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CAUCHY PROBLEMS FOR FIFTH-ORDER KDV EQUATIONS IN WEIGHTED SOBOLEV SPACES

EDDYE BUSTAMANTE, JOSÉ JIMÉNEZ, JORGE MEJÍA

Abstract. In this work we study the initial-value problem for the fifth-order Korteweg-de Vries equation

$$\partial_t u + \partial_x^5 u + u^k \partial_x u = 0, \quad x, t \in \mathbb{R},\ k = 1, 2,$$

in weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(\langle x\rangle^{2r} dx)$. We prove local and global results. For the case $k = 2$ we point out the relationship between decay and regularity of solutions of the initial-value problem.

1. Introduction

In this article we consider the initial-value problem (IVP)

$$\begin{gathered} \partial_t u + \partial_x^5 u + u^k \partial_x u = 0, \quad x, t \in \mathbb{R} \\ u(0) = u_0, \end{gathered} \tag{1.1}$$

with $k = 1, 2$. When $k = 1$ we refer to this problem as the IVP for the fifth-order Korteweg-de Vries (KdV) equation. When $k = 2$ we refer to this problem as the IVP for the modified fifth-order KdV equation.

For $k = 1$ the equation was proposed by Kakutani and Ono as a model for magneto-acoustic waves in plasma physics (see [11]). The equations that we study are included in the class

$$\partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \dots, \partial_x^{2j} u) = 0, \quad x, t \in \mathbb{R}, \quad j \in \mathbb{Z}^+, \tag{1.2}$$

where $P : \mathbb{R}^{2j+1} \to \mathbb{R}$ (or $P : \mathbb{C}^{2j+1} \to \mathbb{C}$) is a polynomial having no constant or linear terms, i.e.

$$P(z) = \sum_{|\alpha|=l_0}^{l_1} a_\alpha z^\alpha \quad \text{with } l_0 \ge 2 \text{ and } z = (z_1, \dots, z_{2j+1}).$$

The class in (1.2) generalizes several models, arising in both mathematics and physics, of higher-order nonlinear dispersive equations.

For many years the well-posedness of these IVP has been studied in the context of the classical Sobolev spaces $H^s(\mathbb{R})$. In particular, fifth-order KdV equations with more general non-linearities, than those we are considering, were studied in [6, 13, 20, 21, 22, 27]. In 1983 Kato [12] studied the IVP for the generalized KdV

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equation in several spaces, besides the classical Sobolev spaces. Among them, Kato considered weighted Sobolev spaces.

In this work we are concerned with the well-posedness of (1.1) in weighted Sobolev spaces. This type of spaces arises in a natural manner when we are interested in determining if the Schwartz space is preserved by the flow of the evolution equation in (1.1).

Kenig, Ponce and Vega [17] studied the IVP associated with equation (1.2) in weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^m dx)$, with m positive integer. Pilod [26] study the case of higher-order dispersive models in the context of weighted Besov and Sobolev spaces.

Some relevant nonlinear evolution equations as the KdV equation, the non-linear Schrödinger equation and the Benjamin-Ono equation, have also been studied in the context of weighted Sobolev Spaces (see [1, 2, 3, 4, 5, 7, 8, 10, 23, 24, 25] and references therein).

We study real valued solutions of (1.1) in the weighted Sobolev spaces

$$Z_{s,r} := H^s(\mathbb{R}) \cap L^2(\langle x\rangle^{2r} dx),$$

where $\langle x\rangle := (1+x^2)^{1/2}$, and $s, r \in \mathbb{R}$.

The relation between the indices s and r for (1.1) can be found, after the following considerations, contained in the work by Kato:

Suppose we have a solution $u \in C([0,\infty); H^s(\mathbb{R}))$ to (1.1) for some $s \geq 2$. We want to estimate (pu, u), where $p(x) := \langle x\rangle^{2r}$ and $(\cdot,\cdot)$ is the inner product in $L^2(\mathbb{R})$. Proceeding formally we multiply the equation in (1.1) by up, integrate over $x \in \mathbb{R}$ and apply integration by parts to obtain

$$\frac{d}{dt}(pu,u) = 5(p^{(1)}\partial_x^2 u, \partial_x^2 u) - 5(p^{(3)}\partial_x u, \partial_x u) + (p^{(5)}u, u) + \frac{2}{k+2}(p^{(1)}u^{k+2}, 1). \quad (1.3)$$

To see that (pu, u) is finite and bounded in t, we must bound the right-hand side in (1.3) in terms of (pu, u) and $\|u\|_{H^s}^2$. The most difficult term to control in the right-hand side in (1.3) is $5(p^{(1)}\partial_x^2 u, \partial_x^2 u)$. Using the interpolation Lemma 2.2 (see section 2), for $\theta \in [0,1]$ and $u \in Z_{s,r}$ we have

$$\|\langle x\rangle^{(1-\theta)r} u\|_{H^{\theta s}} \leq C \|\langle x\rangle^r u\|_{L^2}^{1-\theta} \|u\|_{H^s}^{\theta}.$$

The term $5(p^{(1)}\partial_x^2 u, \partial_x^2 u)$ can be controlled when $\theta s = 2$ if $p^{(1)}(x) \sim \langle x\rangle^{2(1-\theta)r}$. Taking into account that $p^{(1)}(x) \sim \langle x\rangle^{2r-1}$, we must require that $2r-1 = 2(1-\theta)r$ and $\theta s = 2$, which leads to $s = 4r$. In this way the natural weighted Sobolev space to study (1.1) is $Z_{4r,r}$.

Now, we describe the main results of this work. With respect to (1.1) with $k = 1$ we establish local well-posedness (LWP) in $Z_{4r,r}$ for $\frac{5}{16} < r < \frac{1}{2}$ and global well-posedness (GWP) in $Z_{4r,r}$, for $r \geq 1/2$.

In the first case ($\frac{5}{16} < r < \frac{1}{2}$), we use the known linear estimates for the group associated to the linear part of the equation, which were obtained by Kenig, Ponce and Vega in [14, 15, 16], and a pointwise formula for the group, related with fractional weights, which was deduced by Fonseca, Linares, and Ponce in [2]. These ingredients allow us to use a contraction principle in an adequate subspace of $C([0,T]; Z_{4r,r})$ to the integral equation associated to our IVP, to prove local well-posedness in $Z_{4r,r}$.

In the second case ($r \geq \frac{1}{2}$) we use the local well-posedness of (1.1) in the context of the Sobolev spaces $H^{4r}(\mathbb{R})$, which can be obtained in a similar fashion, as it was done by Kenig, Ponce and Vega in [15, 16] for the KdV equation, to get a solution

$u \in C([0,T]; H^{4r}(\mathbb{R}))$. Then we perform a priori estimates on the the differential equation in order to prove that if the initial data belongs to $H^{4r}(\mathbb{R}) \cap L^2(\langle x\rangle^{2r}dx)$ then necessarily $u \in L^\infty([0,T]; L^2(\langle x\rangle^{2r}dx)$. In this step of the proof we apply the interpolation inequality (Lemma 2.2), mentioned before, which was proved in [5]. Finally, we give the proof of the continuous dependence of the solution on the initial data in $Z_{4r,r}$.

With respect to (1.1) with $k = 2$, we establish local and global well-posedness in $Z_{2,1/2}$. For the LWP, again, the idea of the proof is to apply the contraction principle to the integral equation associated to the IVP, in a certain subspace of $C([0,T]; H^2(\mathbb{R}))$, in which we consider additional mixed space-time norms, suggested by the linear estimates of the group. This way, we obtain, firstly, a solution in $C([0,T]; H^2(\mathbb{R}))$. Then, proceeding as in (1.1) with $k = 1$, in the case $r \geq 1/2$, we can affirm that $u \in C([0,T]; Z_{2,1/2})$ and that (1.1) with $k = 2$ is local well-posed in $Z_{2,1/2}$.

To deduce global well-posedness results from local well-posedness results we use the following conservation laws for the solutions of (1.1) (see [14]):

$$I_1(t) := \int_{\mathbb{R}} u^2(t)dx = I_1(0), \quad \text{for } k = 1, 2, \tag{1.4}$$

$$I_2^1(t) := \frac{1}{6}\int_{\mathbb{R}} u^3(t)dx + \frac{1}{2}\int_{\mathbb{R}} (\partial_x^2 u)^2(t)dx = I_2^1(0), \quad \text{for } k = 1, \text{ and,} \tag{1.5}$$

$$I_2^2(t) := \frac{1}{12}\int_{\mathbb{R}} u^4(t)dx + \int_{\mathbb{R}} (\partial_x^2 u)^2(t)dx = I_2^2(0), \quad \text{for } k = 2. \tag{1.6}$$

Isaza, Linares and Ponce [9] showed that there exists a relation between decay and regularity for the solutions of the KdV equation in $L^2(\mathbb{R})$. More precisely, they proved that if $u \in C(\mathbb{R}; L^2(\mathbb{R}))$ is the global solution of the equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0,$$

obtained in the context of the Bourgain spaces (see [18]), and there exists $\alpha > 0$ such that in two different times $t_0, t_1 \in \mathbb{R}$

$$|x|^\alpha u(t_0), |x|^\alpha u(t_1) \in L^2(\mathbb{R}),$$

then $u \in C(\mathbb{R}, H^{2\alpha}(\mathbb{R}))$. To achieve this goal, they chose a functional setting, where the norm $\|\partial_x u\|_{L^\infty(\mathbb{R};L^2([0,T]))}$ of the solution u depends continuously on the initial data in $L^2(\mathbb{R})$.

Following [9], and taking into account that the norm $\|\partial_x^4 u\|_{L^\infty(\mathbb{R};L^2([0,T]))}$ of the solution u of (1.1) with $k = 2$, depends continuously on the initial data in $Z_{2,1/2}$, we prove that if $u \in C([0,T]; Z_{2,1/2})$ is a solution of (1.1) with $k = 2$ and, for some $\alpha > 0$, there exist two different times $t_0, t_1 \in [0,T]$ such that $|x|^{1/2+\alpha}u(t_0)$ and $|x|^{1/2+\alpha}u(t_1)$ are in $L^2(\mathbb{R})$ then $u \in C([0,T]; H^{2+4\alpha}(\mathbb{R}))$.

Before stating in a precise manner the main results of this article, let us explain the notation for mixed space-time norms. For $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ (or $\mathbb{C}$) we have

$$\|f\|_{L_x^p L_T^q} := \Big(\int_{\mathbb{R}}\Big(\int_0^T |f(x,t)|^q dt\Big)^{p/q} dx\Big)^{1/p}.$$

When $p = \infty$ or $q = \infty$ we must do the obvious changes with the essential supremum. When in the space-time norm appears t instead of T, the time interval is $[0, +\infty)$.

