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EXTENDING INFINITY HARMONIC FUNCTIONS BY ROTATION

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Abstract. If $u(\mathbf{x}, y)$ is an infinity harmonic function, i.e., a viscosity solution to the equation $-\Delta_\infty u = 0$ in $\Omega \subset \mathbb{R}^{m+1}$ then the function $v(\mathbf{x}, \mathbf{z}) = u(\mathbf{x}, \|\mathbf{z}\|)$ is infinity harmonic in the set $\{(\mathbf{x}, \mathbf{z}) : (\mathbf{x}, \|\mathbf{z}\|) \in \Omega\}$ (provided $u(\mathbf{x}, -y) = u(\mathbf{x}, y)$).

1. Introduction and statement of results

A function $u : \Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^d$, is said to be infinity harmonic if u is a viscosity solution to the equations

$$-\Delta_\infty u = -\sum_{i,j=1}^{d} u_{x_i} u_{x_j} u_{x_i x_j} = 0,$$

in Ω. In order for u to be a viscosity solution it has to be both a subsolution and a supersolution (that is, infinity subharmonic and infinity superharmonic, respectively) and the requirement for u to be a subsolution to $-\Delta_\infty u = 0$ is that u is upper semicontinuous and if φ is twice continuously differentiable in a neighbourhood of a point $\mathbf{x}_1 \in \Omega$, $u(\mathbf{x}_1) = \varphi(\mathbf{x}_1)$ and $u(\mathbf{x}) \le \varphi(\mathbf{x})$ when $|\mathbf{x} - \mathbf{x}_1| < \delta$ for some $\delta > 0$, then $-\sum_{i,j=1}^{d} \varphi_{x_i} \varphi_{x_j} \varphi_{x_i x_j} \le 0$ at the point $\mathbf{x}_1$. In the requirements for u to be a supersolution the inequalities $\le$ are reversed and u is required to be lower semicontinuous, so that u is a supersolution if and only if $-u$ is a subsolution (because $\Delta_\infty(-\varphi) = -\Delta_\infty \varphi$ if φ is twice continuously differentiable).

This equation arises when one wants to find a Lipschitz continuous function u in Ω satisfying given boundary values on $\partial\Omega$ and one requires that this function in addition is an absolutely minimizing extension in the sense that if Ω_0 is an open bounded subset of Ω and $u = v$ on $\partial\Omega_0$, where v is a continuous function in the closure of Ω_0, then the Lipschitz constant of u in Ω_0 is not larger than the one of v, see e.g. [2] and [5]. In addition infinity harmonic functions and their generalizations appear in several other contexts, see e.g. [3], in particular the value of a "Tug-of-war" game is an infinity harmonic function, see [6].

The purpose of this note is to extend the observation that both the function $u(x, y) = |x|^{4/3} - |y|^{4/3}$ and its extension $v(\mathbf{x}, \mathbf{y}) = \|x\|^{4/3} - \|y\|^{4/3}$ are infinity harmonic in $\mathbb{R}^2$ and $\mathbb{R}^{m+n}$, respectively, where $\|\cdot\|$ is the Euclidean norm.

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More precisely, we show that if $u(\mathbf{x}, y)$ is infinity harmonic, then so is the function $v(\mathbf{x}, \mathbf{z}) = u(\mathbf{x}, \|\mathbf{z}\|)$; that is, we can extend an infinity harmonic function to a higher dimensional space by rotation. Here we formulate this result using coordinates, writing vectors in $\mathbb{R}^{m+1}$ and $\mathbb{R}^{m+n}$ in the form $(\mathbf{x}, y)$ and $(\mathbf{x}, \mathbf{z})$, respectively, where $\mathbf{x} \in \mathbb{R}^m$, $y \in \mathbb{R}$, and $\mathbf{z} \in \mathbb{R}^n$ but note that the property of being infinity harmonic does not depend on the coordinate system, that is, if u is infinity harmonic in Ω then $u \circ T$ is infinity harmonic in $T^{-1}\Omega$ when T is an isometry.

