Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 145, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND UNIQUENESS OF LOCAL WEAK SOLUTIONS FOR THE EMDEN-FOWLER WAVE EQUATION IN ONE DIMENSION

MENG-RONG LI

ABSTRACT. In this article we consider the existence and uniqueness of local weak solutions to the Emden-Fowler type wave equation

$$t^2u_{tt} - u_{xx} = |u|^{p-1}u \quad \text{in } [1,T] \times (a,b)$$

with initial-boundary value conditions in a finite time interval.

1. INTRODUCTION

In this article we focus on the existence and uniqueness of weak solutions in $\mathcal{H}_2 := C^1([1,T), H_0^1(a,b)) \cap C^2([1,T), L^2(a,b))$ for the Emden-Fowler type wave equation

$$t^2u_{tt} - u_{xx} = |u|^{p-1}u \quad \text{in } [1,T) \times (a,b) \tag{1.1}$$

subject to zero boundary values and initial values

$$u(1) = u_0 \in H^2(a,b) \cap H_0^1(a,b), \quad \text{and} \quad u_t(1) = u_1 \in H_0^1(a,b).$$

Here $p > 1$, and a and b are real numbers.

The study of the Emden-Fowler ordinary differential equation is derived from earlier theories concerning gas dynamics in astrophysics developed at the turn of the 20th century. The fundamental problem in the study of stellar structures at that time was to study the equilibrium configuration of the mass of spherical gas clouds. The equation

$$\frac{d}{dt}(t^2\frac{du}{dt}) + t^2u^p = 0, \tag{1.2}$$

is generally referred to as the Lane-Emden equation. Astrophysicists were interested in the behavior of the solution of (1.2) which satisfies the initial condition $u(0) = 1$, $u'(0) = 0$. The mathematical foundation for the investigation of such an equation and also of the more general equation

$$\frac{d}{dt}\big(t^\rho\frac{du}{dt}\big) + t^\sigma u^\gamma = 0, \quad t \geq 0, \tag{1.3}$$

was made by Fowler [15, 16, 17, 18] in a series of four papers from 1914 to 1931. The Emden-Fowler equation also arises in the study of gas dynamics and fluid

---

2010 *Mathematics Subject Classification.* 35L05, 35D30, 35G20, 35G31, 35L70, 35L81.
Key words and phrases. Emden-Fowler wave equation; existence; uniqueness.

Submitted April 20, 2015. Published June 6, 2015.

mechanics [12], there the solutions of physical interest are bounded non-oscillatory ones which possess a positive zero. The zero of such a solution corresponds to an equilibrium state in a fluid with spherical distribution of density and under mutual attraction of its particles. The Emden-Fowler equations also appear in the study of relativistic mechanics, nuclear physics and also in the study of chemically reacting systems [1, 7, 10, 11, 13, 14, 19, 29, 31]. The Emden-Fowler equation (1.3) can be transformed into a first order nonlinear autonomous system, in fact a quadratic system, and information concerning its solutions can be obtained from the associated quadratic systems through phase plane analysis. This approach was first used by Emden in his analysis of the Lane-Emden equation (1.2).

For a general survey on the Thomas-Fermi equation, we refer the reader to March [30]. The first comprehensive study on the generalized Emden-Fowler equation

$$\frac{d^2x}{dt^2} + a(t)|x|^\gamma \operatorname{sgn} x = 0, \quad t \geq 0$$

was made by Atkinson [2, 3, 4, 5, 6].

Recently, in [27], we considered positive solutions of the Emden-Fowler equation $t^2u'' = u^p$ and obtained some results on the non-existence of global solutions, the estimates for the life-spans and the asymptotic behavior of solutions.

About the semilinear wave equation, Jörgens [21] published the first global existence theorem for the equation

$$\Box\, u + f(u) = 0 \quad \text{in } [0,T) \times \Omega, \tag{1.4}$$

in case $\Omega = \mathbb{R}^n, n = 3$ and $f(u) = g(u^2)u$. His result can be applied to the equation $\Box\, u + u^3 = 0$; and Browder [9] generalized Jörgens' result for $n > 2$.