Our results read as follows:

Theorem 1.1. *Let $r > \frac{5}{16}$ and $u_0 \in Z_{4r,r}$. Then there exist $T > 0$ and a unique u, solution of* (1.1) *with $k = 1$ such that*

$$u \in C([0,T]; Z_{4r,r}), \tag{1.7}$$

$$\|\partial_x u\|_{L^4_T L^\infty_x} < \infty, \tag{1.8}$$

$$\|D^{4r}_x \partial_x u\|_{L^\infty_x L^2_T} < \infty, \quad \text{and} \tag{1.9}$$

$$\|u\|_{L^2_x L^\infty_T} < \infty. \tag{1.10}$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $Z_{4r,r}$ such that the data-solution map $\tilde{u}_0 \mapsto \tilde{u}$ from V into the class defined by (1.7)-(1.10) *with T' instead of T is Lipschitz.*

When $5/16 < r < 1/2$, T depends on $\|u_0\|_{Z_{4r,r}}$, and when $r \geq 1/2$ the size of T depends only on $\|u_0\|_{H^{4r}}$.

Let us recall that the operator D is defined through the Fourier transform by the multiplier $|\xi|$.

Remark 1.2.
(a) From the proof of Theorem 1.1 it is clear that if (1.1) is globally well-posed in $H^{4r}(\mathbb{R})$, $r \geq \frac{1}{2}$, then the IVP is also globally well-posed in $Z_{4r,r}$.
(b) Using the regularity property in Theorem 2.1 it follows, from Theorem 1.1, that (1.1) is globally well-posed in $Z_{s,r}$ for $s \geq 4r$ and $r \geq \frac{1}{2}$.
(c) Let us observe that applying the same method used in the proof of Theorem 1.1 it can be seen that (1.1) is locally well-posed in $Z_{s,l}$ with $s \geq 4r$, $l \leq r$ and $r \geq 1/2$.

Theorem 1.3. *Let $r \geq 1/2$ and $u_0 \in Z_{4r,r}$. Then* (1.1) *for the fifth-order KdV equation ($k = 1$) is globally well-posed in $Z_{4r,r}$.*

Theorem 1.4. *Let $u_0 \in Z_{2,1/2}$. Then there exist $T = T(\|u_0\|_{H^2}) > 0$ and a unique u, solution of* (1.1) *for the modified fifth-order KdV equation ($k = 2$), such that*

$$u \in C([0,T]; Z_{2,1/2}), \tag{1.11}$$

$$\|\partial_x^4 u\|_{L^\infty_x L^2_T} < \infty, \tag{1.12}$$

$$\|u\|_{L^{16/5}_x L^\infty_T} < \infty, \tag{1.13}$$

$$\|u\|_{L^4_x L^\infty_T} < \infty. \tag{1.14}$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $Z_{2,1/2}$ such that the data-solution map $\tilde{u}_0 \mapsto \tilde{u}$ from V into the class defined by (1.11)-(1.14) *with T' instead of T is Lipschitz.*

Theorem 1.5. *The initial-value problem* (1.1) *for the modified fifth-order KdV equation ($k = 2$) is globally well-posed in $Z_{2,1/2}$.*

Theorem 1.6. *For $T > 0$ let $u \in C([0,T]; Z_{2,1/2})$ be the solution of the modified fifth-order KdV equation ($k = 2$), obtained in Theorems 1.4 and 1.5. Let us suppose that for $\alpha > 0$ there exist two different times $t_0, t_1 \in [0,T]$, with $t_0 < t_1$, such that $|x|^{1/2+\alpha} u(t_0)$ and $|x|^{1/2+\alpha} u(t_1)$ are in $L^2(\mathbb{R})$. Then $u \in C([0,T]; H^{2+4\alpha}(\mathbb{R}))$.*

This article is organized as follows: in section 2 we recall some linear estimates of the group associated to the linear part of the equation in (1.1), a pointwise estimate for this group, related with fractional weights, and an interpolation inequality in weighted Sobolev spaces. In section 3 we study (1.1) with $k = 1$ and prove Theorems 1.1 and 1.3. In section 4 we consider (1.1) with $k = 2$ and establish Theorems 1.4 and 1.5. In section 5 we give the proof of Theorem 1.6.

Throughout the paper the letter C will denote diverse constants, which may change from line to line, and whose dependence on certain parameters is clearly established in all cases.

2. Preliminary results

In this section we recall some linear estimates for the group associated to the linear part of the equation in (1.1), a pointwise estimate for "fractional weights", and an interpolation inequality in weighted Sobolev spaces. On the other hand, we establish an standard estimate in weighted Sobolev spaces.

Let us consider the linear problem associated with (1.1):

$$\begin{gathered} \partial_t u + \partial_x^5 u = 0, \quad x, t \in \mathbb{R} \\ u(0) = u_0, \end{gathered} \tag{2.1}$$

whose solution is given by the group $\{W(t)\}_{t\in\mathbb{R}}$, i.e.

$$u(x,t) = [W(t)u_0](x) := (S_t * u_0)(x),$$

where $S_t(x)$ is defined by the oscillatory integral

$$S_t(x) = C \int_{\mathbb{R}} e^{ix\xi} e^{-it\xi^5} d\xi.$$

Kenig, Ponce and Vega [14, 15, 16] established the following estimates for the group $\{W(t)\}_{t\in\mathbb{R}}$:

(i) (Homogeneous smoothing effect) There exists a constant C such that

$$\|\partial_x^2 W(t)u_0\|_{L_x^\infty L_t^2} \le C\|u_0\|_{L^2}. \tag{2.2}$$

(ii) (Dual version of estimate (2.2)) There exists a constant C such that

$$\|\partial_x^2 \int_0^t W(t-t')f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \le C\|f\|_{L_x^1 L_T^2}. \tag{2.3}$$

(iii) (Inhomogeneous smoothing effect) There exists a constant C such that

$$\|\partial_x^4 \int_0^t W(t-t')f(\cdot,t')dt'\|_{L_x^\infty L_t^2} \le C\|f\|_{L_x^1 L_t^2}. \tag{2.4}$$

(iv) (Estimate of the maximal function) For any $\rho > \frac{3}{4}$ and $s > \frac{5}{4}$ there exists C such that

$$\|W(t)u_0\|_{L_x^2 L_T^\infty} \le C(1+T)^\rho \|u_0\|_{H^s}. \tag{2.5}$$

(v) There exists a constant C such that, for $u_0 \in H^{1/4}(\mathbb{R})$ (see [19]),

$$\|W(t)u_0\|_{L_x^4 L_T^\infty} \le C\|D^{1/4}u_0\|_{L^2}. \tag{2.6}$$

By interpolation it follows, from (2.5) and (2.6), that for $\rho > \frac{3}{4}$ and $s > \frac{5}{4}$,

$$\|W(t)u_0\|_{L_x^{16/5} L_T^\infty} \le C(1+T)^\rho \|u_0\|_{H^s}. \tag{2.7}$$

(vi) There exists a constant C such that

$$\|D_x^{3/4} W(t)u_0\|_{L_t^4 L_x^\infty} \le C\|u_0\|_{L^2}. \tag{2.8}$$

Using (2.2), (2.5) and (2.6), and proceeding as in the proofs of [16, Theorem 2.1] and [14, Theorem 1.1], it can be established the following theorem.

Theorem 2.1. *Let $s > 5/4$. Then for any $u_0 \in H^s(\mathbb{R})$ there exist a positive value $T = T(\|u_0\|_{H^s})$ (with $T(\rho) \to \infty$ as $\rho \to 0$) and a unique solution u of* (1.1) *with $k = 1$, satisfying*

$$u \in C([0,T]; H^s(\mathbb{R}))\,, \tag{2.9}$$

$$\|\partial_x u\|_{L^4_T L^\infty_x} < \infty\,, \tag{2.10}$$

$$\|D^s_x \partial_x u\|_{L^\infty_x L^2_T} < \infty\,, \tag{2.11}$$

$$\|u\|_{L^2_x L^\infty_T} < \infty\,. \tag{2.12}$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^s(\mathbb{R})$ such that the data-solution map $\tilde{u_0} \mapsto \tilde{u}$ from V into the class defined by (2.9)-(2.12) *with T' instead of T is Lipschitz. Also, if $u_0 \in H^{s'}$ with $s' > s$ then the above results hold with s' instead of s in the same time interval $[0,T]$ (regularity property).*

Let us observe the gain of two derivatives in x in the linear estimate (2.2). However, the condition (2.11) only uses the gain of one derivative in x.

One of the main tools for establishing local well-posedness of (1.1) with $k = 1$ in weighted Sobolev spaces with low regularity is the following pointwise formula, proved by Fonseca, Linares, and Ponce in [2]:

(vii) For $r \in (0,1)$ and $u_0 \in Z_{4r,r}$ we have for all $t \in \mathbb{R}$ and for almost every $x \in \mathbb{R}$:

$$|x|^r[W(t)u_0](x) = W(t)(|x|^r u_0)(x) + W(t)\{\Phi_{t,r}(\widehat{u_0})\}^\vee(x), \tag{2.13}$$

where

$$\|(\Phi_{t,r}(\widehat{u_0})(\xi))^\vee\|_{L^2} \le C_r(1+|t|)(\|u_0\|_{L^2} + \|D^{4r}_x u_0\|_{L^2}). \tag{2.14}$$

With respect to the weight $\langle x\rangle := (1+x^2)^{1/2}$, for $N \in \mathbb{N}$, we will consider a truncated weight w_N of $\langle x\rangle$, such that $w_N \in C^\infty(\mathbb{R})$,

$$w_N(x) = \begin{cases} \langle x\rangle & \text{if } |x| \le N, \\ 2N & \text{if } |x| \ge 3N, \end{cases} \tag{2.15}$$

The function w_N is non-decreasing in $|x|$ and for $j \in \mathbb{N}$ and $x \in \mathbb{R}$, the derivatives $w_N^{(j)}$ of order j of w_N satisfy

$$|w_N^{(j)}(x)| \le \frac{c_j}{w_N^{j-1}(x)}\,, \tag{2.16}$$

where the constant c_j is independent from N.

Fonseca and Ponce [5] deduced the following interpolation inequality, related to the weights $\langle x\rangle$ and w_N.

Lemma 2.2. *Let $a, b > 0$ and $f \in Z_{a,b} \equiv H^a(\mathbb{R}) \cap L^2(\langle x\rangle^{2b}dx)$. Then for any $\theta \in (0,1)$*

$$\|J^{\theta a}(\langle x\rangle^{(1-\theta)b} f)\|_{L^2} \le C\|\langle x\rangle^b f\|_{L^2}^{1-\theta}\|J^a f\|_{L^2}^\theta, \tag{2.17}$$

where $J^a f := (1-\partial_x^2)^{a/2} f$. Moreover, inequality (2.17) *is still valid with $w_N(x)$ as in* (2.15) *instead of $\langle x\rangle$ with a constant C independent of N.*

Finally, in our arguments we will use the following standard estimate, concerning the weights $\langle x\rangle$ and w_N.