Observe that the extension property studied here does not hold for standard harmonic functions, as for example $\log(|x|^2 + |y|^2)$ is harmonic in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ but the function $\log(|x|^2 + \|\mathbf{z}\|^2)$ is not harmonic in $\mathbb{R} \times \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Theorem 1.1. *Assume that $m, n \geq 1$ and that*

- (i) $\Omega_{m,1} \subseteq \mathbb{R}^{m+1}$ *is open and if* $(\mathbf{x}, y) \in \Omega_{m,1}$ *with* $\mathbf{x} \in \mathbb{R}^m$ *and* $y < 0$*, then* $(\mathbf{x}, -y) \in \Omega_{m,1}$*;*
- (ii) $u : \Omega_{m,1} \to \mathbb{R}$ *is infinity harmonic (subharmonic, superharmonic) in* $\Omega_{m,1}$ *and if* $(\mathbf{x}, y) \in \Omega_{m,1}$ *with* $\mathbf{x} \in \mathbb{R}^m$ *and* $y < 0$*, then* $u(\mathbf{x}, -y) = u(\mathbf{x}, y)$*.*

Then v is infinity harmonic (subharmonic, superharmonic) in $\Omega_{m,n}$ where

- (a) $\Omega_{m,n} = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in \mathbb{R}^m,\ \mathbf{z} \in \mathbb{R}^n,\ (\mathbf{x}, \|\mathbf{z}\|) \in \Omega_{m,1}\}$*;*
- (b) $v(\mathbf{x}, \mathbf{z}) = u(\mathbf{x}, \|\mathbf{z}\|)$, $(\mathbf{x}, \mathbf{z}) \in \Omega_{m,n}$*.*

The main property of infinity harmonic functions that we need in order to prove this extension property is "comparison with cones" which is formulated in the form we need here in the following theorem. This well known result and its corollary are stated for subsolutions, but the corresponding results with inequlities reversed and sup replaced by inf hold for supersolutions as well.

Theorem 1.2. *Assume that $d \geq 1$, $\Omega \subset \mathbb{R}^d$ is open, and $w : \Omega \to \mathbb{R}$. Then w is a subsolution to the equation $-\Delta_\infty w = 0$ in Ω if and only if w is locally bounded and*

$$w(\mathbf{x}) \leq w(\mathbf{x}_0) + \sup_{\|\boldsymbol{\xi} - \mathbf{x}_0\| = r} \frac{w(\boldsymbol{\xi}) - w(\mathbf{x}_0)}{r} \|\mathbf{x} - \mathbf{x}_0\|, \quad \|\mathbf{x} - \mathbf{x}_0\| \leq r, \tag{1.1}$$

when $\{\boldsymbol{\xi} : \|\boldsymbol{\xi} - \mathbf{x}_0\| \leq r\} \subset \Omega$.

This theorem is proved in [4] but for completeness and since it is there not formulated in the form above we give a self-contained proof below. Observe that the "only if" part is a consequence of the comparison principle which holds for infinity harmonic functions, see e.g. [1], and of the fact that a cone function $\mathbf{x} \mapsto a + b\|\mathbf{x} - \mathbf{x}_0\|$ is infinity harmonic in $\mathbb{R}^d \setminus \{\mathbf{x}_0\}$.

A consequence of Theorem 1.2 is the well known strong maximum principle, which we will need as well.

Corollary 1.3. *Assume that $d \geq 1$, $\Omega \subset \mathbb{R}^d$ is open and connected, and $w : \Omega \to \mathbb{R}$ is a subsolution to $-\Delta_\infty w = 0$ in Ω. Then either $w(\mathbf{x}) < \sup_{\boldsymbol{\xi} \in \Omega} w(\boldsymbol{\xi})$ for all $\mathbf{x} \in \Omega$ or w is a constant in Ω.*

2. Proofs

Proof of Theorem 1.1. Suppose first that u is infinity subharmonic in $\Omega_{m,1}$. Then u and hence v is upper semicontinuous so that v is locally bounded. Let $(\mathbf{x}_0, \mathbf{z}_0) \in$

$\Omega_{m,n}$ and $r>0$ be such that $\{(\boldsymbol{\xi},\boldsymbol{\zeta})\in\mathbb{R}^{m+n} : \|(\boldsymbol{\xi},\boldsymbol{\zeta})-(\mathbf{x}_0,\mathbf{z}_0)\|\le r\}\subset\Omega_{m,n}$. If we can show that

$$v(\mathbf{x},\mathbf{z})\le v(\mathbf{x}_0,\mathbf{z}_0)+\max_{\|(\boldsymbol{\xi},\boldsymbol{\zeta})-(\mathbf{x}_0,\mathbf{z}_0)\|=r}\frac{v(\boldsymbol{\xi},\boldsymbol{\zeta})-v(\mathbf{x}_0,\mathbf{z}_0)}{r}\|(\mathbf{x},\mathbf{z})-(\mathbf{x}_0,\mathbf{z}_0)\|, \quad (2.1)$$

