.10}$$

then a solution $t \to U(t)$ will be obtained as the unique fixed point of the contractive transformation $\Pi : D \to D$.

With assumptions 1-2 and the definition of D, we are ready to prove the existence and uniqueness of a local solution for Cauchy problem (4.1).

Theorem 4.1. *Given $\bar{z} = (\bar{u}\ , \ \bar{\eta}) \in H^1 \times L^2$, the Cauchy problem* (4.1) *has a unique local solution defined on a time interval* $[0, T]$ *with* $T > 0$.

Proof. We first show that $\Pi : D \to D$ defined above is a strict contraction. Choose $T > 0$ sufficiently small and $U, \tilde{U} \in D$. Define

$$\lambda = \max_{t\in[0,T]} \|U(t) - \tilde{U}(t)\|_{L^\infty}, \quad \tau(\xi) = \sup_{t\in[0,T]} \{t; \theta(t,\xi) \neq \pi\},$$
$$\tilde{\tau}(\xi) = \sup_{t\in[0,T]} \{t; \tilde{\theta}(t,\xi) \neq \pi\}$$

For each $\xi \in \Omega^\delta$, we have $|\tau(\xi) - \tilde{\tau}(\xi)| \leq 2\lambda$. For $t \in [0,T]$, we have

$$\begin{aligned}
&\|\Pi U(t) - \Pi\tilde{U}(t)\|_{L^\infty} \\
&\leq \int_0^t \|F(U(\tau)) - F(\tilde{U}(\tau))\|_{L^\infty} d\tau + \int_0^t \|G(U(\tau)) - G(\tilde{U}(\tau))\|_{L^\infty} d\tau \\
&\leq 2\kappa \int_0^t \|U(\tau) - \tilde{U}(\tau)\|_{L^\infty} d\tau + \kappa \int_0^t \operatorname{meas}(\{\xi; \theta \neq \pi, \tilde{\theta} = \pi\}) d\tau \\
&\quad + \kappa \int_0^t \operatorname{meas}(\{\xi; \tilde{\theta} \neq \pi, \theta = \pi\}) d\tau \\
&\leq 2T\kappa\lambda + \kappa \int_{\Omega^\delta} |\tau(\xi) - \tilde{\tau}(\xi)| d\xi \\
&\leq 2\kappa T\lambda + 2\kappa \operatorname{meas}(\Omega^\delta)\lambda \leq \frac{\lambda}{2},
\end{aligned}$$

where T is chosen as $T \leq 1/(8\kappa)$. This shows that Π is a strict contraction, which yields the desired local solution of Cauchy problem (4.1).

Next we show that the local solutions of the semilinear system (3.16) can be globally extended for all times $t \geq 0$. In the following, we prove that the "extended energy"

$$\tilde{E}(t) = \int_R \Big(u^2 \cos^2 \frac{\theta_1}{2} + \eta^2 \cos^2 \frac{\theta_1}{2} + 1\Big) w(t,\xi) d\xi$$

remains constant in time. We remark that the extended energy $\tilde{E}(t)$ is strictly larger than the total energy

$$E(t) = \int_{\{\theta(t,\xi)\neq\pi\}} \Big(u^2 \cos^2 \frac{\theta}{2} + \eta^2 \cos^2 \frac{\theta}{2} + 1\Big) w(t,\xi)(t,\xi) d\xi$$

in the sense that here the integration ranges over the entire real line.

For future use, we show the following identities

$$\begin{gathered}
u_\xi = u_x \cdot \frac{\partial y}{\partial \xi} = \sec \frac{\theta}{2} \cdot \cos^2 \frac{\theta}{2} \cdot w = \cos \frac{\theta}{2} \cdot w, \\
P_\xi = P_x \cdot \frac{\partial y}{\partial \xi} = P_x \cdot \cos^2 \frac{\theta}{2} \cdot w\,,
\end{gathered} \tag{4.11}$$

hold for all times $t \geq 0$, as long as the solution is defined. Moreover, when $\theta = \pi$, a separate computation yields

$$u_\xi = 0 = \cos \frac{\pi}{2} \cdot w, \quad P_\xi = 0 = P_x \cdot \cos^2 \frac{\pi}{2} \cdot w.$$