Lemma 2.3. *Let $b > 0$ and $n \in \mathbb{N}$. Suppose that $J^n(\langle x\rangle^b u_0) \in L^2(\mathbb{R})$. Then*

$$\|\langle x\rangle^b \partial_x^n u_0\|_{L^2} \le C(b,n)\|J^n(\langle x\rangle^b u_0)\|_{L^2}\,. \tag{2.18}$$

Moreover, the inequality (2.18) is still valid with $w_N(x)$ as in (2.15) instead of $\langle x\rangle$ with a constant $C(b,n)$ independent of N.

The proof of the above lemma follows by induction on n and the Leibniz formula.

3. Well-posedness of (1.1) with $k = 1$

3.1. **Proof of Theorem 1.1.** We consider two cases.
Case: $5/16 < r < 1/2$. Proceeding as in [15, 16], for $u : \mathbb{R}\times[0,T] \to \mathbb{R}$ we define:

$$\lambda_1^T(u) := \max_{[0,T]} \|u(t)\|_{H^{4r}}, \tag{3.1}$$

$$\lambda_2^T(u) := \|\partial_x u\|_{L^4_T L^\infty_x}, \tag{3.2}$$

$$\lambda_3^T(u) := \|D_x^{4r}\partial_x u\|_{L^\infty_x L^2_T}, \tag{3.3}$$

$$\lambda_4^T(u) := (1+T)^{-\rho}\|u\|_{L^2_x L^\infty_T}, \quad \text{with } \rho \text{ a fixed number such that } \rho > \frac{3}{4}. \tag{3.4}$$

Additionally, we introduce

$$\lambda_5^T(u) := \||x|^r u\|_{L^\infty_T L^2_x}. \tag{3.5}$$

Let us consider

$$\Lambda^T(u) := \max_{1\le j\le 5} \lambda_j^T(u), \tag{3.6}$$

$$X_T := \{u \in C([0,T]; H^{4r}(\mathbb{R})) : \Lambda^T(u) < \infty\}. \tag{3.7}$$

Using the linear estimates (2.8), (2.2) and (2.5), Kenig, Ponce and Vega [16], showed that for $u_0 \in H^{4r}(\mathbb{R})$, $T > 0$ and $1 \le i \le 4$,

$$\lambda_i^T(W(t)u_0) \le C\|u_0\|_{H^{4r}}. \tag{3.8}$$

On the other hand, from (2.13) and (2.14), it follows that, for $t \in [0,T]$,

$$\lambda_5^T(W(t)u_0) \le \||x|^r u_0\|_{L^2} + C_r(1+T)(\|u_0\|_{L^2} + \|D_x^{4r}u_0\|_{L^2}). \tag{3.9}$$

In consequence, for $u_0 \in Z_{4r,r}$, the estimates (3.8) and (3.9) imply that

$$\Lambda^T(W(t)u_0) \le \||x|^r u_0\|_{L^2} + C(1+T)\|u_0\|_{H^{4r}}. \tag{3.10}$$

Let us denote by $u := \Phi(v) \equiv \Phi_{u_0}(v)$ the solution of the linear inhomogeneous IVP

$$\begin{gathered} \partial_t u + \partial_x^5 u + v\partial_x v = 0,\\ u(0) = u_0, \end{gathered} \tag{3.11}$$

where $v \in X_T^a := \{w \in X_T : \Lambda^T(w) \le a\}$, for $a > 0$. By Duhamel's formula:

$$\Phi(v)(t) \equiv u(t) = W(t)u_0 - \int_0^t W(t-t')(v\partial_x v)(t')dt'.$$

Taking into account that

$$\Lambda^T(u) \le \Lambda^T(W(t)u_0) + \int_0^T \Lambda^T(W(t-t')(v\partial_x v(t')))dt',$$

from (3.10) it follows that

$$\begin{aligned}\Lambda^T(u) &\le \||x|^r u_0\|_{L^2} + C(1+T)\|u_0\|_{H^{4r}} + C(1+T)(\|v\partial_x v\|_{L^1_T L^2_x} \\ &\quad + \|D_x^{4r}(v\partial_x v)\|_{L^1_T L^2_x}) + \||x|^r v\partial_x v\|_{L^1_T L^2_x}.\end{aligned} \tag{3.12}$$

In [15] (see proof of Lemma 4.1) it was proved that

$$\begin{aligned}&\|v\partial_x v\|_{L^1_T L^2_x} + \|D_x^{4r}(v\partial_x v)\|_{L^1_T L^2_x} \\ &\le CT^{1/2}(1+T)^\rho \lambda_4^T(v)\lambda_3^T(v) + CT^{3/4}\lambda_2^T(v)\lambda_1^T(v) + CT(\lambda_1^T(v))^2 \\ &\le C(T^{1/2}(1+T)^\rho + T^{3/4} + T)(\Lambda^T(v))^2,\end{aligned} \tag{3.13}$$

and let us observe that

$$\begin{aligned}\||x|^r v\partial_x v\|_{L^1_T L^2_x} &\le CT^{3/4}\||x|^r v\partial_x v\|_{L^4_T L^2_x} \\ &\le CT^{3/4}\||x|^r v\|_{L^\infty_T L^2_x}\|\partial_x v\|_{L^4_T L^\infty_x} \\ &\le CT^{3/4}\lambda_5^T(v)\lambda_2^T(v) \le CT^{3/4}(\Lambda^T(v))^2.\end{aligned} \tag{3.14}$$

From (3.12)-(3.14) it follows that

$$\Lambda^T(u) \le \||x|^r u_0\|_{L^2} + C(1+T)\|u_0\|_{H^{4r}} + C(1+T)(T^{1/2}(1+T)^\rho + T^{3/4} + T)(\Lambda^T(v))^2.$$

Taking $a := 2(\||x|^r u_0\|_{L^2} + C(1+T)\|u_0\|_{H^{4r}})$ and T sufficiently small in order to have

$$C(1+T)(T^{1/2}(1+T)^\rho + T^{3/4} + T)a < \frac{1}{2},$$

it can be seen that $\Phi : X_T^a \to X_T^a$. Reasoning as in [16] (proof of Theorem 2.1), for $T > 0$ small enough, $\Phi : X_T^a \to X_T^a$ is a contraction. In consequence, there exists a unique $u \in X_T^a$ such that $\Phi(u) = u$. In other words, for $t \in [0, T]$:

$$u(t) = W(t)u_0 - \int_0^t W(t-t')(u\partial_x u)(t')dt'.$$

To conclude the proof of this case we reason in the same manner as it was done at the end of the proof of [16, Theorem 2.1].

Case: $r \ge 1/2$**.** By Theorem 2.1 there exist $T = T(\|u_0\|_{H^{4r}})$ and a unique u in the class defined by the conditions (2.9)-(2.12) with $s = 4r$, which is a solution of (1.1) with $k = 1$. Let $\{u_{0m}\}_{m\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R})$ such that $u_{0m} \to u_0$ in $H^{4r}(\mathbb{R})$ and let $u_m \in C([0,T]; H^\infty(\mathbb{R}))$ be a solution of the equation in (1.1) corresponding to the initial data u_{0m}. (Without loss of generality we can suppose that u_m is defined in the same interval $[0, T]$ (see regularity property in Theorem 2.1)). By Theorem 2.1 $u_m \to u$ in $C([0,T]; H^{4r}(\mathbb{R}))$. We multiply the equation

$$\partial_t u_m + \partial_x^5 u_m + u_m\partial_x u_m = 0 \tag{3.15}$$

by $u_m w_N^{2r}$, where w_N is the truncated weight defined in (2.15), and for a fixed $t \in [0, T]$, we integrate in $\mathbb{R}$ with respect to x and use integration by parts to obtain

$$\begin{aligned}&\frac{d}{dt}(u_m(t), u_m(t)w_N^{2r}) \\ &= 5(\partial_x^2 u_m(t), \partial_x^2 u_m(t)(w_N^{2r})^{(1)}) - 5(\partial_x u_m(t), \partial_x u_m(t)(w_N^{2r})^{(3)}) \\ &\quad + (u_m(t), u_m(t)(w_N^{2r})^{(5)}) + \frac{2}{3}(1, u_m(t)^3(w_N^{2r})^{(1)}),\end{aligned} \tag{3.16}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R})$.

Integrating the above equation with respect to the time variable in the interval $[0,t]$, we have

$$\begin{aligned}
&(u_m(t), u_m(t)w_N^{2r}) \\
&= (u_{0m}, u_{0m}w_N^{2r}) + 5\int_0^t (\partial_x^2 u_m(t'), \partial_x^2 u_m(t')(w_N^{2r})^{(1)})dt' \\
&\quad - 5\int_0^t (\partial_x u_m(t'), \partial_x u_m(t')(w_N^{2r})^{(3)})dt' + \int_0^t (u_m(t'), u_m(t')(w_N^{2r})^{(5)})dt' \\
&\quad + \frac{2}{3}\int_0^t (1, u_m(t')^3(w_N^{2r})^{(1)})dt'.
\end{aligned} \tag{3.17}$$

Since $u_m \to u$ in $C([0,T]; H^{4r}(\mathbb{R}))$, with $4r \geq 2$, and the weight w_N^{2r} and its derivatives are bounded functions, it follows from equality (3.17), after passing to the limit when $m \to \infty$, that

$$\begin{aligned}
&(u(t), u(t)w_N^{2r}) \\
&= (u_0, u_0 w_N^{2r}) + 5\int_0^t (\partial_x^2 u(t'), \partial_x^2 u(t')(w_N^{2r})^{(1)})dt' \\
&\quad - 5\int_0^t (\partial_x u(t'), \partial_x u(t')(w_N^{2r})^{(3)})dt' \\
&\quad + \int_0^t (u(t'), u(t')(w_N^{2r})^{(5)})dt' + \frac{2}{3}\int_0^t (1, u(t')^3(w_N^{2r})^{(1)})dt' \\
&\equiv I + II + III + IV + V.
\end{aligned} \tag{3.18}$$

Let us estimate the terms on the right-hand side of (3.18). First of all

$$I \leq \|u_0\|^2_{L^2(\langle x\rangle^{2r}dx)}. \tag{3.19}$$

With respect to the term II, using Lemmas 2.3 and 2.2, we have

$$|II| \leq 10r\int_0^t (\partial_x^2 u(t'), \partial_x^2 u(t') w_N^{2r-1}|(w_N)^{(1)}|)dt' \tag{3.20}$$

$$\leq C\int_0^t (\partial_x^2 u(t'), \partial_x^2 u(t') w_N^{2r-1})dt' \leq C\int_0^t \|J^2(w_N^{r-\frac{1}{2}}u(t'))\|^2_{L^2}dt' \tag{3.21}$$

$$\leq C\int_0^t \|J^{4r}u(t')\|^{1/r}_{L^2}\|w_N^r u(t')\|^{2-1/r}_{L^2}dt' \leq C\int_0^t \|w_N^r u(t')\|^{2-1/r}_{L^2}dt' \tag{3.22}$$

$$\leq C\int_0^t (1 + \|w_N^r u(t')\|^2_{L^2})dt' \leq Ct + C\int_0^t (u(t'), u(t')w_N^{2r})dt'. \tag{3.23}$$