For local Lipschitz f, Li [25, 26] proved the nonexistence of global solutions of the initial-boundary value problem for the semilinear wave equation (1.4) in a bounded domain $\Omega \subset \mathbb{R}^n$ under the assumption

$$\bar{E}(0) = \|Du\|_2^2(0) + 2\int_\Omega f(u)(0,x)dx \leq 0,$$

$$\eta f(\eta) - 2(1+2\alpha)\int_0^\eta f(r)dr \leq \lambda_1 \alpha \eta^2 \quad \forall \eta \in \mathbb{R}$$

with $\alpha > 0$, $\lambda_1 := \sup\{\|u\|_2/\|\nabla u\|_2 : u \in H_0^1(\Omega)\}$ and $a'(0) > 0$. There we have obtained a rough estimate for the life-span

$$T \leq \beta_2 := 2\big[1 - \big(1 - k_2 a(0)^{-\alpha}\big)^{1/2}\big]/(k_1k_2)$$

with

$$k_1 := \alpha a(0)^{-\alpha-1}\sqrt{a'(0)^2 - 4E(0)a(0)}, \quad k_2 := (-4\alpha^2 E(0)/k_1^2)^{\alpha/(1+2\alpha)}.$$

For $n = 3$ and $f(u) = -u^3$, there exist global solutions of (1.4) for small initial data [24]; but if $E(0) < 0$ and $a'(0) > 0$ then the solutions are only local, i.e. $T < \infty$ [26]. John [22] showed the nonexistence of solutions of the initial-boundary value problem for the wave equation $\Box\, u = A|u|^p$, $A > 0$, $1 < p < 1 + \sqrt{2}$, $\Omega = \mathbb{R}^3$. This problem was considered by Glassey [20] in the two dimensional case $n = 2$; for $n > 3$ Sideris [32] showed the nonexistence of global solutions under the conditions $\|u_0\|_1 > 0$ and $\|u_1\|_1 > 0$. According to this result Strauss [33, p. 27] guessed that the solutions for the above mentioned wave equation are global for $\Omega = \mathbb{R}^n$, $p \geq p_0(n) = \lambda$, which is the positive root of the quadratic equation $(n-1)\lambda^2 - (n+1)\lambda - 2 = 0$.

We want to extend our results [27] on ordinary differential equations and the wave equation [25] to the equation (1.4), therefore we will deal with the existence and uniqueness theme of Emden-Fowler type wave equation (1.1) with zero boundary values and initial values $u(1) = u_0 \in H^2(a,b) \cap H_0^1(a,b)$ and $u_t(1) = u_1 \in H_0^1(a,b)$, where $p > 1$, and a, b are real numbers.

We are not aware of any other paper discussing this theme. We make a substitution $t = e^s$, $u(t,x) = v(s,x)$ to avoid degeneration of the equation (1.1) which can be transferred into a nonlinear wave equation with negative linear damping (2.1) below. The main difficulties in constructing our existence result for equation (2.1) are the use of the Banach Fixed Point Theorem in a suitable solution space and the control of the boundedness of successive approximations solutions of equation (2.1). We shall set-up the fundamental lemmas in section 2 and prove the main result in section 3.

2. Fundamental lemmas

To obtain the existence of solutions to (1.1) with zero boundary values, $u(t,a) = u(t,b) = 0$, and initial values $u(1) = u_0 \in H^2(a,b) \cap H_0^1(a,b)$, $u_t(1) = u_1 \in H_0^1(a,b)$, we need a fundamental Lemma from [28, p. 95], [23, p. 96].

Lemma 2.1. *For $f \in W^{1,1}([t_0,T), L^2(a,b))$ the linear wave equation*

$$\begin{gathered}\Box\, u := u_{tt} - u_{xx} = f(t,x) \quad \text{in } [1,T) \times (a,b)\\ u(t_0,\cdot) := u_0 \in H^2(a,b) \cap H_0^1(a,b),\\ u_t(t_0,\cdot) := u_1 \in H_0^1(a,b),\end{gathered}$$

possesses exactly one solution $u \in \mathcal{H}_2 := C^1([t_0,T), H_0^1(a,b)) \cap C^2([t_0,T), L^2(a,b))$ with $u(t) \in H^2(a,b)$ for all $t \in [t_0,T]$. Furthermore,

$$\frac{d}{dt}\int_a^b (u_t^2 + |\nabla u|^2)dx - 2\int_a^b u_t f(t,x)dx = 0 \quad \text{a.e. in } [t_0,T).$$

To prove the existence of a local weak solution of (1.1) in $\mathcal{H}_2$, we make the substitution $s = \ln t$, $u(t,x) = v(s,x)$, then (1.1) can be transformed into

$$v_{ss} - v_{xx} = v_s + |v|^{p-1}v := -h(v), \tag{2.1}$$

$$v(0,x) = u(1,x) = u_0(x) := v_0(x), \tag{2.2}$$

$$v_s(0,x) = u_1(x) := v_1(x). \tag{2.3}$$

For $T > 0$, $S = \ln T$ and $v \in \bar{\mathcal{H}}_2 = C^1([0,S), H_0^1(a,b)) \cap C^2([0,S), L^2(a,b))$, we want to prove that $h(v) \in W^{1,1}([0,S), L^2(a,b))$, thus we build the following Lemma.