when $\|(\mathbf{x},\mathbf{z})-(\mathbf{x}_0,\mathbf{z}_0)\|\le r$ then we have shown that v satisfies comparison with cones from above and it follows from Theorem 1.2 that v is a subsolution to $-\Delta_\infty v=0$ in $\Omega_{m,n}$. If u is infinity superharmonic the same argument can be applied to $-u$ and $-v$ and we can conclude that v is infinity superharmonic. Thus the claims of the theorem follow provided we can show that inequality (2.1) holds under the assumption that u is infinity subharmonic.

Note that we may, without loss of generality, assume that $(\mathbf{x},y)\in\Omega_{m,1}$ if and only if $(\mathbf{x},-y)\in\Omega_{m,1}$ and $u(\mathbf{x},y)=u(\mathbf{x},-y)$, because we can, if needed, extend u to $\{(\mathbf{x},y)\in\mathbb{R}^{m+1} : (\mathbf{x},-y)\in\Omega_{m,1}\}$ by $u(\mathbf{x},y)=u(\mathbf{x},-y)$ as the the property of being a subsolution is a local one and the function $(\mathbf{x},y)\mapsto(\mathbf{x},-y)$ is an isometry. Suppose that $\|(\boldsymbol{\xi},\mu)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|\le r$. We want to show that $(\boldsymbol{\xi},\mu)\in\Omega_{m,1}$ and because we may assume that $\mu\ge 0$ we can take $\boldsymbol{\zeta}=\frac{\mu}{\|\mathbf{z}_0\|}\mathbf{z}_0$ if $\mathbf{z}_0\ne\mathbf{0}$ and otherwise take $\boldsymbol{\zeta}$ to be an arbitrary vector in $\mathbb{R}^n$ so that $\|\boldsymbol{\zeta}\|=\mu$. Then $|\mu-\|\mathbf{z}_0\||=\|\boldsymbol{\zeta}-\mathbf{z}_0\|$ so that $\|(\boldsymbol{\xi},\boldsymbol{\zeta})-(\mathbf{x}_0,\mathbf{z}_0\|\le r$ which by the defintion of $\Omega_{m,n}$, our choice of r and by the fact that $\|\boldsymbol{\zeta}\|=\mu$ implies that $(\boldsymbol{\xi},\mu)\in\Omega_{m,1}$.

Suppose that $\boldsymbol{\xi}_0$ and μ_0 are such that $\|(\boldsymbol{\xi}_0,\mu_0)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|=r$ and

$$\max_{\|(\boldsymbol{\xi},\mu)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|=r}\big(u(\boldsymbol{\xi},\mu)-u(\mathbf{x}_0,\|\mathbf{z}_0\|)\big)=u(\boldsymbol{\xi}_0,\mu_0)-u(\mathbf{x}_0,\|\mathbf{z}_0\|). \quad (2.2)$$

Since u is upper semicontinuous, such a point $(\boldsymbol{\xi}_0,\mu_0)$ exists.

If $\mu_0<0$ and $\|(\boldsymbol{\xi}_0,-\mu_0)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|<r$ then, since $u(\boldsymbol{\xi}_0,-\mu_0)=u(\boldsymbol{\xi}_0,\mu_0)$, the maximum value is obtained in an interior point in the ball with center $(\mathbf{x}_0,\|\mathbf{z}_0\|)$ and radius r which by the strong maximum principle implies that u is a constant in this ball. In this case we may take μ_0 so that $\mu_0\ge 0$. If on the other hand $\mu_0<0$ and $\|(\boldsymbol{\xi}_0,-\mu_0)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|=r$, then it follows again from the assumption that $u(\boldsymbol{\xi}_0,-\mu_0)=u(\boldsymbol{\xi}_0,\mu_0)$ that μ_0 can be replaced by $-\mu_0$ so that we may without loss of generality assume that $\mu_0\ge 0$.