Using inequality (2.16) for the derivatives of w_N it can be seen that

$$|(w_N^{2r})^{(3)}| \leq Cw_N^{2r-3} \quad \text{and} \quad |(w_N^{2r})^{(5)}| \leq Cw_N^{2r-5}. \tag{3.24}$$

In this manner we can bound the term III as follows:

$$|III| \leq C\int_0^t (\partial_x u(t'), \partial_x u(t') w_N^{2r-3})dt'. \tag{3.25}$$

If $2r - 3 \leq 0$, since $u \in C([0,T]; H^{4r}(\mathbb{R}))$ with $4r \geq 2$, it is clear that

$$|III| \leq Ct. \tag{3.26}$$

If $2r-3>0$, we apply Lemmas 2.3 and 2.2 to conclude that

$$\begin{aligned}|III| &\leq C\int_0^t \|J(w_N^{r-\frac{3}{2}}u(t'))\|_{L^2}^2 dt' \\ &\leq C\int_0^t \|J^{4r}u(t')\|_{L^2}^{\frac{1}{2r}} \|w_N^{\frac{r(r-3/2)}{(r-1/4)}}u(t')\|_{L^2}^{\frac{4r-1}{2r}} dt' \\ &\leq C\int_0^t \|w_N^{\frac{r(r-3/2)}{(r-1/4)}}u(t')\|_{L^2}^{2-\frac{1}{2r}} dt' \leq C\int_0^t \|w_N^r u(t')\|_{L^2}^{2-\frac{1}{2r}} dt' \\ &\leq C\int_0^t (1+\|w_N^r u(t')\|_{L^2}^2) dt' \leq Ct + C\int_0^t (u(t'), u(t')w_N^{2r}) dt'.\end{aligned} \tag{3.27}$$

In any case the estimate (3.27) holds. In a similar manner it can be shown the following estimate for the term IV:

$$|IV| \leq C\int_0^t (u(t'), u(t')w_N^{2r}) dt'. \tag{3.28}$$

With respect to the term V we have:

$$\begin{aligned}|V| &\leq C\int_0^t \|u(t')\|_{L^\infty}(u(t'), u(t')w_N^{2r-1}) dt' \\ &\leq C\int_0^t \|u(t')\|_{H^{4r}}(u(t'), u(t')w_N^{2r}) dt' \\ &\leq C\int_0^t (u(t'), u(t')w_N^{2r}) dt'\,.\end{aligned} \tag{3.29}$$

From equality (3.18) and the estimates (3.19)-(3.29) it follows that, for $t\in[0,T]$,

$$(u(t), u(t)w_N^{2r}) \leq \|u_0\|_{L^2(\langle x\rangle^{2r}dx)}^2 + Ct + C\int_0^t (u(t'), u(t')w_N^{2r}) dt'\,.$$

Gronwall's inequality enables us to conclude that, for $t\in[0,T]$,

$$\begin{aligned}&(u(t), u(t)w_N^{2r}) \\ &\leq \|u_0\|_{L^2(\langle x\rangle^{2r}dx)}^2 + Ct + C\int_0^t (\|u_0\|_{L^2(\langle x\rangle^{2r}dx)}^2 + Ct')e^{C(t-t')} dt'\,.\end{aligned} \tag{3.30}$$

Passing to the limit in (3.30) when $N\to\infty$ we obtain, for $t\in[0,T]$,

$$\begin{aligned}&\|u(t)\|_{L^2(\langle x\rangle^{2r}dx)}^2 \\ &\leq \|u_0\|_{L^2(\langle x\rangle^{2r}dx)}^2 + Ct + C\int_0^t (\|u_0\|_{L^2(\langle x\rangle^{2r}dx)}^2 + Ct')e^{C(t-t')} dt' \leq C(T),\end{aligned} \tag{3.31}$$

which implies that $u\in L^\infty([0,T];L^2(\langle x\rangle^{2r}dx))$.

Now let us see that $u\in C([0,T];L^2(\langle x\rangle^{2r}dx))$. For that we follow an argument contained in [1] and [8]. From (3.31) it is clear that there is a positive constant M such that, for all $t\in[0,T]$,

$$\|u(t)\|_{L_w^2}^2 \leq \|u_0\|_{L_w^2}^2 + Mt, \tag{3.32}$$

where the notation $L_w^2 := L^2(\langle x\rangle^{2r}dx)$ was used.

Taking into account that $u \in C([0,T];L^2)$ and using (3.32), it can be seen that, for $\phi \in L^2_w$, the function $t \mapsto (\phi, u(t))_{L^2_w}$ is continuous from $[0,T]$ to $\mathbb{C}$. From this fact and (3.32) it follows that

$$\begin{aligned}\|u(t)-u(0)\|^2_{L^2_w} &= \|u(t)\|^2_{L^2_w} + \|u(0)\|^2_{L^2_w} - 2\operatorname{Re}(u(0),u(t))_{L^2_w}\\ &\le \|u(0)\|^2_{L^2_w} + Mt + \|u(0)\|^2_{L^2_w} - 2\operatorname{Re}(u(0),u(t))_{L^2_w} \to 0\end{aligned}$$

as $t \to 0^+$, which proves that $u : [0,T] \to L^2(\langle x\rangle^{2r}dx)$ is continuous at $t=0$.

The continuity of u at a point $t_0 \in (0,T]$ is a consequence from the continuity of u at $t=0$ and from the fact that the functions $v_1(x,t) := u(x,t_0+t)$ and $v_2(x,t) := u(-x,t_0-t)$ are also solutions of the fifth-order KdV equation. In this manner, we had proved that if $u_0 \in Z_{4r,r}$ $(r \ge \frac{1}{2})$ there exist $T = T(\|u_0\|_{H^{4r}}) > 0$ and a unique $u \in C([0,T];Z_{4r,r})$, solution of (1.1), with $k=1$, belonging to the class defined by the conditions (2.9)-(2.12) with $s=4r$.

Finally, let us prove that if $\widetilde{u}_m \in C([0,T];Z_{4r,r})$ is the solution of the fifth-order KdV equation, corresponding to the initial data $\widetilde{u}_{m0}$, where $\widetilde{u}_{m0} \to u_0$ in $Z_{4r,r}$ when $m \to \infty$, then $\widetilde{u}_m \to u$ in $C([0,T];Z_{4r,r})$. By Theorem 2.1 we have that $\widetilde{u}_m \to u$ in $C([0,T];H^{4r})$. In consequence we only must prove that $\widetilde{u}_m \to u$ in $C([0,T];L^2(\langle x\rangle^{2r}dx))$. Let $v_m := \widetilde{u}_m - u$ and $v_{m0} := \widetilde{u}_{m0} - u_0$. Proceeding in a similar manner as it was done when we established that $u \in L^\infty([0,T];L^2(\langle x\rangle^{2r}dx))$ and taking into account that $v_m \to 0$ in $C([0,T];H^{4r})$ it can be seen that, for $t \in [0,T]$,

$$\|v_m(t)\|^2_{L^2(w_N^{2r}dx))} \le \|v_{m0}\|^2_{L^2(\langle x\rangle^{2r}dx)} + C_m t + C\int_0^t \|v_m(t')\|^2_{L^2(w_N^{2r}dx))} dt' ,$$

where $\lim_{m\to\infty} C_m = 0$. Hence, by Gronwall's inequality, we have for $t \in [0,T]$ and $N \in \mathbb{N}$ that

$$\|v_m(t)\|^2_{L^2(w_N^{2r}dx))} \le (\|v_{m0}\|^2_{L^2(\langle x\rangle^{2r}dx)} + C_m T)e^{CT} .$$

From this inequality it follows, after passing to the limit when $N \to \infty$, that

$$v_m \to 0 \quad \text{in } C([0,T];L^2(\langle x\rangle^{2r}dx)) .$$

The proof of Theorem 1.1 is complete.

3.2. **Proof of Theorem 1.3.** Taking into account Remarks 1.2(a) and 1.2(b) it is sufficient to show that (1.1) for the fifth-order KdV equation is globally well-posed in $H^2(\mathbb{R})$.

To see this, first of all, we prove that if $u \in C([0,T];H^2(\mathbb{R}))$ is a solution of (1.1) then, for all $t \in [0,T]$,

$$\|u(t)\|^2_{H^2} \le K \equiv K(\|u_0\|_{H^2}), \tag{3.33}$$

where K depends only on $\|u_0\|_{H^2(\mathbb{R})}$. Let us observe that

$$\int_{\mathbb{R}} (\partial_x u)^2(t)dx \le \frac{1}{2}\Big[\int_{\mathbb{R}} (\partial_x^2 u)^2(t)dx + \int_{\mathbb{R}} u^2(t)dx\Big]. \tag{3.34}$$

Using the definition of the H^2-norm, inequality (3.34) and the conservation laws (1.4) and (1.5) it follows that

$$\begin{aligned} \|u(t)\|_{H^2}^2 &= \int_{\mathbb{R}} u^2(t)dx + \int_{\mathbb{R}} (\partial_x u)^2(t)dx + \int_{\mathbb{R}} (\partial_x^2 u)^2(t)dx \\ &\le \frac{3}{2}\int_{\mathbb{R}} u^2(t)dx + \frac{3}{2}\int_{\mathbb{R}} (\partial_x^2 u)^2(t)dx \\ &= \frac{3}{2} I_1(t) + 3I_2^1(t) - \frac{1}{2}\int_{\mathbb{R}} u^3(t)dx \\ &= \frac{3}{2}\|u_0\|_{L^2}^2 + 3\Big[\frac{1}{2}\|\partial_x^2 u_0\|_{L^2}^2 + \frac{1}{6}\int_{\mathbb{R}} u_0^3 dx\Big] - \frac{1}{2}\int_{\mathbb{R}} u^3(t)dx. \end{aligned} \tag{3.35}$$

Now, from the Sobolev lemma, we have

$$\int_{\mathbb{R}} u_0^3 dx \le \|u_0\|_{L^\infty} \int_{\mathbb{R}} u_0^2 dx \le C\|u_0\|_{H^2}^3. \tag{3.36}$$

On the other hand, the Sobolev lemma, the conservation law (1.4) and Young's inequality imply that

$$\begin{aligned} \Big|\int_{\mathbb{R}} u^3(t)dx\Big| &\le \|u(t)\|_{L^\infty}\|u(t)\|_{L^2}^2 \\ &\le C\|u(t)\|_{H^1}\|u(t)\|_{L^2}^2 = C\|u(t)\|_{H^1}\|u_0\|_{L^2}^2 \\ &\le \frac{1}{2}\|u(t)\|_{H^1}^2 + \frac{C^2}{2}\|u_0\|_{L^2}^4 \\ &\le \frac{1}{2}\|u(t)\|_{H^1}^2 + \frac{C^2}{2}\|u_0\|_{H^2}^4. \end{aligned} \tag{3.37}$$

Therefore, from (3.35)–(3.37), we have

$$\|u(t)\|_{H^2}^2 \le \frac{3}{2}\|u_0\|_{H^2}^2 + C\|u_0\|_{H^2}^3 + \frac{C^2}{4}\|u_0\|_{H^2}^4 + \frac{1}{4}\|u(t)\|_{H^2}^2,$$

and from the above inequality

$$\|u(t)\|_{H^2}^2 \le C(\|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^3 + \|u_0\|_{H^2}^4) \equiv K, \tag{3.38}$$

which proves (3.33).