Thus the identity in (4.11) still holds for the cases $\theta = \pi$. Then we obtain

$$\begin{gathered}
(uP)_\xi = u_\xi P + uP_\xi = w(P \cdot \cos \frac{\theta}{2} + uP_x \cdot \cos^2 \frac{\theta}{2}), \\
(u^3)_\xi = 3u^2 u_\xi = 3wu^2 \cdot \cos \frac{\theta}{2},
\end{gathered}$$

$$(u^2)_\xi = 2uu_\xi = 2uw \cdot \cos\frac{\theta}{2}.$$

Differentiating the extended energy $\tilde{E}(t)$ with respect to the variable t, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_R \tilde{E}(t)d\xi = \frac{d}{dt}\int_R \Big(u^2\cos^2\frac{\theta}{2} + \eta^2\cos^2\frac{\theta}{2} + 1\Big) w d\xi \\
&= \int_R \Big[\Big(u^2\cos^2\frac{\theta}{2} + \eta^2\cos^2\frac{\theta}{2} + 1\Big)\frac{\partial w}{\partial t} + \Big(2uu_t\cdot\cos^2\frac{\theta}{2} - u^2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\frac{\partial\theta}{\partial t}\Big)w \\
&\quad + \Big(2\eta\eta_t\cdot\cos^2\frac{\theta}{2} - \eta^2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\frac{\partial\theta}{\partial t}\Big)w\Big]d\xi \\
&= \int_{\{\theta(\xi)\neq\pi\}} \Big\{2\Big(u^2\cos^2\frac{\theta}{2} + \eta^2\cos^2\frac{\theta}{2} + 1\Big)\Big(u^2 - Au + \frac{\eta^2}{2} + \eta - P\Big) \\
&\quad - 2uP_x\cdot\cos\frac{\theta}{2} - 2\eta(\eta+1) - (u^2+\eta^2)\sin\frac{\theta}{2} \\
&\quad \times\big[-\csc\frac{\theta}{2} + (2u^2 - 2Au + \eta^2 + 2\eta - 2P)\cos\frac{\theta}{2}\cdot\cot\frac{\theta}{2}\big]\Big\}\cos\frac{\theta}{2} w d\xi \\
&= \int_R w\Big\{3u^2\cos\frac{\theta}{2} - 2Au\cos\frac{\theta}{2} - 2P\cos\frac{\theta}{2} - 2uP_x\cos^2\frac{\theta}{2}\Big\}d\xi \\
&= \int_R \partial_\xi(u^3 - Au^2 - 2uP)d\xi = 0.
\end{aligned} \tag{4.12}$$

In the sense that $\cos\frac{\theta}{2} = 0$ whenever $\theta = \pi$, thus we are again integrating over the entire real line R on the fourth identity of (4.12). This implies that the extended energy $\tilde{E}(t)$ is consistent, namely

$$\tilde{E}(t) = \int_R \Big(u^2\cos^2\frac{\theta_1}{2} + \eta^2\cos^2\frac{\theta_1}{2} + 1\Big)w(t,\xi)d\xi = \tilde{E}(0) = E_0. \tag{4.13}$$

From (4.11) and (4.13), we can obtain the bound

$$\sup_{\xi\in\mathbb{R}}|u^2(t,\xi)| \le 2\int_R |uu_\xi|d\xi \le 2\int_R |u|\cdot|\cos\frac{\theta}{2}|wd\xi \le E_0. \tag{4.14}$$

This provides a priori bound on $\|u(t)\|_{L^\infty}$, similarly we can derive an a priori bound on $\|\eta(t)\|_{L^\infty}$. Also from the estimation (4.13) and the definitions (3.13) and (3.14), we obtain

$$\begin{aligned}
&\|P(t)\|_{L^\infty}, \|P_x(t)\|_{L^\infty} \\
&\le \|G\|_{L^\infty}\|\big(u^2 + \frac{1}{2}u_x^2 + \eta^2\big)(t)\|_{L^1} + \frac{A}{2}\big(\|G\|_{L^2}^2 + \|u\|_{L^2}^2\big) + \frac{1}{2}\big(\|G\|_{L^2}^2 + \|\eta\|_{L^2}^2\big) \\
&\le \frac{1}{2}E_0 + \frac{A}{2}(\frac{1}{4} + E_0) + \frac{1}{2}(\frac{1}{4} + E_0) \\
&\le \frac{2+A}{2}E_0 + \frac{A+1}{8}.
\end{aligned} \tag{4.15}$$

