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HOMOCLINIC AND QUASI-HOMOCLINIC SOLUTIONS FOR DAMPED DIFFERENTIAL EQUATIONS

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Abstract. We study the existence and multiplicity of homoclinic solutions for the second-order damped differential equation

$$\ddot{u} + c\dot{u} - L(t)u + W_u(t,u) = 0,$$

where $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t. Under certain assumptions on L and W, we obtain infinitely many homoclinic solutions when the nonlinearity $W(t,u)$ is sub-quadratic or super-quadratic by using critical point theorems. Some recent results in the literature are generalized, and the open problem proposed by Zhang and Yuan is solved. In addition, with the help of the Nehari manifold, we consider the case where $W(t,u)$ is indefinite and prove the existence of at least one nontrivial quasi-homoclinic solution.

1. Introduction

In this article, we study the existence and multiplicity of homoclinic solutions for the damped problem

$$\ddot{u} + c\dot{u} - L(t)u + W_u(t,u) = 0, \tag{1.1}$$

where $c \geq 0$ is a constant, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix for all $t \in \mathbb{R}$ and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. We say that a solution u of (1.1) is a nontrivial homoclinic solution (to 0) if $u(t) \not\equiv 0$, $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \to 0$ as $|t| \to \infty$ (see [12]).

When $c = 0$, equation (1.1) is reduced to the second-order Hamiltonian system

$$\ddot{u} - L(t)u + W_u(t,u) = 0. \tag{1.2}$$

In the past two decades, by critical point theory the existence and multiplicity of homoclinic solutions of (1.2) have been extensively investigated by many authors; see e.g. [7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30] and the references therein.

When $c \neq 0$, little work has been done about the existence and multiplicity of homoclinic solutions for (1.1). In the recent paper [1], by solving a minimum or constrained minimum problem, the authors considered the existence of fast heteroclinic solutions of the damped ODE

$$\ddot{u} + c\dot{u} + f(u) = 0,$$

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which arises from investigating travelling waves with speed c of Fisher-Kolmogorov's equation. In [4], the authors studied the existence of homoclinic solutions of problem (1.1) when $c = c(t)$ is continuous and $W(t, u)$ is singular, where the upper and lower solution method and fixed point theorems are used in the proof. In [31], by using a minimizing method, Zhang and Yuan obtained the existence of homoclinic solutions for (1.1) in the case where $W(t, u)$ is sub-quadratic and $c \geq 0$. But, they didn't obtain the multiplicity of homoclinic solutions for (1.1), although $W(t, u)$ is even with respect to u.

Inspired by the works mentioned above, we discuss the existence and multiplicity of homoclinic solutions for (1.1). In Theorems 1.10–1.14, we are interested in the case where $c \geq 0$. To the best of our knowledge, even for $c = 0$, our results when $W(t, u)$ is indefinite and sub-quadratic at infinity are new. We mention an open problem proposed in [31]. The open problem is how to prove the existence of homoclinic solutions for (1.1) by the mountain pass theorem when $c > 0$ and W is super-quadratic at infinity. This problem is solved in Theorems 1.16–1.17. Finally, by using the Nehari manifold, we also obtain the existence of homoclinic solutions for (1.1) in the case where $W(t, u)$ is indefinite.

The first aim of this paper is to study the existence of infinitely many homoclinic solutions for (1.1) under conditions similar to those introduced in [31]. Consider two scenarios on L:

- (L0) $L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there is a continuous function $l : \mathbb{R} \to \mathbb{R}^+$ such that
$$\langle L(t)u, u\rangle \geq l(t)|u|^2, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N,$$
where $l(t) \to \infty$ as $|t| \to \infty$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^N$;
- (L0') There exist two constants $-\infty < l_1 < l_2 < +\infty$ such that
$$l_1|u|^2 \leq \langle L(t)u, u\rangle \leq l_2|u|^2, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N.$$

For convenience, we first describe some weighted spaces. If $1 \leq p < +\infty$, let
$$L^p(e^{ct}) = \big\{u \in L^p_{\rm loc}(\mathbb{R}, \mathbb{R}^N) : |u|_p = \Big(\int_{\mathbb{R}} e^{ct}|u(t)|^p dt\Big)^{1/p} < +\infty\big\}.$$
Then, $L^p(e^{ct})$ is a reflexive Banach space (see [19]).

When $c > 0$ and L satisfies (L0), we define the weighted space
$$E = \big\{u \in H^1_{\rm loc}(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} e^{ct}(|\dot{u}|^2 + \langle L(t)u, u\rangle)dt < +\infty\big\}$$
endowed with the norm
$$\|u\| = (u, u)^{1/2} = \Big(\int_{\mathbb{R}} e^{ct}(|\dot{u}|^2 + \langle L(t)u, u\rangle)dt\Big)^{1/2}.$$
Then, it is not difficult to prove that E is a Hilbert space.