Now we show how to extend the local solution u to any time interval. From the proof of Theorem 2.1 it can be seen that the size of the time interval of the solution $u \in C([0,T]; H^2(\mathbb{R}))$ of (1.1) is such that

$$T \ge \min\big\{1, \frac{1}{C\|u_0\|_{H^2}^2}\big\}.$$

Reasoning as in the proof of Theorem 2.1 we obtain a solution $u \in C([T, T + t_0]; H^2(\mathbb{R}))$ of the IVP

$$\begin{gathered} \partial_t v + \partial_x^5 v + v\partial_x v = 0, \quad x, t \in \mathbb{R} \\ v(T) = u(T), \end{gathered}$$

such that

$$t_0 \ge \min\big\{1, \frac{1}{C\|u(T)\|_{H^2}^2}\big\}.$$

In this manner we obtain a solution $u \in C([0, T + t_0]; H^2(\mathbb{R}))$ of (1.1). By the a priori estimate (3.33) we have that

$$\frac{1}{\|u(t)\|_{H^2}^2} \geq \frac{1}{K},$$

for $t \in [0, T + t_0]$, and therefore

$$t_0 \geq \min\big\{1, \frac{1}{CK}\big\}.$$

We repeat this argument $n + 1$ times to obtain a solution $u \in C([0, T + t_0 + \cdots + t_n]; H^2(\mathbb{R}))$ with

$$t_j \geq \min\big\{1, \frac{1}{CK}\big\}, \quad j = 0, \dots, n.$$

Since $\sum_{j=0}^{n} t_j \to \infty$ as $n \to \infty$ then we can extend the solution to any time interval. The proof is complete.

4. Well-posedness of (1.1) with $k = 2$

4.1. **Proof of Theorem 1.4.** For $T > 0$, let us define the space

$$Y_T := \{u \in C([0, T]; H^2(\mathbb{R})) : \|\partial_x^4 u\|_{L_x^\infty L_T^2} < \infty, \|u\|_{L_x^{16/5} L_T^\infty} < \infty, \|u\|_{L_x^4 L_T^\infty} < \infty\}, \tag{4.1}$$

and, for $u \in Y_T$, let us consider the norms

$$\lambda_1^T(u) := \max_{[0,T]} \|u(t)\|_{H^2}, \tag{4.2}$$

$$\lambda_2^T(u) := \|\partial_x^4 u\|_{L_x^\infty L_T^2}, \tag{4.3}$$

$$\lambda_3^T(u) := \|u\|_{L_x^{16/5} L_T^\infty}, \tag{4.4}$$

$$\lambda_4^T(u) := \|u\|_{L_x^4 L_T^\infty}, \tag{4.5}$$

$$\Lambda^T(u) := \max_{1 \leq i \leq 4} \lambda_i^T(u). \tag{4.6}$$

For $a > 0$, let Y_T^a be the closed ball in Y_T defined by

$$Y_T^a := \{u \in Y_T : \Lambda^T(u) \leq a\}. \tag{4.7}$$

We shall prove that there exist $T > 0$ and $a > 0$ such that the operator $\Psi : Y_T^a \to Y_T^a$ defined by

$$\Psi(v) = W(t)u_0 - \int_0^t W(t - t')(v^2 \partial_x v)(t') dt'$$

is a contraction.

Also the linear estimates in section 2, we will need some nonlinear estimates in order to prove that Ψ is a contraction.

First of all we establish these nonlinear estimates. Let $u \in Y_T$:

(i) Using interpolation we have

$$\begin{aligned} \|u^2 \partial_x u\|_{L_x^1 L_T^2} &\leq \|u^2\|_{L_x^{8/5} L_T^\infty} \|\partial_x u\|_{L_x^{8/3} L_T^2} \\ &\leq \|u\|_{L_x^{16/5} L_T^\infty}^2 \|u\|_{L_x^2 L_T^2}^{3/4} \|\partial_x^4 u\|_{L_x^\infty L_T^2}^{1/4} \\ &\leq C T^{3/4} \|u\|_{L_x^{16/5} L_T^\infty}^2 \|u\|_{L_T^\infty L_x^2}^{3/4} \|\partial_x^4 u\|_{L_x^\infty L_T^2}^{1/4}. \end{aligned} \tag{4.8}$$

(ii) By (2.4) and (4.8) it follows that

$$\begin{aligned}
&\|\partial_x^4 \int_0^t W(t-t')(u^2\partial_x u)(t')dt'\|_{L_x^\infty L_T^2} \\
&\le C\|u^2\partial_x u\|_{L_x^1 L_T^2} \\
&\le CT^{3/4}\|u\|^2_{L_x^{16/5}L_T^\infty}\|u\|^{3/4}_{L_T^\infty L_x^2}\|\partial_x^4 u\|^{1/4}_{L_x^\infty L_T^2}.
\end{aligned} \tag{4.9}$$

(iii) By (2.3) and (4.8) it follows that

$$\begin{aligned}
&\|\partial_x^2 \int_0^t W(t-t')(u^2\partial_x u)(t')dt'\|_{L_T^\infty L_x^2} \\
&\le C\|u^2\partial_x u\|_{L_x^1 L_T^2} \\
&\le CT^{3/4}\|u\|^2_{L_x^{16/5}L_T^\infty}\|u\|^{3/4}_{L_T^\infty L_x^2}\|\partial_x^4 u\|^{1/4}_{L_x^\infty L_T^2}.
\end{aligned} \tag{4.10}$$

Now we prove that there exist $T > 0$ and $a > 0$ such that $\Psi(Y_T^a) \subset Y_T^a$. Let $v \in Y_T^a$. Then by (4.10),

$$\begin{aligned}
&\lambda_1^T(\Psi(v)) \\
&\le \lambda_1^T(W(t)u_0) + \lambda_1^T\Big(\int_0^t W(t-t')(v^2\partial_x v)(t')dt'\Big) \\
&\le \|u_0\|_{H^2} + C\Big(\sup_{[0,T]}\|\int_0^t W(t-t')(v^2\partial_x v)(t')dt'\|_{L^2} \\
&\quad + \sup_{[0,T]}\|\partial_x^2\int_0^t W(t-t')(v^2\partial_x v)(t')dt'\|_{L^2}\Big) \\
&\le \|u_0\|_{H^2} + CT\sup_{[0,T]}\|v(t)\|^3_{H^2} + CT^{3/4}\|v\|^2_{L_x^{16/5}L_T^\infty}\|v\|^{3/4}_{L_T^\infty L_x^2}\|\partial_x^4 v\|^{1/4}_{L_x^\infty L_T^2} \\
&\le \|u_0\|_{H^2} + CT^{3/4}(T^{1/4}+1)(\Lambda^T(v))^3.
\end{aligned} \tag{4.11}$$

From (2.2) and (4.9) it follows that

$$\begin{aligned}
\lambda_2^T(\Psi(v)) &\le \|\partial_x^4 W(t)u_0\|_{L_x^\infty L_T^2} + \|\partial_x^4\int_0^t W(t-t')(v^2\partial_x v)(t')dt'\|_{L_x^\infty L_T^2} \\
&\le C\|\partial_x^2 u_0\|_{L^2} + CT^{3/4}\|v\|^2_{L_x^{16/5}L_T^\infty}\|v\|^{3/4}_{L_T^\infty L_x^2}\|\partial_x^4 v\|^{1/4}_{L_x^\infty L_T^2} \\
&\le C\|u_0\|_{H^2} + CT^{3/4}(\Lambda^T(v))^3.
\end{aligned} \tag{4.12}$$

Using (2.7), the Leibniz rule and interpolation, we obtain

$$\begin{aligned}
&\lambda_3^T(\Psi(v)) \\
&\le \|W(t)u_0\|_{L_x^{16/5}L_T^\infty} + \|\int_0^t W(t-t')(v^2\partial_x v)dt'\|_{L_x^{16/5}L_T^\infty} \\
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho\int_0^T \|v^2\partial_x v(t')\|_{H^2}dt' \\
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho\int_0^T \|v^2\partial_x v(t')\|_{L^2}dt' \\
&\quad + C(1+T)^\rho\int_0^T \|\partial_x^2(v^2\partial_x v)(t')\|_{L^2}dt'
\end{aligned}$$

$$
\begin{aligned}
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho T(\Lambda^T(v))^3 + C(1+T)^\rho\Big(\int_0^T \|(\partial_x v)^3(t')\|_{L^2}dt' \\
&\quad + \int_0^T \|(v\partial_x v\partial_x^2 v)(t')\|_{L^2}dt' + \int_0^T \|(v^2\partial_x^3 v)(t')\|_{L^2}dt'\Big) \\
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho T(\Lambda^T(v))^3 + C(1+T)^\rho T^{1/2}\|v^2\partial_x^3 v\|_{L^2_T L^2_x} \\
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho T(\Lambda^T(v))^3 \\
&\quad + C(1+T)^\rho T^{1/2}\|v^2\|_{L^{16/7}_x L^8_T}\|\partial_x^3 v\|_{L^{16}_x L^{8/3}_T} \\
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho T(\Lambda^T(v))^3 \\
&\quad + C(1+T)^\rho T^{1/2}\|v\|_{L^{16/5}_x L^\infty_T}\|v\|_{L^8_x L^8_T}\|v\|^{1/4}_{L^4_x L^\infty_T}\|\partial_x^4 v\|^{3/4}_{L^\infty_x L^2_T} \\
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho T(\Lambda^T(v))^3 \\
&\quad + C(1+T)^\rho T^{1/2}\lambda_3^T(v)T^{1/8}\lambda_1^T(v)\lambda_4^T(v)^{1/4}\lambda_2^T(v)^{3/4} \\
&\le C(1+T)^\rho\|u_0\|_{H^2} + C(1+T)^\rho T(\Lambda^T(v))^3 + C(1+T)^\rho T^{5/8}(\Lambda^T(v))^3. \qquad (4.13)
\end{aligned}
$$