Then by (4.14), (4.15) and the fourth equation in (3.16), we deduce that there exists a constant B, depending only on the total energy E_0, such that

$$|\frac{\partial w}{\partial t}| \le Bw,$$

which yields

$$e^{-Bt} \le w(t) \le e^{Bt}.$$

From the third equation in (3.16), we know that $0 \le \theta(t) \le 2\pi$.

All the above estimates show that a priori bounds (4.2)-(4.5) which we needed to construct a local solution with a constant C exist over any given time interval $[0, T]$. This completes the proof that the local solution can be extended globally for all times $t \ge 0$. □

5. Global dissipative solutions for the two-component Camassa-Holm system

In this section, we show that the global solution of the system (3.16) yields a global dissipative solution of system (2.1), in the original variables (t, x). In the following, we shall show the continuous dependence of solutions to system (2.1). Recalling that we have obtained the local existence theorem by representing the solution of (4.1) as the fixed point of a contraction in a suitable space, this yields uniqueness and continuous dependence with respect to convergence on the initial data in $L^\infty \times L^\infty$.

Theorem 5.1. *Let $\bar{z}_n = (\bar{u}_n, \bar{\eta}_n)$ be a sequence of initial data with $\|\bar{z}_n - \bar{z}\|_{H^1} \to 0$. Then, for any $T > 0$, the corresponding solutions $z_n(t, \xi) = (u_n, \eta_n)(t, \xi)$ converge to $z(t, \xi) = (u, \eta)(t, \xi)$ uniformly with $(t, \xi) \in [0, T] \times R$.*

Proof. Let (u, η, θ, w) and $(\tilde{u}, \tilde{\eta}, \tilde{\theta}, \tilde{w})$ be any two solutions of (3.16), with the initial condition (3.13). Let E_0 be an upper bound for the energies of the two solutions. Suppose that at time $t = 0$, there exists a constant δ_0,

$$\|z(0) - \tilde{z}(0)\|_{L^\infty} \le \delta_0, \quad \|\theta(0, \xi) - \tilde{\theta}(0, \xi)\|_{L^2} \le \delta_0.$$

Next, for $t \in [0, T]$, we will establish an a-priori bound depending only on δ_0, T and E_0 on

$$\|z(t) - \tilde{z}(t)\|_{L^\infty}. \tag{5.1}$$

Define the set

$$\Lambda = \{\xi \in \mathbb{R}; \theta(T, \xi) = \pi\} \cup \{\xi \in \mathbb{R}; \tilde{\theta}(T, \xi) = \pi\},$$

thus $\alpha^* = \mathrm{meas}(\Lambda)$ is a uniformly bounded number depending only on T and E_0.

Let $\tau(\xi) = \inf\{t \in [0, T]; \min\{\theta(t, \xi), \tilde{\theta}(t, \xi)\} = \pi\}$ such that $\tau(\xi)$ is the first time when one of the two solutions reaches the value π. We now construct a measure-preserving mapping: $[0, \alpha^*] \to \Lambda$, which is denoted as $\alpha \to \xi(\alpha)$ with the additional property:

$$\alpha \le \alpha' \text{ if and only if } \tau(\xi(\alpha)) \ge \tau(\xi(\alpha')). \tag{5.2}$$