Lemma 2.2. *For $T > 0$, $S = \ln T$, $v \in \bar{\mathcal{H}}_2$, we have $h(v) \in W^{1,1}([0,S), L^2(a,b))$; that is,*

$$\int_0^S (\|h(v)\|_2 + \|\frac{\partial}{\partial s}h(v)\|_2)ds < \infty \quad \text{if } \|v\|_{\bar{H}2} < \infty.$$

Proof. By the definition of $h(v)$ we have

$$\|h(v)\|_{W^{1,1}}^2 = \Big(\int_0^S (\|h(v)\|_2 + \|\frac{\partial}{\partial s}h(v)\|_2)ds\Big)^2$$

$$
\begin{aligned}
&\le \int_0^S 1\,ds \int_0^S \Big(\|h(v)\|_2 + \|\frac{\partial}{\partial s}h(v)\|_2\Big)^2 ds \\
&\le 2S \int_0^S \Big(\|h(v)\|_2^2 + \|\frac{\partial}{\partial s}h(v)\|_2^2\Big) ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(\int_0^S (\|h(v)\|_2 + \|\frac{\partial}{\partial s}h(v)\|_2) ds\Big)^2 \\
&\le 2S \int_0^S \Big(\int_a^b (v_s + |v|^{p-1}v)^2 dx + \int_a^b (v_{ss} + p|v|^{p-1}v_s)^2 dx\Big) ds \\
&\le 4S \int_0^S \int_a^b \Big(v_s^2 + |v|^{2p} + v_{ss}^2 + p|v|^{2p-2}v_s^2\Big)\,dx\,ds \\
&= 4S\Big(\int_0^S \int_a^b (v_s^2 + v_{ss}^2)\,dx\,ds + I + II\Big),
\end{aligned}
$$

where

$$
I = \int_0^S \int_a^b |v|^{2p}\,dx\,ds, \quad II = p\int_0^S \int_a^b |v|^{2p-2}v_s^2\,dx\,ds.
$$

The boundedness of $\|h(v)\|_{W^{1,1}}^2$ is equivalent to show the boundedness of these two integrals I and II for some small S, near zero, and this can be deduced using the Sobolev inequality, since that for any fixed $s \in [0,S]$ and $x \in [a,b]$ we have the following estimates

$$
|v|^{2p}(s,x) \le \Big(\int_a^x |v_x|(s,\eta)d\eta\Big)^{2p} \le (x-a)^p\Big(\int_a^x v_x^2(s,\eta)d\eta\Big)^p, \tag{2.4}
$$

$$
\begin{aligned}
\int_a^b |v|^{2p}(s,x)dx &\le \int_a^b \Big(\int_a^x |v_x|(s,\eta)d\eta\Big)^{2p} dx \\
&\le \int_a^b (x-a)^p\Big(\int_a^x v_x^2(s,\eta)d\eta\Big)^p dx \\
&\le \frac{1}{p+1}(b-a)^{p+1}\Big(\int_a^b v_x^2(s,\eta)d\eta\Big)^p \\
&\le \frac{1}{p+1}(b-a)^{p+1}\Big(\max_{s\in[0,S]}\int_a^b v_x^2(s,\eta)d\eta\Big)^p,
\end{aligned} \tag{2.5}
$$

$$
I \le \frac{1}{p+1}(b-a)^{p+1}S\Big(\max_{s\in[0,S]}\int_a^b v_x^2(s,\eta)d\eta\Big)^p, \tag{2.6}
$$

$$
\begin{aligned}
|v|^{2p-2}v_s^2(s,x) &\le 2|v|^{2p-2}(s,x)\Big(v_1^2(x) + \Big(\int_0^s v_{ss}(\xi,x)d\xi\Big)^2\Big) \\
&\le 2|v|^{2p-2}(s,x)\Big(v_1^2 x + s\int_0^s v_{ss}^2(\xi,x)d\xi\Big),
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
II &= p\int_a^b \int_0^S |v|^{2p-2}v_s^2(s,x)\,ds\,dx \\
&\le 2p\int_a^b \int_0^S |v|^{2p-2}(s,x)\Big(v_1^2(x) + s\int_0^s v_{ss}^2(\xi,x)d\xi\Big)\,ds\,dx \\
&= III + IV,
\end{aligned} \tag{2.8}
$$

where

$$III = 2p\int_a^b\int_0^S v_1^2(x)|v|^{2p-2}(s,x)\,ds\,dx,$$
$$IV = 2p\int_a^b\int_0^S s|v|^{2p-2}(s,x)\Big(\int_0^s v_{ss}^2(\xi,x)d\xi\Big)\,ds\,dx.$$