When $c > 0$ and L satisfies (L0'), then E is a Hilbert space if we take $l_1 > 0$. When $c = 0$ and L satisfies (L0) or (L0') with $l_1 > 0$, consider the space
$$E_0 = \big\{u \in H^1(\mathbb{R}, \mathbb{R}^N) : \|u\| = \Big(\int_{\mathbb{R}} |\dot{u}|^2 + \langle L(t)u, u\rangle dt\Big)^{1/2} < +\infty\big\}.$$
Then, it is easy to show that E_0 is also a Hilbert space.

Remark 1.1. In [31], the authors defined the weighted Sobolev space

$$E_c = \big\{u \in H^1(\mathbb{R}, \mathbb{R}^N) : \|u\| = \Big(\int_{\mathbb{R}} e^{ct}(|\dot{u}|^2 + \langle L(t)u, u\rangle)dt\Big)^{1/2} < +\infty\big\}.$$

It seems that E_c is not complete when $c > 0$. For example, take $c = 1$, $L(t) = e^{\theta|t|}$. Any function $u \in H^1_{\rm loc}(\mathbb{R}, \mathbb{R}^N)$ such that

$$\int_{\mathbb{R}} e^t|\dot{u}|^2 + e^{t+\theta|t|}|u|^2 dt < +\infty$$

(e.g. $u \in C^\infty$ with $u(t) = 1$ for $t < 0$ and $u(t) = 0$ for $t > 1$) can be approximated by functions in $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ in the strong topology induced by the previous norm, even if might not belong to $H^1(\mathbb{R}, \mathbb{R}^N)$, hence to E_c. We thank an anonymous referee for making this remark.

Definition 1.2. Given a function $g \in C(\mathbb{R}, \mathbb{R})$ and $g \not\equiv 0$, a solution u of (1.1) is called a nontrivial g-quasi-homoclinic solution, if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \not\equiv 0$, $u(t) \to 0$ as $t \to +\infty$ and $g(t)u(t) \to 0$ as $t \to -\infty$.

Remark 1.3. If $g(t) \equiv 1$, then a g-quasi-homoclinic solution is a usual homoclinic solution. Particularly, $c = 0$ for $g(t) = e^{ct/2}$ is the case. In this paper, we always take $g(t) = e^{ct/2}$.

Remark 1.4. In [31], Zhang and Yuan introduced the so-called fast homoclinic solution, that is, a homoclinic solution u of (1.1) with $u \in E_c$ and $c > 0$. However, in [9], the authors pointed out that solutions in E_c are not suitable because a solution in E_c is not fast homoclinic to 0 as $t \to -\infty$. In fact, if u is a solution of (1.1) in E, then $u(t) \to 0$ as $t \to +\infty$ and $e^{ct/2}u(t) \to 0$ as $t \to -\infty$. It seems to be reasonable to call such a solution as a g-quasi-homoclinic one.

In [31], the following conditions are assumed:

- (W0) $W(t, u) = \alpha(t)|u|^\varsigma$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\alpha(t_0) > 0$ for some $t_0 \in \mathbb{R}$, $\alpha \in L^{2/(2-\varsigma)}(e^{ct})$ and $1 < \varsigma < 2$ is a constant;
- (W0') $W(t, u) = \beta(t)V(u)$, where $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\beta(t_0') > 0$ for some $t_0' \in \mathbb{R}$ and $\beta \in L^2(e^{ct})$, $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $V(0) = 0$. Furthermore, there exist constants $M_1 > 0, M_2 > 0, 1 < \xi < 2$ and $0 < r_1 \le 1$ such that

$$V(u) \ge M_2|u|^\xi,\ \forall u \in \mathbb{R}^N \text{ with } |u| \le r_1, \tag{1.3}$$

$$|V_u(u)| \le M_1, \forall u \in \mathbb{R}^N. \tag{1.4}$$

Theorem 1.5 ([31, Theorem 1.1])**.** *If $c > 0$ and the assumptions* (L0) *and* (W0) *are satisfied, then* (1.1) *has at least one nontrivial fast homoclinic solution. If $c = 0$ and the assumptions (L0) and (W0) hold, then* (1.1) *has at least one nontrivial homoclinic solution.*

Theorem 1.6 ([31, Theorem 1.2])**.** *If $c > 0$ and the assumptions* (L0) *and* (W0') *are satisfied, then* (1.1) *has at least one nontrivial fast homoclinic solution. If $c = 0$ and the assumptions (L0) and (W0') hold, then* (1.1) *has at least one nontrivial homoclinic solution.*

Theorem 1.7 ([31, Theorem 1.3])**.** *For $c > 0$, under the conditions* (W0) *and* (L0') *with $l_1 > -\frac{c^2}{4}$,* (1.1) *has at least one nontrivial fast homoclinic solution. For $c = 0$,* *if* (W0) *and* (L0') *with* $l_1 > 0$ *are satisfied, then* (1.1) *has at least one nontrivial homoclinic solution.*