According to the mapping $[0, \alpha^*] \to \Lambda$, we define the distance function

$$\begin{aligned} &J((u, \eta, \theta, w), (\tilde{u}, \tilde{\eta}, \tilde{\theta}, \tilde{w})) \\ &= (\|u - \tilde{u}\|_{L^\infty} + \|\eta - \tilde{\eta}\|_{L^\infty} + \|\theta - \tilde{\theta}\|_{L^2} + \|w - \tilde{w}\|_{L^2}) \\ &\quad + K_0 \int_0^{\alpha^*} e^{K\alpha} (|\theta(\xi(\alpha)) - \tilde{\theta}(\xi(\alpha))|) d\alpha. \end{aligned} \tag{5.3}$$

For convenience, we set

$$J(t) = J((u, v, \theta, w), (\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{w}))(t) = J^*(t) + K_0 J^{\#}(t), \tag{5.4}$$

where

$$J^*(t) = \Big(\|u - \tilde{u}\|_{L^\infty} + \|\eta - \tilde{\eta}\|_{L^\infty} + \|\theta - \tilde{\theta}\|_{L^2} + \|w - \tilde{w}\|_{L^2}\Big),$$

$$J^{\#}(t)=\int_0^{\alpha^*} e^{K\alpha}\big(|\theta(\xi(\alpha))-\tilde\theta(\xi(\alpha))|\big)d\alpha. \tag{5.5}$$

In the following we show that, for suitable constants K_0, K, M depending only on T and E_0, the inequality

$$\frac{d}{dt}J(t)\le MJ(t) \tag{5.6}$$

holds. Moreover, this will imply

$$J(t)\le e^{Mt}J(0),\quad t\in[0,T],$$

which provides an a-priori estimate on the distance at (5.1).

For each fixed $t\in[0,T]$, we define the sets

$$\begin{gathered}\Gamma(t)=\{\xi\in\Lambda:\theta(t,\xi)\ne\pi,\tilde\theta(t,\xi)=\pi\}\cup\{\xi\in\Lambda:\tilde\theta(t,\xi)\ne\pi,\theta(t,\xi)=\pi\},\\ \Gamma^+(t)=\{\xi\in\Lambda:\theta(t,\xi)=\tilde\theta(t,\xi)=\pi\},\\ \Gamma^-(t)=\{\xi\in\Lambda:\theta(t,\xi),\tilde\theta(t,\xi)\ne\pi\}=\{\xi\in\Lambda:\tau(t)>t\},\end{gathered}$$

with the following properties

$$\Gamma(t)\cap\Gamma^+(t)=\Gamma(t)\cap\Gamma^-(t)=\Gamma^+(t)\cap\Gamma^-(t)=\Phi,\quad \Gamma(t)\cup\Gamma^+(t)\cup\Gamma^-(t)=\Lambda$$

for each $t\in[0,T]$. Set $m(t)=\operatorname{meas}(\Gamma^-(t))$, such that

$$\Gamma^-(t)=\{\xi(\alpha);\alpha\in[0,m(t)]\}. \tag{5.7}$$

From the equations in (3.16), we have the estimate

$$\begin{aligned}&\frac{d}{dt}\Big(\|u-\tilde u\|_{L^\infty}+\|\eta-\tilde\eta\|_{L^\infty}+\|\theta-\tilde\theta\|_{L^2}+\|w-\tilde w\|_{L^2}\Big)\\ &\le\kappa\Big(\|u-\tilde u\|_{L^\infty}+\|\eta-\tilde\eta\|_{L^\infty}+\|\theta-\tilde\theta\|_{L^2}+\|w-\tilde w\|_{L^2}+\operatorname{meas}(\Gamma(t))\Big).\end{aligned} \tag{5.8}$$

Moreover, from (5.7) we can deduce that

$$\begin{aligned}&\frac{d}{dt}\int_0^{\alpha^*} e^{K\alpha}(|\theta(t,\xi(\alpha))-\tilde\theta(t,\xi(\alpha))|)d\alpha\\ &=\int_{\Gamma(t)\cup\Gamma^+(t)\cup\Gamma^-(t)} e^{K\alpha(\xi)}\cdot\frac{\partial}{\partial t}(|\theta(t,\xi(\alpha))-\tilde\theta(t,\xi(\alpha))|)d\alpha\\ &=\int_{\Gamma(t)} e^{K\alpha(\xi)}\cdot\frac{\partial}{\partial t}(|\theta(t,\xi(\alpha))-\tilde\theta(t,\xi(\alpha))|)d\xi\\ &\quad+\int_0^{m(t)} e^{K\alpha(\xi)}\cdot\frac{\partial}{\partial t}(|\theta(t,\xi(\alpha))-\tilde\theta(t,\xi(\alpha))|)d\alpha.\end{aligned} \tag{5.9}$$

Indeed, the integral over $\Gamma^+(t)$ is zero.