Since u is infinity subharmonic we can apply Theorem 1.2 and using (2.2) we obtain

$$\begin{aligned}u(\mathbf{x},\|\mathbf{z}\|)&\le u(\mathbf{x}_0,\|\mathbf{z}_0\|)+\max_{\|(\boldsymbol{\xi},\mu)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|=r}\frac{u(\boldsymbol{\xi},\mu)-u(\mathbf{x}_0,\|\mathbf{z}_0\|)}{r}\\&\quad\times\|(\mathbf{x},\|\mathbf{z}\|)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|\\&=u(\mathbf{x}_0,\|\mathbf{z}_0\|)+\frac{u(\boldsymbol{\xi}_0,\mu_0)-u(\mathbf{x}_0,\|\mathbf{z}_0\|)}{r}\|(\mathbf{x},\|\mathbf{z}\|)-(\mathbf{x}_0,\|\mathbf{z}_0\|)\|.\end{aligned} \quad (2.3)$$

By the definition of v, the triangle inequality, and by the fact that $u(\boldsymbol{\xi}_0,\mu_0)-u(\mathbf{x}_0,\|\mathbf{z}_0\|)\ge 0$ (use Corollary 1.3) we conclude from inequality (2.3) that

$$v(\mathbf{x},\mathbf{z})\le v(\mathbf{x}_0,\mathbf{z}_0)+\frac{u(\boldsymbol{\xi}_0,\mu_0)-u(\mathbf{x}_0,\|\mathbf{z}_0\|)}{r}\|(\mathbf{x},\mathbf{z})-(\mathbf{x}_0,\mathbf{z}_0)\|. \quad (2.4)$$

As above we choose $\boldsymbol{\zeta}_0=\frac{\mu_0}{\|\mathbf{z}_0\|}\mathbf{z}_0$ if $\mathbf{z}_0\ne\mathbf{0}$ and otherwise we take $\boldsymbol{\zeta}_0$ to be an arbitrary vector in $\mathbb{R}^n$ such that $\|\boldsymbol{\zeta}_0\|=\mu_0$ so that $\|\boldsymbol{\zeta}_0\|=\mu_0$ as we have $\mu_0\ge 0$. Hence it follows from the definition of v that

$$u(\boldsymbol{\xi}_0,\mu_0)-u(\mathbf{x}_0,\|\mathbf{z}_0\|)=v(\boldsymbol{\xi}_0,\boldsymbol{\zeta}_0)-v(\mathbf{x}_0,\mathbf{z}_0). \quad (2.5)$$

Our choice of $\boldsymbol{\zeta}_0$ implies in addition that $\|\boldsymbol{\zeta}_0 - \mathbf{z}_0\| = |\mu_0 - \|\mathbf{z}_0\||$ from which it follows that

$$\|(\boldsymbol{\xi}_0, \boldsymbol{\zeta}_0) - (\mathbf{x}_0, \mathbf{z}_0)\| = \|(\boldsymbol{\xi}_0, \mu_0) - (\mathbf{x}_0, \|\mathbf{z}_0\|)\| = r.$$

This result combined with (2.5) shows that

$$\frac{u(\boldsymbol{\xi}_0, \mu_0) - u(\mathbf{x}_0, \|\mathbf{z}_0\|)}{r} \le \max_{\|(\boldsymbol{\xi},\boldsymbol{\zeta})-(\mathbf{x}_0,\mathbf{z}_0)\|=r} \frac{v(\boldsymbol{\xi}, \boldsymbol{\zeta}) - v(\mathbf{x}_0, \mathbf{z}_0)}{r}.$$

This inequality combined with (2.4) shows that inequality (2.1) holds and the proof is complete. □

Proof of Theorem 1.2. Assume first that w is a subsolution to $-\Delta_\infty w = 0$ in Ω. By definition w is upper semicontinuous and hence locally bounded. Suppose $\{\boldsymbol{\xi} : \|\boldsymbol{\xi} - \mathbf{x}_0\| \le r\} \subset \Omega$ for some $\mathbf{x}_0 \in \mathbb{R}^d$ and $r > 0$. Let $\alpha \in (0,1)$, choose β to be

$$\beta = \frac{1}{r^\alpha} \max_{\|\boldsymbol{\xi}-\mathbf{x}_0\|=r} (w(\boldsymbol{\xi}) - w(\mathbf{x}_0)),$$

and define

$$\psi(\mathbf{x}) = w(\mathbf{x}_0) + \beta\|\mathbf{x} - \mathbf{x}_0\|^\alpha, \quad \mathbf{x} \in \mathbb{R}^d.$$