By (2.4) and (2.5) we obtain

$$\begin{aligned}
III &\le 2p\int_0^S\Big(\int_a^b |v|^{2p}(s,x)dx\Big)^{\frac{p-1}{p}}\Big(\int_a^b v_1^{2p}(x)dx)^{1/p}ds\\
&\le 2p\int_0^S\Big(\Big(\max_{s\in[0,S]}\int_a^b v_x^2(s,\eta)d\eta\Big)^p\Big)^{\frac{p-1}{p}}\Big(\Big(\int_a^b v_1'(\eta)^2d\eta\Big)^p\Big)^{1/p}ds\\
&= \frac{4p^2}{p+1}(b-a)^{p+1}S\Big(\int_a^b v_1'(\eta)^2d\eta\Big)\Big(\max_{s\in[0,S]}\int_a^b v_x^2(s,\eta)d\eta\Big)^{p-1}
\end{aligned}$$

and

$$\begin{aligned}
IV &\le 2pS\int_a^b\int_0^S\Big((x-a)^{p-1}(\int_a^x v_x^2(s,\eta)d\eta)^{p-1}\Big)\Big(\int_0^s v_{ss}^2(\xi,x)d\xi\Big)\,ds\,dx\\
&\le 2pS(b-a)^{p-1}\Big(\max_{s\in[0,S]}\int_a^b v_x^2(s,x)dx\Big)^{p-1}\\
&\quad\times\int_a^b\int_0^S\Big(\int_0^s v_{ss}^2(\xi,x)d\xi\Big)\,ds\,dx\\
&\le 2pS^2(b-a)^{p-1}\Big(\max_{s\in[0,S]}\int_a^b v_x^2(s,x)dx\Big)^{p-1}\int_a^b\int_0^S v_{ss}^2(\xi,x)d\xi dx.
\end{aligned}$$

□

The following Lemma is easy to check, we omit its proof.

Lemma 2.3. *Suppose that X is a Banach space and $f_n : [t_0, T) \to X$ are differentiable functions and the sequence $f_n(t)$ converges uniformly to $f(t)$. If the sequence $df_n(t)/dt$ converges to $g(t)$, then $f : [t_0, T) \to X$ is differentiable and $df(t)/dt = g(t)$ in X.*

3. Existence of solutions for the Emden-Fowler type wave equation

From the three lemmas above, we can obtain the following local existence result.

Theorem 3.1. *Suppose that $p > 1$, $u_0 \in H^2(a,b)\cap H_0^1(a,b)$ and $u_1 \in H_0^1(a,b)$, then the initial-boundary value problem for the semilinear wave equation* (1.1) *with $u(1,x) = u_0(x)$, $u_t(1,x) = u_1(x)$ and $u(t,a) = 0 = u(t,b)$ on $[1,T]$, possesses exactly one solution in $\mathcal{H}_2$ for some $T > 1$.*

Proof. Proof the existence of a local solution in

$$\mathcal{H}_1 := C([1,T), H_0^1(a,b))\cap C^1([1,T), L^2(a,b)).$$

By using the substitution $s = \ln t$, $u(t,x) = v(s,x)$, equation (1.1) can be transformed to

$$\begin{gathered}
v_{ss} - v_{xx} = v_s + |v|^{p-1}v := -h(v),\\
v(0,x) = u(1,x) = u_0(x) := v_0(x),
\end{gathered}$$

$$v_s(0,x) = u_1(x) := v_1(x).$$

(1) For $T > 0$ and $v \in \bar{\mathcal{H}}_2 = C^1([0,S), H_0^1(a,b)) \cap C^2([0,S), L^2(a,b))$, by Lemma 2.2 we have that $h(v) \in W^{1,1}([0,S), L^2(a,b))$.