Choosing $\delta>0$ which depends only on T, E_0 sufficiently small, we have

$$|\theta(t,\xi)-\tilde\theta(t,\xi)|\le\delta$$

for $\xi\in\Gamma(t)$, which implies

$$\frac{\partial}{\partial t}|\theta(t,\xi)-\tilde\theta(t,\xi)|\le-\frac12.$$

On the other hand, choosing a constant κ large enough such that $|\theta(t,\xi)-\tilde\theta(t,\xi)|\ge\delta$, we obtain

$$\frac{\partial}{\partial t}|\theta(t,\xi)-\tilde\theta(t,\xi)|\le-\frac12+\kappa|\theta(t,\xi)-\tilde\theta(t,\xi)|.$$

Finally, for $\xi \in \Gamma^-(t)$, we have

$$\frac{\partial}{\partial t}|\theta(t,\xi)-\tilde{\theta}(t,\xi)| \le \kappa\cdot\big(\|u-\tilde{u}\|_{L^\infty}+\|\eta-\tilde{\eta}\|_{L^\infty}+\|\theta-\tilde{\theta}\|_{L^2}+\|w-\tilde{w}\|_{L^2} + \operatorname{meas}(\Gamma(t))+|\theta(t,\xi)-\tilde{\theta}(t,\xi)|\big).$$

Therefore,

$$\begin{aligned}
&\int_0^{m(t)} e^{K\alpha}\cdot\frac{\partial}{\partial t}(|\theta(t,\xi(\alpha))-\tilde{\theta}(t,\xi(\alpha))|)d\alpha\\
&\le \kappa(J^*(t)+\operatorname{meas}(\Gamma(t)))\int_0^{m(t)} e^{K\alpha}d\alpha+\kappa\int_0^{m(t)} e^{K\alpha}\Big(|\theta(t,\xi(\alpha))-\tilde{\theta}(t,\xi(\alpha))|\Big)d\alpha\\
&\le \kappa(J^*(t)+\operatorname{meas}(\Gamma(t)))\int_0^{m(t)} e^{K\alpha}d\alpha+\kappa\int_{\Gamma^-(t)} e^{K\alpha(\xi)}\big(|\theta(t,\xi)-\tilde{\theta}(t,\xi)|\big)d\xi.
\end{aligned} \tag{5.10}$$

Now, (5.8) can be rewritten in the form

$$\frac{d}{dt}J^*(t)\le \kappa\cdot(J^*(t)+\operatorname{meas}(\Gamma(t))). \tag{5.11}$$

Notice that $\xi\in\Gamma(t)$ implies $\alpha(\xi)\ge m(t)$, together (5.9) and (5.10) imply

$$\begin{aligned}
\frac{d}{dt}J^{\#}(t) &\le -\frac{1}{2}\int_{\Gamma(t)} e^{K\alpha(\xi)}d\xi+\kappa\int_{\Gamma(t)\cup\Gamma^-(t)} e^{K\alpha(\xi)}(|\theta(t,\xi)-\tilde{\theta}(t,\xi)|)d\xi\\
&\quad+\kappa(J^*(t)+\operatorname{meas}(\Gamma(t)))\cdot\int_0^{m(t)} e^{K\alpha}d\alpha\\
&\le -\frac{1}{2}e^{Km(t)}\operatorname{meas}(\Gamma(t))+\kappa J^{\#}(t)+\kappa J^*(t)\int_0^{\alpha^*} e^{K\alpha}d\alpha\\
&\quad+\kappa\operatorname{meas}(\Gamma(t))e^{Km(t)}\int_0^{m(t)} e^{K(\alpha-m(t))}d\alpha\\
&\le -\frac{1}{4}e^{Km(t)}\operatorname{meas}(\Gamma(t))+\kappa J^{\#}(t)+\frac{\kappa}{K}e^{K\alpha^*}J^*(t).
\end{aligned} \tag{5.12}$$