By definition, $w(\mathbf{x}) \le \psi(\mathbf{x})$ when $\mathbf{x} = \mathbf{x}_0$ and when $\|\mathbf{x} - \mathbf{x}_0\| = r$ so that if we have $\max_{\|x-x_0\|\le r}\big(w(\mathbf{x}) - \psi(\mathbf{x})\big) > 0$ and define $\varphi(\mathbf{x}) = \psi(\mathbf{x}) + \max_{\|x-x_0\|\le r}\big(w(\mathbf{x}) - \psi(\mathbf{x})\big)$, then there is a point $\mathbf{x}_1$ such that $0 < \|\mathbf{x}_1 - \mathbf{x}_0\| < r$, $\varphi(\mathbf{x}_1) = w(\mathbf{x}_1)$ and $w(\mathbf{x}) \le \varphi(\mathbf{x})$ when $\|\mathbf{x} - \mathbf{x}_1\| < \min\{\|\mathbf{x}_1 - \mathbf{x}_0\|, r - \|\mathbf{x}_1 - \mathbf{x}_0\|\}$. But since w is a subsolution it follows from the definition of a subsolution that $-\Delta_\infty\varphi(\mathbf{x}_1) \le 0$ which is a contradiction since a calculation shows that $-\Delta_\infty\varphi(\mathbf{x}) = -\beta^3\alpha^3(\alpha - 1)\|\mathbf{x} - \mathbf{x}_0\|^{3\alpha-4} > 0$ when $\mathbf{x} \ne \mathbf{x}_0$. Thus we know that $w(x) \le \psi(x)$ when $\|\mathbf{x} - \mathbf{x}_0\| \le r$ and taking the limit $\alpha \uparrow 1$ we get inequality (1.1).

Assume next that w is locally bounded and (1.1) holds when $\{\boldsymbol{\xi} : \|\boldsymbol{\xi} - \mathbf{x}_0\| \le r\} \subset \Omega$. Then if follows from (1.1) that w is upper semicontinuous. If w is not a subsolution to $-\Delta_\infty w = 0$ in Ω, then there is a point $\mathbf{x}_1 \in \Omega$ and a function φ which is twice continuously differentiable in the set $\{\mathbf{x} : \|\boldsymbol{\xi} - \mathbf{x}_1\| < \delta\} \subset \Omega$ where $\delta > 0$, such that $w(\mathbf{x}) \le \varphi(\mathbf{x})$ when $\|\mathbf{x} - \mathbf{x}_1\| < \delta$, $w(\mathbf{x}_1) = \varphi(\mathbf{x}_1)$, and $-\Delta_\infty\varphi(\mathbf{x}_1) > 0$. The Taylor expansion of φ is

$$\varphi(\mathbf{x}) = \varphi(\mathbf{x}_1) + \langle \mathbf{p}, \mathbf{x} - \mathbf{x}_1\rangle + \tfrac{1}{2}\langle A(\mathbf{x} - \mathbf{x}_1), \mathbf{x} - \mathbf{x}_1\rangle + o(\|\mathbf{x} - \mathbf{x}_1\|^2), \tag{2.6}$$

where $\mathbf{p} = D\varphi(\mathbf{x}_1)$, $A = D^2\varphi(\mathbf{x}_1)$ and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^d$. Note that since $-\Delta_\infty\varphi(\mathbf{x}_1) > 0$ we have $\langle A\mathbf{p}, \mathbf{p}\rangle < 0$ and therefore also $\mathbf{p} \ne \mathbf{0}$. We define $\mathbf{p}_0 = \frac{1}{\|\mathbf{p}\|}\mathbf{p}$ and introduce new coordinates $(t, \mathbf{y})$ by writing $\mathbf{x} = \mathbf{x}_1 + t\mathbf{p}_0 + \mathbf{y}$ where $\langle \mathbf{p}, \mathbf{y}\rangle = 0$. Then we have

$$\begin{aligned}\langle A(t\mathbf{p}_0 + \mathbf{y}), t\mathbf{p}_0 + \mathbf{y}\rangle &\le t^2\langle A\mathbf{p}_0, \mathbf{p}_0\rangle + 2t\|A\mathbf{p}_0\|\|\mathbf{y}\| + \|A\|\|\mathbf{y}\|^2 \\ &\le -2c_1t^2 + \tfrac{1}{2}c_2\|\mathbf{y}\|^2,\end{aligned} \tag{2.7}$$

where c_1 and c_2 are positive constants, (in fact we can choose $c_1 = -\frac{1}{3}\langle A\mathbf{p}_0, \mathbf{p}_0\rangle$ and $c_2 = 2(\frac{\|A\mathbf{p}_0\|^2}{c_1} + \|A\|)$).