According to Lemma 2.1, let $w := Tv$ be the solution of initial-boundary value problem for the equation

$$\begin{gathered} \Box\, w + h(v) = 0, \\ w(0,\cdot) := v_0(\cdot) \in H^2(a,b) \cap H_0^1(a,b), \\ w_s(0,\cdot) := u_1(\cdot) = v_1(\cdot) \in H_0^1(a,b), \end{gathered}$$

we have $w \in \bar{\mathcal{H}}_2, w(s) \in H^2(a,b)$ for all $s \in [0,S)$ and

$$\frac{d}{ds}\|Dw\|_2^2(s) + 2\int_a^b w_s h(v)(s,x)dx = 0.$$

Suppose that $v_2 := su_0$, then by Lemma 2.2 we get $-h(v_2) = -h(su_0) = u_0 + su_0|su_0|^{p-1} \in W^{1,1}([0,S), L^2(a,b))$ and therefore, there exists a function $v_3 \in \bar{\mathcal{H}}_2$ which satisfies the initial-boundary value problem for the equation

$$\begin{gathered} \Box\, w + h(v_2) = 0, \\ w(0,\cdot) := u_0(\cdot) \in H^2(a,b) \cap H_0^1(a,b), \\ w_s(0,\cdot) := v_1(\cdot) \in H_0^1(a,b). \end{gathered}$$

Let $v_{m+1} := Tv_m, m \geq 2$ be the solution of the initial-boundary value problem for the linear equation

$$\begin{gathered} \Box\, v_{m+1} + h(v_m) = 0 \quad \text{in } [0,S) \times (a,b), \\ v_{m+1}(0,\cdot) = u_0(\cdot) \in H^2(a,b) \cap H_0^1(a,b), \\ (v_{m+1})_s(0,\cdot) = v_1(\cdot) \in H_0^1(a,b). \end{gathered}$$

Therefore, by Lemma 2.1, we have $v_{m+1}(s) \in H^2(a,b)$ for all $s \in [0,S)$, $v_{m+1} \in \bar{\mathcal{H}}_2, m \in \mathbb{N}$ and

$$\frac{d}{ds}\int_a^b |Dv_{m+1}(s,x)|^2 dx + 2\int_a^b (v_{m+1})_s h(v_m(s,x))dx = 0 \quad \text{a.e. in } [0,S), \tag{3.1}$$

where $|Dv|^2 := v_s^2 + |v_x|^2$. Set $A_{m+1}(s) := \|Dv_{m+1}(s)\|_2$. Then by (2.5) we find that

$$(A_{m+1}(s)^2)' \leq 2A_{m+1}(s)\|h(v_m(s))\|_2 \quad \text{a.e. in } [0,S)), \tag{3.2}$$

$$A_{m+1}(s) \leq A_{m+1}(0) + \int_0^s \|h(v_m)(r)\|_2 dr \tag{3.3}$$

and

$$\begin{aligned} &A_{m+1}(s) \\ &\leq \|u_1\|_2 + \|u_0'\|_2 + \int_0^s \|((v_m)_s + |v_m|^{p-1}v_m)\|_2 dr \\ &\leq \|u_1\|_2 + \|u_0'\|_2 + \int_0^s (\|(v_m)_s\|_2 + \|v_m^p\|_2) dr \\ &\leq \|u_1\|_2 + \|u_0'\|_2 + \int_0^s (\|(v_m)_s\|_2 + \sqrt{\frac{1}{p+1}}(b-a)^{\frac{p+1}{2}}\|(v_m)_x\|_2^p) dr \end{aligned} \tag{3.4}$$

for every $m-1 \in \mathbb{N}$, almost everywhere in $[0,S)$, besides $A_{m+1}(s)=0$.

(2) Since that $h(sv_0)=h(v_2) \in W^{1,1}([0,S),L^2(a,b))$, we get $v_3 \in \bar{\mathcal{H}}_2$ and by the inequality (3.3), we obtain $-h(v_2)=u_0+su_0|su_0|^{p-1}$,

$$
\begin{aligned}
&A_3(s)\\
&\le A_3(0)+\int_0^s \|h(ru_0)\|_2 dr\\
&\le \|u_1\|_2+\|u_0'\|_2+\int_0^s \|u_0+ru_0|ru_0|^{p-1}\|_2 dr\\
&\le \|u_1\|_2+\|u_0'\|_2+\int_0^s \Big(\int_a^b (u_0+ru_0|ru_0|^{p-1})^2 dx\Big)^{1/2} dr\\
&\le \|u_1\|_2+\|u_0'\|_2+\sqrt{2}\int_0^s \Big(\int_a^b (u_0^2+r^{2p}u_0^{2p})dx\Big)^{1/2} dr\\
&\le \|u_1\|_2+\|u_0'\|_2+\sqrt{2}\int_0^s \Big(\Big(\int_a^b u_0^2 dx\Big)^{1/2}+r^p(\int_a^b u_0^{2p}(x)dx)^{1/2}\Big) dr\\
&= \|u_1\|_2+\|u_0'\|_2+\sqrt{2}s\big(\|u_0\|_2+\frac{1}{p+1}s^p\|u_0\|_{2p}^p\big)\\
&\le \|u_1\|_2+\|u_0'\|_2+\sqrt{2}S\big(\|u_0\|_2+\frac{1}{p+1}S^p\|u_0\|_{2p}^p\big).
\end{aligned} \tag{3.5}
$$