We choose the constant $K=4\kappa$ in the above inequality such that

$$\kappa\int_0^{m(t)} e^{K(\alpha-m(t))}d\alpha\le\frac{\kappa}{K}=\frac{1}{4}.$$

From (5.11) and (5.12), choosing $K_0=4k$, we obtain

$$\begin{aligned}
&\frac{d}{dt}(J^*(t)+4\kappa J^{\#}(t))\\
&\le \kappa\cdot(J^*(t)+\operatorname{meas}(\Gamma(t)))+4\kappa(-\frac{1}{4}\operatorname{meas}(\Gamma(t))+\kappa J^{\#}(t)+\frac{\kappa}{K}e^{K\alpha^*}J^*(t))\\
&\le \kappa J^*(t)+4\kappa^2J^{\#}(t)+\kappa e^{4\kappa\alpha^*}J^*(t),
\end{aligned}$$

with J^* and $J^{\#}$ are defined in (5.5). With $M=\kappa+\kappa e^{4\kappa\alpha^*}$, our claim (5.6) is satisfied. □

Next we revert to the original variables (t,x), and show that the global solution of system (3.16) yields a global dissipative solution of the original system (2.1).

Let us begin with a global solution (u, η, θ, w) of system (3.16). Define

$$q(t,\xi) = \bar{q}(\xi) + \int_0^t u(\tau,\xi)d\tau. \tag{5.13}$$

Then for each fixed ξ, the function $t \mapsto q(t,\xi)$ provides a solution to the Cauchy problem

$$\begin{gathered} \frac{\partial}{\partial t} q(t,\xi) = u(t,\xi), \\ q(0,\xi) = \bar{q}(\xi). \end{gathered} \tag{5.14}$$

We claim that, if $q(t,\xi) = x$, a solution of system (2.1) can be obtained by setting

$$z(t,x) = z(t,\xi), \tag{5.15}$$

where $z(t,x) = (u,\ \eta)(t,x)$, $z(t,\xi) = (u,\ \eta)(t,\xi)$. The main result reads as follows.

Theorem 5.2. *If* (u, η, θ, w) *is a global solution to the Cauchy problem* (3.16)-(3.13)*, then the function* $z = z(t,x)$ *defined by* (5.12)-(5.15) *provides a global dissipative solution of system* (2.1)*.*

Proof. Using the uniform bound $|u(t,\xi)| \le E_0^{1/2}$, from (5.12) we obtain

$$\bar{q}(\xi) - E_0^{1/2} t \le q(t,\xi) \le \bar{q}(\xi) + E_0^{1/2} t, \quad t \ge 0.$$

The definition of ξ in (3.2) yields

$$\lim_{\xi \to \pm\infty} \bar{q}(t,\xi) = \pm\infty.$$

Then the image of the continuous map $(t,\xi) \to (t, q(t,\xi))$ covers the entire plane $[0,\infty] \times R$. It is clear that the map $\xi \mapsto q(t,\xi)$ is non-decreasing. Then the map $(t,x) \mapsto z(t,x)$ at (5.15) is well defined, for all $t \ge 0$ and $x \in \mathbb{R}$.

For every fixed t, we claim

$$\begin{aligned} \operatorname{meas}(\{q(t,\xi); \theta(t,\xi) = \pi\}) &= \int_{\{\theta(t,\xi)=\pi\}} q_\xi(t,\xi)d\xi \\ &= \int_{\{\theta(t,\xi)=\pi\}} w \cos^2 \frac{\theta}{2}(t,\xi)d\xi = 0, \end{aligned}$$

which implies that, in the x-variable, the image of the singular set, where $\theta = \pi$, has measure zero.