Applying (2.6) we have that

$$
\begin{aligned}
&\lambda_4^T(\Psi(v)) \\
&\le \|W(t)u_0\|_{L^4_x L^\infty_T} + \int_0^T \|W(t-t')(v^2\partial_x v)(t')\|_{L^4_x L^\infty_T}dt' \\
&\le C\|D_x^{1/4}u_0\|_{L^2} + C\int_0^T \|D_x^{1/4}(v^2\partial_x v(t'))\|_{L^2_x}dt' \qquad (4.14) \\
&\le C\|D_x^{1/4}u_0\|_{L^2} + C\int_0^T \|(v^2\partial_x v)(t')\|_{L^2}dt' + C\int_0^T \|\partial_x(v^2\partial_x v)(t')\|_{L^2}dt' \\
&\le C\|u_0\|_{H^2} + CT(\Lambda^T(v))^3.
\end{aligned}
$$

From (4.11)–(4.14) we obtain

$$
\begin{aligned}
\Lambda^T(\Psi(v)) \le C(1+T)^\rho\|u_0\|_{H^2} + CT^{5/8}[T^{1/8}(T^{1/4}+1) \\
+ (1+T)^\rho(1+T^{3/8}) + T^{3/8}](\Lambda^T(v))^3.
\end{aligned} \tag{4.15}
$$

Let us take $a := 2C(1+T)^\rho\|u_0\|_{H^2}$ and T in such a way that

$$
CT^{5/8}[T^{1/8}(T^{1/4}+1) + (1+T)^\rho(1+T^{3/8}) + T^{3/8}]a^2 \le \frac{1}{2}. \tag{4.16}
$$

Since $\Lambda^T(v) \le a$, from (4.15) and (4.16), we have that

$$
\Lambda^T(\Psi(v)) \le \frac{a}{2} + \frac{1}{2}(\Lambda^T(v)) \le \frac{a}{2} + \frac{a}{2} = a,
$$

i.e. $\Psi(Y^a_T) \subset Y^a_T$. Now we find an additional condition on the size of T in order to guarantee that the operator $\Psi : Y^a_T \to Y^a_T$ is a contraction. Let $v, w \in Y^a_T$. Then

$$
\begin{aligned}
\Psi(w) - \Psi(v) = \int_0^t W(t-t')(v^2\partial_x(v-w))(t')dt' \\
+ \int_0^t W(t-t')((v+w)(v-w)\partial_x w)(t')dt'.
\end{aligned}
$$

Therefore, proceeding as before,

$$
\begin{aligned}
&\lambda_1^T(\Psi(w)-\Psi(v))\\
&\le \lambda_1^T\Big(\int_0^t W(t-t')(v^2\partial_x(v-w))(t')dt'\Big)\\
&\quad+\lambda_1^T\Big(\int_0^t W(t-t')((v+w)(v-w)\partial_x w)(t')dt'\Big)\\
&\le CT(\lambda_1^T(v))^2\lambda_1^T(v-w)+CT^{3/4}(\lambda_3^T(v))^2\lambda_1^T(v-w)^{3/4}\lambda_2^T(v-w)^{1/4}\\
&\quad+CT\lambda_1^T(v+w)\lambda_1^T(v-w)\lambda_1^T(w)\\
&\quad+CT^{3/4}\lambda_3^T(v+w)\lambda_3^T(v-w)\lambda_1^T(w)^{3/4}\lambda_2^T(w)^{1/4}\\
&\le C[(T+T^{3/4})\Lambda^T(v)^2+T\Lambda^T(v+w)\Lambda^T(w)\\
&\quad+T^{3/4}\Lambda^T(v+w)\Lambda^T(w)]\Lambda^T(v-w)\\
&\le C(T+T^{3/4})(\Lambda^T(v)^2+\Lambda^T(w)^2)\Lambda^T(v-w),
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&\lambda_2^T(\Psi(w)-\Psi(v))\\
&\le CT^{3/4}\|v\|^2_{L_x^{16/5}L_T^\infty}\|v-w\|^{3/4}_{L_T^\infty L_x^2}\|\partial_x^4(v-w)\|^{1/4}_{L_x^\infty L_T^2}\\
&\quad+CT^{3/4}\|v+w\|_{L_x^{16/5}L_T^\infty}\|v-w\|_{L_x^{16/5}L_T^\infty}\|w\|^{3/4}_{L_T^\infty L_x^2}\|\partial_x^4 w\|^{1/4}_{L_x^\infty L_T^2}\\
&\le CT^{3/4}\Lambda^T(v)^2\Lambda^T(v-w)+CT^{3/4}\Lambda^T(v+w)\Lambda^T(v-w)\Lambda^T(w)\\
&\le CT^{3/4}(\Lambda^T(v)^2+\Lambda^T(w)^2)\Lambda^T(v-w),
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\lambda_3^T(\Psi(w)-\Psi(v))\\
&\le C(1+T)^\rho T\lambda_1^T(v)^2\lambda_1^T(v-w)\\
&\quad+C(1+T)^\rho T^{1/2}\|v^2\|_{L_x^{16/7}L_T^8}\|\partial_x^3(v-w)\|_{L_x^{16}L_T^{8/3}}\\
&\quad+C(1+T)^\rho T\lambda_1^T(v+w)\lambda_1^T(v-w)\lambda_1^T(w)\\
&\quad+C(1+T)^\rho T^{1/2}\|(v+w)(v-w)\|_{L_x^{16/7}L_T^8}\|\partial_x^3 w\|_{L_x^{16}L_T^{8/3}}\\
&\le C(1+T)^\rho T\lambda_1^T(v)^2\lambda_1^T(v-w)\\
&\quad+C(1+T)^\rho T^{1/2}\lambda_3^T(v)T^{1/8}\lambda_1^T(v)\lambda_4^T(v-w)^{1/4}\lambda_2^T(v-w)^{3/4}\\
&\quad+C(1+T)^\rho T\lambda_1^T(v+w)\lambda_1^T(v-w)\lambda_1^T(w)\\
&\quad+C(1+T)^\rho T^{1/2}\lambda_3^T(v+w)T^{1/8}\lambda_1^T(v-w)\lambda_4^T(w)^{1/4}\lambda_2^T(w)^{3/4}\\
&\le CT^{5/8}(1+T)^\rho(1+T^{3/8})(\Lambda^T(v)^2+\Lambda^T(w)^2)\Lambda^T(v-w),
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
&\lambda_4^T(\Psi(w)-\Psi(v))\\
&\le CT\lambda_1^T(v)^2\lambda_1^T(v-w)+CT\lambda_1^T(v+w)\lambda_1^T(v-w)\lambda_1^T(w)\\
&\le CT(\Lambda^T(v)^2+\Lambda^T(w)^2)\Lambda^T(v-w).
\end{aligned}
\tag{4.20}
$$

From (4.17) to (4.20), it follows that

$$
\begin{aligned}
&\Lambda^T(\Psi(w)-\Psi(v))\\
&\le C(2T+2T^{3/4}+T^{5/8}(1+T)^\rho(1+T^{3/8}))(\Lambda^T(v)^2+\Lambda^T(w)^2)\Lambda^T(v-w)\\
&\le C(2T+2T^{3/4}+T^{5/8}(1+T)^\rho(1+T^{3/8}))2a^2\lambda^T(v-w).
\end{aligned}
$$

If $a := 2C(1+T)^{\rho}\|u_0\|_{H^2}$ and T are taken satisfying (4.16) and the additional condition

$$C(2T + 2T^{3/4} + T^{5/8}(1+T)^{\rho}(1+T^{3/8}))2a^2 < 1,$$

then $\Psi : Y_T^a \to Y_T^a$ is a contraction. Hence, there exists a unique $u \in Y_T^a$ such that $\Psi(u) = u$.

From this point, proceeding in a similar way as it was done in [22] we conclude that, given $u_0 \in H^2(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^2}) > 0$ and a unique u, solution of (1.1) for the modified fifth-order KdV equation ($k = 2$), such that

$$u \in C([0,T]; H^2(\mathbb{R})) \tag{4.21}$$

and u satisfies the conditions (1.12), (1.13) and (1.14). Moreover, for any $T' \in (0,T)$ there exists a neighborhood V_1 of u_0 in $H^2(\mathbb{R})$, such that the data-solution map $\tilde{u}_0 \mapsto u_0$ from V_1 into the class defined by (4.21), (1.12), (1.13) and (1.14) with T' instead of T is Lipschitz. If additionally, we have that $u_0 \in Z_{2,1/2}$, then reasoning as in the proof of Theorem 1.1 (case $r \geq 1/2$) we obtain that $u \in C([0,T]; L^2(\langle x\rangle)dx)$, and that there exists a neighborhood V of u_0 in $Z_{2,1/2}$ such that the data-solution map $\tilde{u}_0 \mapsto u_0$ from V into the class defined by (1.11) to (1.14) with T' instead of T is Lipschitz. The proof is complete.

4.2. **Proof of Theorem 1.5.** From Theorem 1.4 to prove that (1.1) (with $k = 2$) is globally well-posed in $Z_{2,1/2}$, it is sufficient to establish that this IVP is globally well-posed in $H^2(\mathbb{R})$. Reasoning as in the proof of Theorem 1.3 it is enough to show that if $u \in C([0,T]; H^2(\mathbb{R}))$ is a solution of (1.1) with $k = 2$, then for every $t \in [0,T]$

$$\|u(t)\|_{H^2}^2 \leq K \equiv K(\|u_0\|_{H^2}), \tag{4.22}$$

where K only depends on $\|u_0\|_{H^2(\mathbb{R})}$.

From (3.34) and the conservation law (1.4) it is clear that (4.22) holds if we prove that, for every $t \in [0,T]$,

$$\|\partial_x^2 u(t)\|_{L^2}^2 \leq K \equiv K(\|u_0\|_{H^2}).$$

By the conservation law (1.6) we have that

$$\|\partial_x^2 u(t)\|_{L^2}^2 = \frac{1}{12}\int_{\mathbb{R}} u_0^4 dx + \int_{\mathbb{R}} (\partial_x^2 u_0)^2 dx - \frac{1}{12}\int_{\mathbb{R}} u^4(t) dx,$$

and, since the last term in the right-hand side of the above equality is non-positive, we obtain that

$$\|\partial_x^2 u(t)\|_{L^2}^2 \leq \frac{1}{12}\int_{\mathbb{R}} u_0^4 dx + \int_{\mathbb{R}} (\partial_x^2 u_0)^2 dx \leq \frac{1}{12}\|u_0\|_{L^\infty}^2 \int_{\mathbb{R}} u_0^2 dx + \int_{\mathbb{R}} (\partial_x^2 u_0)^2 dx.$$

Taking into account that $\|u_0\|_{L^\infty}^2 \leq C\|u_0\|_{H^2}^2$, we have

$$\begin{aligned}\|\partial_x^2 u(t)\|_{L^2}^2 &\leq C\|u_0\|_{H^2}^2 \int_{\mathbb{R}} u_0^2 dx + \int_{\mathbb{R}} (\partial_x^2 u_0)^2 dx \\ &\leq C\|u_0\|_{H^2}^2\|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^2 \equiv K(\|u_0\|_{H^2}).\end{aligned}$$

The proof is complete.