Set

$$
\begin{gathered}
\|v\|_{\infty,S} := \sup\{\|Dv(s)\|_2 : 0 \le s \le S\},\\
M := 1+2(\|u_1\|_2+\|u_0'\|_2),\\
S := \frac{1}{2}\min\Big\{\frac{1}{1+\sqrt{p}(b-a)^{\frac{p}{2}+1}M^{p-1}}, \frac{(p+1)^{1/p}}{1+\|u_0\|_{2p}},\\
\frac{1+\|u_1\|_2+\|u_0'\|_2}{2(1+\|u_0\|_2)+M(1+(b-a)^{\frac{p+1}{2}}M^{p-1})}\Big\}.
\end{gathered}
$$

Then using (3.5), we obtain

$$
\begin{aligned}
A_3(s)=\|Dv_3\|_2(s) &\le A_3(0)+\int_0^s \|h(ru_0)\|_2(r)dr\\
&\le \|u_1\|_2+\|u_0'\|_2+\sqrt{2}(S-0)(\|u_0\|_2+\frac{1}{p+1}(S-0)^p\|u_0\|_{2p}^p) \le M.
\end{aligned}
$$

Consequently $\|v_3\|_{\infty,S} \le M$.

Suppose that $\|v_m\|_{\infty,S} \le M$, then by the definition of S and (3.4),

$$
\begin{aligned}
&A_{m+1}(s)\\
&\le \|u_1\|_2+\|u_0'\|_2+\int_0^s \Big(\|(v_m)_s\|_2+\sqrt{\frac{1}{p+1}}(b-a)^{\frac{p+1}{2}}\|(v_m)_x\|_2^p\Big)(r)dr \le M.
\end{aligned}
$$

Thus we get $\|v_{m+1}\|_{\infty,S} \le M$ for all $m \in \mathbb{N}$.

(3) We claim thatv_m is a Cauchy sequence in

$$
\bar{\mathcal{H}}_1 := C([0,S),H_0^1(a,b)) \cap C^1([0,S),L^2(a,b)).
$$

By Lemma 2.1 and (3.3),

$$
\|D(v_{m+1}-v_m)(s)\|_2(s)
$$

$$
\begin{aligned}
&\le \int_0^s \|h(v_m)-h(v_{m-1})\|_2 dr\\
&\le \int_0^s \Big(\|(v_m-v_{m-1})_s\|_2+\frac{\sqrt{p}}{2}(b-a)^p K_v\|v_m-v_{m-1}\|_2\Big)dr,
\end{aligned}
$$

where $K_v=\|(v_m)_x\|_2^{p-1}+\|(v_{m-1})_x\|_2^{p-1}$, therefore

$$
\begin{aligned}
&\|D(v_{m+1}-v_m)(s)\|_2(s)\\
&\le s\Big(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1}\Big)\|D(v_m-v_{m-1})(s)\|_2\\
&\le S\Big(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1}\Big)\|v_m-v_{m-1}\|_{\infty,S}\quad \forall s\in[0,S),
\end{aligned}
\tag{3.6}
$$

and

$$
\|v_{m+1}-v_m\|_{\infty,S}\le S\Big(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1}\Big)\|v_m-v_{m-1}\|_{\infty,S}\,.
$$

It follows that

$$
\|v_{m+k}-v_m\|_{\infty,S}\le\frac{(S(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1}))^{m-2}\|v_3-v_2\|_{\infty,S}}{1-S(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1})}\to 0 \tag{3.7}
$$

as $m\to\infty$. Since

$$
S\Big(1+\sqrt{\frac{p}{2}}(b-a)^{\frac{p}{2}+1}M^{p-1}\Big)\le\frac12\,\frac{1+\sqrt{\frac{p}{2}}(b-a)^{\frac{p}{2}+1}M^{p-1}}{1+\sqrt{p}(b-a)^{\frac{p}{2}+1}M^{p-1}}\le\frac12.
$$