By changing the variable of integration, we compute

$$\begin{aligned} &\int_R (u^2 + u_x^2 + \eta^2)(t,x)dx \\ &= \int_{\{\cos\theta \ne -1\}} (u^2 \cos^2 \frac{\theta}{2} + \eta^2 \cos^2 \frac{\theta}{2} + 1) w(t,\xi)d\xi \le E_0. \end{aligned} \tag{5.16}$$

This implies the uniform Hölder continuity with exponent $1/2$ of z as a function of x. From the first and second equations in (3.16) and the bounds on $\|P\|_{L^\infty}$, $\|P_x\|_{L^\infty}$, we can obtain that the map $t \mapsto z(t, q(t))$ is uniformly Lipschitz continuous along the characteristic curve $t \mapsto q(t)$. Hence, $z = z(t,x)$ is globally Hölder continuous for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

We know that the map $t \to z(t)$ is Lipschitz continuous with values in $L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is a reflexive space, we can deduce that the map $t \mapsto q(t)$ is differentiable for almost every (a.e.) time $t \ge 0$. Note that the right hand side of the first equation in (2.7) lies in $L^2(\mathbb{R})$, to establish the equality, one may proceed as follows.

For each smooth function with compact support $\varphi \in C_c^\infty$, at a. e. time $t \geq 0$, we have

$$\begin{aligned} &\frac{d}{dt}\int u(t,x)\varphi(x)dx = \int (-uu_x - P_x)(t,x)\varphi(x)dx \\ &= \int [u^2(t,x)\varphi'(x) - P_x(t,x)\varphi(x) + u(t,x)u_x(t,x)\varphi(x)]dx. \end{aligned} \tag{5.17}$$

Let us set

$$\tau(\xi) = \inf\{t > 0; \theta(t) = \pi\} \tag{5.18}$$

for each $\xi \in \mathbb{R}$. Note that, for a. e. time $t \geq 0$

$$\operatorname{meas}(\{\xi; \tau(\xi) = t\}) = 0. \tag{5.19}$$

Choosing a time t such that (5.19) holds. Integrating with respect to the variable ξ and thus we obtain from (3.5) that

$$\begin{aligned} &\frac{d}{dt}\int u(t,\xi)\varphi(q(t,\xi))(w\cdot\cos^2\frac{\theta}{2})(t,\xi)d\xi \\ &= \int \{u_t\varphi w\cos^2\frac{\theta}{2} + u\varphi' q_t w\cos^2\frac{\theta}{2} + u\varphi w_t\cos^2\frac{\theta}{2} - u\varphi w\theta_t\frac{\sin\theta}{2}\}d\xi \\ &= \int_{\theta(t,\xi)\neq\pi} \Big\{-P_x\varphi w\cos^2\frac{\theta}{2} + u^2\varphi' w\cos^2\frac{\theta}{2} + u\varphi(2u^2 - 2Au + \eta^2 + 2\eta - 2P) \\ &\quad\times\cos\frac{\theta}{2}w\cos^2\frac{\theta}{2} - u\varphi w\big[-\csc\frac{\theta}{2} + (2u^2 - 2Au + \eta^2 + 2\eta - 2P) \\ &\quad\cos\frac{\theta}{2}\cot\frac{\theta}{2}\big]\frac{\sin\theta}{2}\Big\}d\xi \\ &= \int_{\theta(t,\xi)\neq\pi} [-P_x\varphi + u^2\varphi' + uu_x\varphi]w\cos^2\frac{\theta}{2}d\xi. \end{aligned}$$

It can be seen (5.17) holds, and therefore we can conclude that $z = (u,\eta)$ is a global solution of the two-component Camassa-Holm system in the sense of Definitions 2.1 and 2.2. $\square$

Next we prove that global dissipative solutions of the two-component Camassa-Holm system (2.1) construct a semigroup. To do this some relevant properties are first given.

Property 1. The total energy is a non-increasing function of time, namely $\|z(t)\|_{H^1(\mathbb{R})} \leq \|z(t')\|_{H^1(\mathbb{R})}$ if $0 \leq t' \leq t$.