5. Relation between decay and regularity the solutions of (1.1) with $k=2$ (Proof of Theorem 1.6)

First we assume that $\alpha \in (0,1/8]$. The general case follows by an iterative argument as it was done by Isaza, Linares and Ponce in [9]. Let us suppose that $t_0=0$ and let $u_0=u(0)$. For $x\geq 0$ and $N\in\mathbb{N}$ let us define $\varphi_{N,\alpha}\in C^5([0,\infty))$ such that

$$\varphi_{N,\alpha}(x)=\begin{cases}(1+x^2)^{\alpha+1/2}-1 & \text{if } x\in[0,N],\\ (2N^2)^{\alpha+1/2} & \text{if } x\geq 10N,\end{cases}$$

$\varphi^{(1)}_{N,\alpha}(x)\geq 0$ and $|\varphi^{(j)}_{N,\alpha}(x)|\leq C$ for $j=2,3,4,5$, with C independent of N.

Let $\phi_N\equiv\phi_{N,\alpha}$ be the odd extension of $\varphi_{N,\alpha}$ to $\mathbb{R}$. Since $C_0^\infty(\mathbb{R})$ is dense in $Z_{2,1/2}$, there exist a sequence $\{u_{0m}\}_{m\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R})$ such that

$$\|u_0-u_{0m}\|_{Z_{2,1/2}}\to 0 \tag{5.1}$$

as $m\to\infty$.

Let u_m be the solution of the modified fifth-order KdV equation such that $u_m(0)=u_{0m}$. By Theorems 1.4 and 1.5 we have that

$$\|u_m-u\|_{C([0,T];Z_{2,1/2})}\to 0, \tag{5.2}$$

$$\|\partial_x^4u_m-\partial_x^4u\|_{L^\infty_xL^2_T}\to 0, \tag{5.3}$$

as $m\to\infty$. Since $u_{0m}\in H^s(\mathbb{R})$ for each $s\in\mathbb{R}$, it can be seen (regularity property), that $u_m(t)\in H^s(\mathbb{R})$ for each $s\in\mathbb{R}$ and each $t\in[0,T]$.

Now we multiply the equation $\partial_tu_m+\partial_x^5u_m+u_m^2\partial_xu_m=0$ by $u_m\phi_N$, integrate in x over $\mathbb{R}$ and apply integration by parts to obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}u_m^2\phi_N\,dx-5\int_{\mathbb{R}}(\partial_x^2u_m)^2\phi_N^{(1)}dx+5\int_{\mathbb{R}}(\partial_xu_m)^2\phi_N^{(3)}dx\\&-\int_{\mathbb{R}}u_m^2\phi_N^{(5)}dx-\frac12\int_{\mathbb{R}}u_m^4\phi_N^{(1)}dx=0.\end{aligned} \tag{5.4}$$

(In the above equation we use the notation $u_m\equiv u_m(t)$).

From convergence in (5.2), since $\alpha\leq 1/2$, it is clear that, for $t\in[0,T]$,

$$\begin{aligned}\Big|-\frac12\int_{\mathbb{R}}u_m^4(t)\phi_N^{(1)}dx\Big|&\leq C\|u_m(t)\|^2_{L^\infty}\int_{\mathbb{R}}u_m^2(t)\langle x\rangle^{2\alpha}dx\\&\leq C\|u_m(t)\|^2_{H^2}\int_{\mathbb{R}}u_m^2(t)\langle x\rangle^{2\alpha}dx\\&\leq C\sup_{t\in[0,T]}\|u_m(t)\|^2_{H^2}\|u_m(t)\|^2_{L^2(\langle x\rangle dx)}\leq C.\end{aligned} \tag{5.5}$$

On the other hand, it is also clear that

$$\Big|-\int_{\mathbb{R}}u_m^2\phi_N^{(5)}dx\Big|\leq C\quad\text{and}\quad\Big|5\int_{\mathbb{R}}(\partial_xu_m)^2\phi_N^{(3)}dx\Big|\leq C. \tag{5.6}$$

Integrating (5.4) in t over the interval $[0,t_1]$ and taking into account the inequalities (5.5) and (5.6) we can conclude that

$$5\int_0^{t_1}\int_{\mathbb{R}}(\partial_x^2u_m)^2\phi_N^{(1)}\,dx\,dt\leq\|u_m^2(t_1)\phi_N\|_{L^1}+\|u_m^2(0)\phi_N\|_{L^1}+Ct_1.$$

Hence

$$\limsup_{m\to\infty}\int_0^{t_1}\int_{\mathbb{R}}(\partial_x^2 u_m)^2\phi_N^{(1)}\,dx\,dt \le \frac{1}{5}\|u^2(t_1)\phi_N\|_{L^1}+\frac{1}{5}\|u^2(0)\phi_N\|_{L^1}+Ct_1\le M, \tag{5.7}$$

where $M = M(\|\langle x\rangle^{\frac{1}{2}+\alpha}u(0)\|_{L^2}+\|\langle x\rangle^{\frac{1}{2}+\alpha}u(t_1)\|_{L^2})$.

Taking into account that $\phi_N^{(1)}$ is a bounded function, the convergence in (5.2) implies that

$$\int_0^{t_1}\int_{\mathbb{R}}(\partial_x^2 u_m)^2\phi_N^{(1)}\,dx\,dt \to \int_0^{t_1}\int_{\mathbb{R}}(\partial_x^2 u)^2\phi_N^{(1)}\,dx\,dt,$$

as $m\to\infty$. Therefore, from (5.7), we obtain that

$$\int_0^{t_1}\int_{\mathbb{R}}(\partial_x^2 u)^2\phi_N^{(1)}\,dx\,dt \le M. \tag{5.8}$$

Since $\phi_N^{(1)}$ is an even function, $\phi_N^{(1)}(x)\ge 0$ and, for $x>1$,

$$\phi_N^{(1)}(x)\to(2\alpha+1)(1+x^2)^{\alpha-\frac{1}{2}}x\sim\langle x\rangle^{2\alpha},$$

as $N\to\infty$, applying Fatou's Lemma in (5.8), we have that

$$\int_0^{t_1}\int_{|x|\ge 1}(\partial_x^2 u)^2\langle x\rangle^{2\alpha}\,dx\,dt\le CM,$$

and taking into account that

$$\int_0^{t_1}\int_{|x|\le 1}(\partial_x^2 u)^2\langle x\rangle^{2\alpha}\,dx\,dt\le C,$$

we obtain that

$$\int_0^{t_1}\int_{\mathbb{R}}(\partial_x^2 u)^2\langle x\rangle^{2\alpha}\,dx\,dt\le C+CM<\infty. \tag{5.9}$$

From (5.9) it follows that

$$\partial_x^2 u(t)\in L^2(\langle x\rangle^{2\alpha}dx),\quad \text{a.e. } t\in[t_0,t_1]. \tag{5.10}$$

Let us define, for $t\in[t_0,t_1]$, $w(t):=\partial_x^2 u(t)$ and $w_m(t):=\partial_x^2 u_m(t)$. Then we have that

$$\partial_t w_m+\partial_x^5 w_m+u_m^2\partial_x w_m+6u_m\partial_x u_m w_m+2(\partial_x u_m)^3=0. \tag{5.11}$$

For $x\ge 0$ and $N\in\mathbb{N}$ let us define $\tilde\varphi_{N,\alpha}\in C^5([0,\infty))$ such that

$$\tilde\varphi_{N,\alpha}(x)=\begin{cases}(1+x^2)^\alpha-1 & \text{if } x\in[0,N],\\ (2N^2)^\alpha & \text{if } x\ge 10N,\end{cases}$$

$\tilde\varphi_{N,\alpha}^{(1)}(x)\ge 0$ and $|\tilde\varphi_{N,\alpha}^{(j)}(x)|\le C$ for $j=1,\dots,5$, with C independent of N.

Let $\tilde\phi_N\equiv\tilde\phi_{N,a}$ be the odd extension of $\tilde\varphi_{N,\alpha}$ to $\mathbb{R}$. Multiplying equation (5.11) by $w_m\tilde\phi_N$, integrating in x over $\mathbb{R}$, using integration by parts, and then integrating in t over an interval $[t_0^*,t_1^*]\subset[t_0,t_1]$ such that $\partial_x^2 u(t_0^*),\partial_x^2 u(t_1^*)\in L^2(\langle x\rangle^{2\alpha}dx)$, we obtain

$$\begin{aligned}
&\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x^2 w_m)^2\tilde{\phi}_N^{(1)}\,dx\,dt\\
&=\frac{1}{5}\int_{\mathbb{R}} w_m^2(t_1^*)\tilde{\phi}_N dx-\frac{1}{5}\int_{\mathbb{R}} w_m^2(t_0^*)\tilde{\phi}_N dx\\
&\quad+\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x w_m)^2(t)\tilde{\phi}_N^{(3)}\,dx\,dt-\frac{1}{5}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m^2(t)\tilde{\phi}_N^{(5)}\,dx\,dt\\
&\quad-\frac{1}{5}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} u_m^2(t)w_m^2(t)\tilde{\phi}_N^{(1)}\,dx\,dt\\
&\quad+2\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} u_m(t)\partial_x u_m(t)w_m^2(t)\tilde{\phi}_N\,dx\,dt\\
&\quad+\frac{4}{5}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x u_m)^3(t)w_m(t)\tilde{\phi}_N\,dx\,dt.
\end{aligned}\tag{5.12}$$

From (5.12) we shall prove that

$$\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x^2 w)^2(t)\langle x\rangle^{2\alpha-1}\,dx\,dt<\infty.$$

Let us observe that

$$\begin{aligned}
&\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x w_m)^2(t)\tilde{\phi}_N^{(3)}\,dx\,dt\\
&=-\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m(t)\partial_x^2 w_m(t)\tilde{\phi}_N^{(3)}\,dx\,dt+\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m^2(t)\tilde{\phi}_N^{(5)}\,dx\,dt.
\end{aligned}$$

Let K be a constant independent of N such that $|\tilde{\phi}_N^{(3)}|\le K\tilde{\phi}_N^{(1)}$. Then

$$\begin{aligned}
&\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x w_m)^2(t)\tilde{\phi}_N^{(3)}\,dx\,dt\\
&\le\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}\Big(Kw_m^2(t)+\frac{(\partial_x^2 w_m)^2(t)}{K}\Big)|\tilde{\phi}_N^{(3)}|\,dx\,dt\\
&\quad+\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m^2(t)\tilde{\phi}_N^{(5)}\,dx\,dt\\
&\le\frac{K}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m^2(t)|\tilde{\phi}_N^{(3)}|\,dx\,dt+\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x^2 w_m)^2(t)\tilde{\phi}_N^{(1)}\,dx\,dt\\
&\quad+\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m^2(t)\tilde{\phi}_N^{(5)}\,dx\,dt.
\end{aligned}\tag{5.13}$$