(ii) We prove the uniqueness of the solutions in $\mathcal{H}_1$. Suppose that w is the limit of v_m, and $v\in\bar{\mathcal{H}}_1$ is an another solution for (2), then

$$
\begin{aligned}
&\frac{d}{ds}\int_a^b |Dv_{m+1}(s,x)-Dv(s,x)|^2dx\\
&\le 2\int_a^b |((v_{m+1})_s-v_s)(h(v_m)-h(v))|dx\\
&\le 2\|D(v_{m+1}-v)\|_2(s)\|h(v_m)-h(v)\|_2(s),
\end{aligned}
$$

$$
\begin{aligned}
\|D(v_{m+1}-v)(s)\|_2(s)&\le\int_0^s\|h(v_m)-h(v)\|_2(r)dr\\
&\le s\Big(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1}\Big)\|D(v_m-v)(s)\|_2\\
&\le S\Big(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1}\Big)\|v_m-v\|_{\infty,S}\quad\forall s\in[0,S).
\end{aligned}
$$

Thus

$$
\|v_{m+1}-v\|_{\infty,S}\le S\Big(1+\sqrt{\frac{p}{2}}(b-a)^{1+\frac{p}{2}}M^{p-1}\Big)\|v_m-v\|_{\infty,S}.
$$

It follows that

$$
\|w-v\|_{\infty,S}\le\|w-v_{m+1}\|_{\infty,S}+\|v_{m+1}-v\|_{\infty,S}\to 0
$$

as $m\to\infty$, so $w\equiv v$ in $\bar{\mathcal{H}}_1$.

(iii) Now we show the local existence u in $\mathcal{H}_2$. Let S and M be the same as above.

(1) For a $S>0$ and $v_m \in \bar{\mathcal{H}}_2$, we have $h(v_m) \in W^{1,1}([0,S), L^2(a,b))$. According to Lemma 2.1, we have $v_{m+1}(s) \in H^2(a,b)$ for all $s \in [0,S)$, $m \in \mathbb{N}$ and

$$\frac{d}{ds}\int_a^b |Dv_{m+1}(s,x)|^2dx = -2\int_a^b (v_{m+1})_s h(v_m)dx \quad \text{a.e. in } [0,S)$$

also

$$\begin{aligned}
&\frac{d}{ds}\|Dv_{m+1}\|_2^2(s) + \frac{d^2}{ds^2}\int_a^b v_{m+1}^2 dx\\
&= -2\int_a^b (v_{m+1})_s h(v_m(s,x))dx + 2\int_a^b (v_{m+1}(v_{m+1})_{ss} + (v_{m+1})_s^2)dx\\
&= -2\int_a^b (v_{m+1})_s(s,x)h(v_m)dx\\
&\quad + 2\int_a^b \Big(v_{m+1}[(v_{m+1})_{xx} - h(v_m)] + (v_{m+1})_s^2\Big)(s,x)dx\\
&= -2\int_a^b (v_{m+1} + (v_{m+1})_s)h(v_m)(s,x)dx + 2\int_a^b \big((v_{m+1})_s^2 - (v_{m+1})_x^2\big)(s,x)dx,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\frac{d}{ds}\|Dv_{m+1}\|_2^2(s) + \frac{d^2}{ds^2}\int_a^b v_{m+1}(s,x)^2dx\Big|\\
&\le 2(\|v_{m+1}\|_2 + \|(v_{m+1})_s\|_2)\|h(v_m)\|_2(s) + 2\|Dv_{m+1}\|_2^2(s),
\end{aligned} \tag{3.8}$$

for every $m \ge 2$, almost everywhere in $[0,S)$.

(2) We claim that v_m is a Cauchy sequence in $\bar{\mathcal{H}}_2$. By similar arguments as in establishing inequalities (3.6)–(3.8), we obtain

$$\begin{aligned}
&\Big|\frac{d}{ds}\|D(v_{m+k} - v_m)\|_2^2(s) + \frac{d^2}{ds^2}\int_a^b (v_{m+k} - v_m)(s,x)^2dx\Big|\\
&\le 2(\|v_{m+k} - v_m\|_2 + \|(v_{m+k} - v_m)_s\|_2)\|h(v_{m+k-1}) - h(v_{m-1})\|_2(s)\\
&\quad + 2\|D(v_{m+k} - v_m)\|_2^2(s)\\
&\to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

By (i), (ii) and Lemma 2.3, we obtain the assertions of Theorem 3.1. $\square$

References

[1] R. Aris; *Introduction to the Analysis of Chemical Reactors*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

[2] F. V. Atkinson; *On linear perturbation of nonlinear differential equations*, Canad. J. Math., 6 (1954), 561-571.

[3] F. V. Atkinson; *The asymptotic solutions of second order differential equations*, Ann. Mat. Pura. Appl., 37 (1954), 347-378.

[4] F. V. Atkinson; *On second order nonlinear oscillation*, Pacific J. Math., 5 (1955),.643-647.

[5] F. V. Atkinson; *On asymptotically linear second order oscillations*, J. Rational Mech. Anal., 4 (1955), 769-793.