Proof. By (5.16), we have

$$\begin{aligned} \|z(t)\|_{H^1} &= \int_R (u^2 + u_x^2 + \eta^2)(t,x)dx \\ &= \int_{\{\cos\theta\neq-1\}} \Big(u^2\cos^2\frac{\theta}{2} + \eta^2\cos^2\frac{\theta}{2} + 1\Big)w(t,\xi)d\xi \\ &= E_0 - \int_{\{\tau(\xi)\leq t\}} \Big(u^2\cos^2\frac{\theta}{2} + \eta^2\cos^2\frac{\theta}{2} + 1\Big)w(t,\xi)d\xi \\ &\leq E_0 - \int_{\{\tau(\xi)\leq t\}} w(t,\xi)d\xi \end{aligned}$$

$$\leq E_0 - \int_{\{\tau(\xi)\leq t'\}} w(t,\xi)d\xi = \|z(t')\|_{H^1}$$

with $\tau(\xi)$ given in (5.18). $\square$

Property 2. Given a sequence of initial data $\bar{z}_n$ such that $\bar{z}_n \to \bar{z}$ in $H^1 \times L^2$, the corresponding solutions $z_n(t,x) \to z(t,x)$ uniformly for (t,x) in bounded sets.

Theorem 5.3. *Let $\bar{z}_n \in H^1 \times [L^2 \cap L^\infty]$ be an initial data, and $z(t) = S_t\bar{z}$ the corresponding global solution of system* (2.1) *constructed in Theorem 5.2. Then the mapping S: $H^1 \times [L^2 \cap L^\infty] \times [0,\infty) \to H^1$ is a semigroup.*

Proof. Let $(t,\xi) \to (u,\eta,\theta,w)(t,\xi)$ be the corresponding solutions to systems (3.16) and (3.13). For the fixed $\tau > 0$ and all $t \in \mathbb{R}^+$, one needs to prove that

$$S_t(S_\tau\bar{z}) = S_{\tau+t}\bar{z}.$$

A new energy variable p is defined as

$$\frac{d}{d\xi}p(\xi) = \begin{cases} w(\tau,\xi) & \text{if } \theta(\tau,\xi) \neq \pi, \\ 0 & \text{if } \theta(\tau,\xi) = \pi, \end{cases} \tag{5.20}$$

with initial data

$$p(\xi_0) = 0. \tag{5.21}$$

Choose the value ξ_0 such that $q(\tau,\xi_0) = 0$. By setting $\hat{z} = S_\tau\bar{z}$, we define

$$\begin{aligned} \hat{z}(t,p) &= z(\tau+t,\xi(p)), \\ \hat{\theta}(t,p) &= \theta(\tau+t,\xi(p)), \\ \hat{w}(t,p) &= \frac{w(\tau+t,\xi(p))}{w(\tau,\xi(p))}, \end{aligned} \tag{5.22}$$

such that $p \to \xi(p)$ provides an a.e. inverse to the mapping in (5.20)-(5.21), namely,

$$\xi(p^*) = \sup\{s; p(s) \leq p^*\}.$$

Recalling the identities (3.6) and (3.5), one has

$$\frac{\partial}{\partial\xi}q(\tau,\xi)\cdot u_x^2(\tau,q(\tau,p(\xi))) = w(\tau,\xi) = \frac{d}{d\xi}p(\xi).$$

By an integration and using (5.20), one gets that

$$\int_0^{q(\tau,\xi)} u_x^2(\tau,x)dx = p(\xi).$$

Now it can be shown that the functions in (5.22) provide a solution to system (3.16). The identities $w(\tau+t,\xi)d\xi = \frac{\hat{w}(t,p(\xi))}{w(t,p(\xi))}\cdot\frac{dp(\xi)}{d\xi}\cdot dp = \hat{w}(t,p(\xi))dp$ imply that the corresponding integral source terms in (3.16) satisfy

$$\hat{P}(t,p) = P(\tau+t,\xi(p)), \quad \hat{P}_x(t,p) = P_x(\tau+t,\xi(p)). \tag{5.23}$$

Since the last equation in (3.16) is linear with respect to the variable w, then we can draw the conclusion that the functions in (5.22) provide a solution to system (3.16). In summary, the global dissipative solutions of system (2.1) construct a semigroup. $\square$

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