Taking into account that the fifth term on the right-hand side of (5.12) is not positive, from (5.12) and (5.13) it follows that

$$
\begin{aligned}
&\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x^2 w_m)^2(t)\tilde{\phi}_N^{(1)}\,dx\,dt\\
&\le \frac{1}{5}\int_{\mathbb{R}} w_m^2(t_1^*)\tilde{\phi}_N dx - \frac{1}{5}\int_{\mathbb{R}} w_m^2(t_0^*)\tilde{\phi}_N dx\\
&\quad + \frac{K}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m^2(t)|\tilde{\phi}_N^{(3)}|\,dx\,dt + \frac{3}{10}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} w_m^2(t)\tilde{\phi}_N^{(5)}\,dx\,dt\\
&\quad + 2\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}} u_m(t)\partial_x u_m(t) w_m^2(t)\tilde{\phi}_N\,dx\,dt\\
&\quad + \frac{4}{5}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x u_m)^3(t) w_m(t)\tilde{\phi}_N\,dx\,dt\\
&\equiv \frac{1}{5}\int_{\mathbb{R}} w_m^2(t_1^*)dx - \frac{1}{5}\int_{\mathbb{R}} w_m^2(t_0^*)\tilde{\phi}_N dx + I + II + III + IV.
\end{aligned}
\tag{5.14}
$$

Since $\tilde{\phi}_N^{(3)}$ and $\tilde{\phi}_N^{(5)}$ are bounded functions, the convergence in (5.2) implies that

$$
I + II \le C. \tag{5.15}
$$

We now estimate $III + IV$. Using the convergence (5.2), and the boundedness of the function $\tilde{\phi}_N^{(1)}$, we obtain, for $\alpha \in (0, 1/4]$, that

$$
\begin{aligned}
III + IV &\le C\int_{t_0^*}^{t_1^*}\|u_m(t)\|_{H^2}^3\|u_m(t)\tilde{\phi}_N\|_{L^\infty}dt\\
&\quad + C\int_{t_0^*}^{t_1^*}\|u_m(t)\|_{H^2}^2\int_{\mathbb{R}}|w_m(t)||\partial_x u_m(t)||\tilde{\phi}_N|\,dx\,dt\\
&\le C\int_{t_0^*}^{t_1^*}\|u_m(t)\tilde{\phi}_N\|_{H^1}dt + C\int_{t_0^*}^{t_1^*}\|w_m(t)\|_{L^2}\|\partial_x u_m(t)\tilde{\phi}_N\|_{L^2}dt\\
&\le C\int_{t_0^*}^{t_1^*}(\|u_m(t)\tilde{\phi}_N\|_{H^1} + \|\partial_x u_m(t)\tilde{\phi}_N\|_{L^2})dt\\
&\le C\int_{t_0^*}^{t_1^*}(\|u_m(t)\tilde{\phi}_N\|_{L^2} + \|\partial_x u_m(t)\tilde{\phi}_N\|_{L^2} + \|u_m(t)\tilde{\phi}_N^{(1)}\|_{L^2})dt\\
&\le C\int_{t_0^*}^{t_1^*}(\|u_m(t)\|_{L^2(\langle x\rangle^{4\alpha}dx)} + \|\partial_x u_m(t)\|_{L^2(\langle x\rangle^{4\alpha}dx)})dt + C\\
&\le C + C\int_{t_0^*}^{t_1^*}\|\partial_x u_m(t)\|_{L^2(\langle x\rangle^{4\alpha}dx)}dt.
\end{aligned}
\tag{5.16}
$$

Using Lemmas 2.3 and 2.2 and the convergence in (5.2), for $\alpha \in (0, 1/8]$ and $t \in [t_0^*, t_1^*] \subset [0, T]$, it follows that

$$
\begin{aligned}
\|\partial_x u_m(t)\|_{L^2(\langle x\rangle^{4\alpha}dx)} &= \|\langle x\rangle^{2\alpha}\partial_x u_m(t)\|_{L^2}\\
&\le \|\langle x\rangle^{1/4}\partial_x u_m(t)\|_{L^2} \le C\|J(\langle x\rangle^{1/4}u_m(t))\|_{L^2}\\
&\le C\|\langle x\rangle^{1/2}u_m(t)\|_{L^2}^{1/2}\|J^2 u_m(t)\|_{L^2}^{1/2} \le C.
\end{aligned}
\tag{5.17}
$$

From (5.14)-(5.17) we conclude that, if $\alpha \in (0, 1/8]$ then

$$\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x^2 w_m)^2(t)\tilde{\phi}_N^{(1)}\,dx\,dt \le \frac{1}{5}\int_{\mathbb{R}} w_m^2(t_1^*)\tilde{\phi}_N - \frac{1}{5}\int_{\mathbb{R}} w_m^2(t_0^*)\tilde{\phi}_N + C, \tag{5.18}$$

where $C = C(T)$ is a constant indepent of N and m.

Let us observe that from the convergence in (5.3) we can conclude that

$$\sup_{x\in\mathbb{R}}\int_0^T |\partial_x^2 w_m(x,t) - \partial_x^2 w(x,t)|^2 dt = \|\partial_x^4 u_m - \partial_x^4 u\|^2_{L_x^\infty L_T^2} \to 0,$$

as $m \to \infty$. Hence, denoting by $|A|$ the measure of a set $A \subset \mathbb{R}$, we have

$$\begin{aligned}
&\Big|\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}[(\partial_x^2 w_m)^2(x,t) - (\partial_x^2 w)^2(x,t)]\tilde{\phi}_N^{(1)}\,dx\,dt\Big| \\
&\le C\int_{supp\,\tilde{\phi}_N^{(1)}}\int_{t_0^*}^{t_1^*} |\partial_x^2 w_m - \partial_x^2 w||\partial_x^2 w_m + \partial_x^2 w|dtdx \\
&\le C\int_{supp\,\tilde{\phi}_N^{(1)}} \|\partial_x^2 w_m(x,\cdot) - \partial_x^2 w(x,\cdot)\|_{L_T^2}\|\partial_x^2 w_m(x,\cdot) + \partial_x^2 w(x,\cdot)\|_{L_T^2}\,dx \\
&\le C\|\partial_x^2 w_m - \partial_x^2 w\|_{L_x^\infty L_T^2}\|\partial_x^2 w_m + \partial_x^2 w\|_{L_x^\infty L_T^2}|\operatorname{supp}\tilde{\phi}_N^{(1)}| \\
&\le C\|\partial_x^2 w_m - \partial_x^2 w\|_{L_x^\infty L_T^2} \to 0,
\end{aligned}$$

as $m \to \infty$. Therefore, passing to the limit in (5.18), as $m \to \infty$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x^2 w)^2(t)\tilde{\phi}_N^{(1)}\,dx\,dt \\
&\le \frac{1}{5}\int_{\mathbb{R}} w^2(t_1^*)\tilde{\phi}_N - \frac{1}{5}\int_{\mathbb{R}} w^2(t_0^*)\tilde{\phi}_N + C \\
&\le C\int_{\mathbb{R}} w^2(t_1^*)\langle x\rangle^{2\alpha}dx + C\int_{\mathbb{R}} w^2(t_0^*)\langle x\rangle^{2\alpha} + C \equiv M.
\end{aligned} \tag{5.19}$$

Since $\tilde{\phi}_N^{(1)}$ is an even function, $\tilde{\phi}_N^{(1)} \ge 0$ and, for $x \ge 1$,

$$\tilde{\phi}_N^{(1)}(x) \to 2\alpha\langle x\rangle^{2\alpha-1}\langle x\rangle' \sim \langle x\rangle^{2\alpha-1},$$

as $N \to \infty$, applying Fatou's Lemma in (5.19) we can conclude that

$$\int_{t_0^*}^{t_1^*}\int_{|x|\ge 1}(\partial_x^2 w)^2(t)\langle x\rangle^{2\alpha-1}\,dx\,dt \le CM.$$

On the other hand, since $2\alpha - 1 \le 0$,

$$\begin{aligned}
\int_{t_0^*}^{t_1^*}\int_{|x|\le 1}(\partial_x^2 w)^2(t)\langle x\rangle^{2\alpha-1}\,dx\,dt &\le \int_{t_0^*}^{t_1^*}\int_{|x|\le 1}(\partial_x^2 w)^2(t)\,dx\,dt \\
&\le 2\|\partial_x^2 w\|^2_{L_x^\infty L_T^2} = 2\|\partial_x^4 u\|^2_{L_x^\infty L_T^2} < \infty.
\end{aligned}$$

In consequence

$$\int_{t_0^*}^{t_1^*}\int_{\mathbb{R}}(\partial_x^2 w)^2(t)\langle x\rangle^{2\alpha-1}\,dx\,dt < \infty. \tag{5.20}$$

From (5.20) it follows that

$$\partial_x^2 w(t)\langle x\rangle^{\alpha-1/2} \in L^2(\mathbb{R}), \quad \text{a.e. } t \in [t_0^*, t_1^*].$$

This fact and (5.10) imply that

$$\langle x\rangle^{\alpha} w(t) \in L^2 \text{ and } J^2(\langle x\rangle^{\alpha-1/2} w(t)) \in L^2, \quad \text{a.e. } t \in [t_0^*, t_1^*].$$

Let us define $f := \langle x\rangle^{\alpha-1/2} w(t)$. Then

$$\langle x\rangle^{1/2} f \in L^2 \text{ and } J^2(f) \in L^2.$$

Using Lemma 2.2 with $a = 2$ and $b = 1/2$ we conclude that, for $\theta \in [0,1]$,

$$\|J^{2\theta}(\langle x\rangle^{(1-\theta)/2} f)\|_{L^2} \le C \|\langle x\rangle^{1/2} f\|_{L^2}^{(1-\theta)} \|J^2(f)\|_{L^2}^{\theta}.$$

Taking $\theta = 2\alpha$, we have that $\|J^{4\alpha} w(t)\|_{L^2} < \infty$, a.e. $t \in [t_0^*, t_1^*]$, i.e., $w(t) \in H^{4\alpha}$, a.e. $t \in [t_0^*, t_1^*]$, i.e., $u(t) \in H^{2+4\alpha}$.

The rest of the proof is as in [9], in consequence we omit the details.

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Eddye Bustamante M.
Departamento de Matemáticas, Universidad Nacional de Colombia, A. A. 3840 Medellín, Colombia
E-mail address: `eabusta0@unal.edu.co`

José Jiménez U.
Departamento de Matemáticas, Universidad Nacional de Colombia, A. A. 3840 Medellín, Colombia
E-mail address: `jmjimene@unal.edu.co`

Jorge Mejía L.
Departamento de Matemáticas, Universidad Nacional de Colombia, A. A. 3840 Medellín, Colombia
E-mail address: `jemejia@unal.edu.co`