[6] F. V. Atkinson; *On second order differential inequalities*, Proc. Roy. Soc. Edinburgh Sect. A, (1973).

[7] A. M. Arthur, P. D. Robinson; *Complementary variational principle for $\nabla^2\phi = f(\phi)$ with applications to the Thomas-Fermi and Liouville equations*, Proc. Cambridge Philos. Soc., 65 (1969), 535-542.

[8] R. Bellman; *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.

[9] F. E. Browder; *On non-linear wave equations*, M. Z. 80 (1962), 249-264.

[10] S. Chandrasekhar; *Introduction to the Study of Stellar Structure*, University of Chicago Press, Chicago, 1939.

[11] S. Chandrasekhar; *Principles of Stellar Dynamics*, University of Chicago Press, Chicago, 1942.

[12] R. Conti, D. Graffi, G. Sansone; *The Italian contribution to the theory of nonlinear ordinary differential equations and to nonlinear mechanics during the years 1951-1961*, Qualitative Methods in the Theory of Nonlinear Vibrations, Proc. Internat. Sympos. Nonlinear Vibrations, vol. II, (1961), 172-189.

[13] H. T. Davis; *Introduction to Nonlinear Differential and Integral Equations*, U. S. Atomic Energy Commission, Washington, D. C., 1960.

[14] A. Das, C. V. Coffman; *A class of eigenvalues of the fine-structure constant and internal energy obtained from a class of exact solutions of the combined Klein-Gordon-Maxwell - Einstein field equations*, J. Math. Phys., 8 (1967), 1720-1735.

[15] R. H. Fowler; *The form near infinity of real, continuous solutions of a certain differential equation of the second order*, Quart. J. Math., 45 (1914), 289-350.

[16] R. H. Fowler; *The solution of Emden's and similar differential equations*, Monthly Notices Roy, Astro. Soc., 91 (1930), 63-91.

[17] R. H. Fowler; *Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations*, Proc. London Math. Soc.,13 (1914), 341-371.

[18] R. H. Fowler; *Further studies of Emden's and similar differential equations*, Quart. J. Math., 2 (1931), 259-288.

[19] G. R. Gavalas; *Nonlinear Differential Equations of Chemically Reacting Systems*, Springer-Verlag, New York, 1968.

[20] R. Glassey; *Finite-time blow-up for solutions of nonlinear wave equations*, M. Z. 177 (1981), 323-340.

[21] K. Jörgens; *Das Anfangswertproblem im Großen für eine Klasse nichtlinearer Wellengleichungen*, M. Z. 77 (1961), 295-307.

[22] F. John; *Blow-up for quasilinear wave equations in three space dimensions*, Comm. Pure. Appl. Math. 36 (1981).29-51.

[23] A. Haraux; *Nonlinear Evolution Equations - Global Behavior of Solutions*, Lecture Notes in Math. Springer 1981.

[24] S. Klainerman; *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math.33, (1980), 43-101.

[25] M. R. Li; *Estimates for the life-span of the solutions of some semilinear wave equations*, Communications on Pure and Applied Analysis, Vol.7, No. 2, (2008), 417-432.

[26] M. R. Li; *Nichtlineare Wellengleichungen 2. Ordnung auf beschränkten Gebieten.* PhD - Dissertation Tübingen 1994.

[27] M. R. Li, H. Y. Yao, Y. T. Li; *Asymptotic behavior of positive solutions of the nonlinear differential equation* $t^2u'' = u^n$, Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 250, 1-9.

[28] J. Lions, E. Magenes; *Non-Homogenous Boundary Value Problems and Applications*, Springer Verlag 1972, Vol. 2.

[29] Z. Nehari; *On a nonlinear differential equation arising in nuclear physics*, Proc. Roy. Irish Academy Sect. A, 62 (1963), 117-135.

[30] N. H. March; *The Thomas-Fermi approximation in quantum mechanics*, Advances in Phys. 6 (1957), 1-101.

[31] V. N. Shevyelo; *Problems, methods, and fundamental results in the theory of oscillation of solutions of nonlinear non-autonomous ordinary differential equations*, Proc. 2nd All-Union Conf on Theoretical and Appl. Mech., Moscow, (1965), 142-157.

[32] T. Sideris; *Nonexistence of global solutions to semi-linear wave equations in high dimensions*, J. Differential Equations 52 (1982), 303-345.

[33] W. A. Strauss; *Nonlinear Wave Equations*, A.M.S. Providence, 1989.

Meng-Rong Li

Department of Mathematical Sciences, National Chengchi University, Taipei, Taiwan

E-mail address: liwei@math.nccu.edu.tw