Since $w(\mathbf{x}) \le \varphi(\mathbf{x})$ when $\|\mathbf{x} - \mathbf{x}_1\| < \delta$ and $w(\mathbf{x}_1) = \varphi(\mathbf{x}_1)$ we can by (2.6) and (2.7) choose r so that

$$w(\mathbf{x}_1 + t\mathbf{p}_0 + \mathbf{y}) \le w(\mathbf{x}_1) + \|\mathbf{p}\|t - c_1t^2 + c_2\|\mathbf{y}\|^2, \quad \|t\mathbf{p}_0 + \mathbf{y}\| \le \frac{3}{2}r, \quad \langle \mathbf{p}, \mathbf{y}\rangle = 0, \tag{2.8}$$

and

$$0 < r < \min\Big\{\frac{2\delta}{3}, \frac{\|\mathbf{p}\|}{c_1 + 2c_2}\Big\}. \tag{2.9}$$

Now we choose $\mathbf{x}_0 = \mathbf{x}_1 - \frac{r}{2}\mathbf{p}_0$ and we will show that the inequality in (1.1) does not hold if $\mathbf{x} = \mathbf{x}_1$. Since $\|\mathbf{x}_1 - \mathbf{x}_0\| = \frac{r}{2}$ we have to show that

$$w(\mathbf{x}_1) > \frac{1}{2}w(\mathbf{x}_0) + \frac{1}{2}\max_{\|\boldsymbol{\zeta}-\mathbf{x}_0\|=r} w(\boldsymbol{\zeta}). \tag{2.10}$$

With the aid of the upper bounds for both terms on the right-hand side in this inequality that we get from (2.8) we see that it suffices to show that

$$-\|\mathbf{p}\|\frac{r}{2} - \frac{c_1 r^2}{4} + \max_{(t+\frac{r}{2})^2+\|\mathbf{y}\|^2=r^2}\Big(\|\mathbf{p}\|t - c_1 t^2 + c_2\|\mathbf{y}\|^2\Big) < 0.$$

This inequality holds because

$$\begin{aligned}
&\max_{(t+\frac{r}{2})^2+\|\mathbf{y}\|^2=r^2}\Big(\|\mathbf{p}\|t - c_1 t^2 + c_2\|\mathbf{y}\|^2\Big)\\
&= \max_{-\frac{3}{2}r\le t\le\frac{r}{2}}\Big(\|\mathbf{p}\|t - c_1 t^2 + c_2\Big(r^2 - \big(t+\frac{r}{2}\big)^2\Big)\Big)\\
&= \|\mathbf{p}\|\frac{r}{2} - c_1\frac{r^2}{4},
\end{aligned}$$

where we used the fact that assumption (2.9) implies that the function to be maximized is increasing in the interval $[-\frac{3r}{2}, \frac{r}{2}]$. Thus we get the desired contradiction (2.10) and the proof is completed. □

Proof of Corollary 1.3. Let $S = \sup_{\boldsymbol{\xi}\in\Omega} w(\boldsymbol{\xi})$. Suppose there is a point $\mathbf{x} \in \Omega$ such that $w(\mathbf{x}) = S$ and w is not a constant. Then there are, because Ω is open and connected, points $\mathbf{x}_0$ and $\mathbf{x}_1 \in \Omega$ such that $\{\boldsymbol{\xi} : \|\boldsymbol{\xi} - \mathbf{x}_0\| \le 2\|\mathbf{x}_1 - \mathbf{x}_0\|\} \subset \Omega$ and

$$w(\mathbf{x}_0) < S = w(\mathbf{x}_1). \tag{2.11}$$

Since S is the supremum we have $\sup_{\|\boldsymbol{\xi}-\mathbf{x}_0\|=2\|\mathbf{x}_1-\mathbf{x}_0\|}(w(\boldsymbol{\xi}) - w(\mathbf{x}_0)) \le S - w(\mathbf{x}_0)$. But then it follows from (1.1) that

$$w(\mathbf{x}_1) \le w(\mathbf{x}_0) + \frac{1}{2}(S - w(\mathbf{x}_0)) < S,$$

which is a contradiction by (2.11). □

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