Theorem 1.8 ([31, Theorem 1.4])**.** *For* $c > 0$*, under the conditions* (W0') *and* (L0') *with* $l_1 > 0$*,* (1.1) *has at least one nontrivial fast homoclinic solution. For* $c = 0$*, if* (W0') *and* (L0') *with* $l_1 > 0$ *are satisfied, then* (1.1) *has at least one nontrivial homoclinic solution.*

Remark 1.9. In [31], the embedding of E_c in $L^2(e^{ct})$ is not compact under the conditions of Theorems 1.7–1.8. The authors seem to have to show that the functional I is weakly lower semi-continuous. In addition, under the assumptions of Theorems 1.5 and 1.7, they did not give whether or not (1.1) has infinitely many homoclinic solutions, although $W(t, u)$ is even with respect to u. One of the aims of the paper is to study the existence of infinitely many homoclinic or g-quasi-homoclinic solutions for (1.1) under the same conditions.

Theorem 1.10. *If* $c > 0$ *and the assumptions* (L0) *and* (W0) *hold, then* (1.1) *has infinitely many* g*-quasi-homoclinic solutions. If* $c = 0$ *and the conditions* (L0) *and* (W0) *are satisfied, then* (1.1) *has infinitely many homoclinic solutions.*

Theorem 1.11. *If* $c > 0$ *and the assumptions* (L0) *and* (W0') *with* $V(u) = V(-u)$ *for all* $u \in \mathbb{R}^N$ *hold, then* (1.1) *has infinitely many* g*-quasi-homoclinic solutions. If* $c = 0$ *and the conditions* (L0) *and* (W0') *with* $V(u) = V(-u)$ *for all* $u \in \mathbb{R}^N$ *are satisfied, then* (1.1) *has infinitely many homoclinic solutions.*

Remark 1.12. Assumption (W0) or (W0') implies that $W(t, u)$ can be sign-changing. For $c = 0$, in [26, 20, 28], the authors investigate the existence of infinitely many homoclinic solutions of (1.1) with sub-quadratic $W(t, u)$. However, they require that $W(t, u) \geq 0$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$.

Theorem 1.13. *If* $c > 0$*, under the conditions* (W0) *and* (L0') *with* $l_1 > 0$*, then* (1.1) *has infinitely many* g*-quasi-homoclinic solutions. If* $c = 0$ *and the assumptions* (W0) *and* (L0') *with* $l_1 > 0$ *hold, then* (1.1) *has infinitely many homoclinic solutions.*

Theorem 1.14. *If* $c > 0$*, under the conditions* (L0') *with* $l_1 > 0$ *and* (W0') *with* $V(u) = V(-u)$ *for all* $u \in \mathbb{R}^N$*, then* (1.1) *has infinitely many* g*-quasi-homoclinic solutions. If* $c = 0$ *and the assumptions* (L0') *with* $l_1 > 0$ *and* (W0') *with* $V(u) = V(-u)$ *for all* $u \in \mathbb{R}^N$ *hold, then* (1.1) *has infinitely many homoclinic solutions.*

Remark 1.15. When $c = 0$, the authors in [7, 11, 10, 23, 21, 27, 29] considered the case where $W(t, u)$ is sub-quadratic as $|u| \to \infty$. They obtained that (1.1) has infinitely many homoclinic solutions. But they all require that $\alpha(t)$ has a positive lower bound. Obviously, for $c = 0$, we have $\inf_{t\in\mathbb{R}} |\alpha(t)| = 0$ since $\alpha \in L^{2/(2-\varsigma)}(e^{ct})$ is continuous. Hence, $W(t, u) = \alpha(t)|u|^{\varsigma}$ does not satisfy the conditions in [7, 10, 11, 21, 23, 27, 29]. Therefore, our results complement theirs.

Next, for $c > 0$, we consider the case where $W(t, u)$ is super-quadratic growth as $|u| \to \infty$. In this case, the problem is quite different from the sub-quadratic case, because E can not be continuously embedded into $L^\infty(\mathbb{R}, \mathbb{R}^N)$. In order to overcome the difficulty, we have to strengthen the condition (L0). Suppose that W and L satisfy the following hypotheses:

(W1) $W_u(t, u) = o(|u|)$ as $u \to 0$ uniformly for $t \in \mathbb{R}$;

(W2) There exists a $\mu > 2$ such that
$$0 < \mu W(t,u) \le \langle W_u(t,u), u\rangle, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N\backslash\{0\};$$
(W3) There are constants $a_1 > 0$ and $p \in (2,+\infty)$ such that
$$|W_u(t,u)| \le a_1(1+|u|^{p-1}), \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N.$$
(L1) There is $r \in (2p-2,+\infty)$ such that
$$\frac{1}{l(t)e^{(r-2)ct}} \in L^1(\mathbb{R},\mathbb{R}),$$
where $l(t)$ is defined in (L0).

Theorem 1.16. *Suppose* (L0), (L1), (W1)–(W3) *hold. Then* (1.1) *has at least one nontrivial g-quasi-homoclinic solution.*

If the following symmetric condition holds, we can consider multiplicity results of homoclinic solutions for (1.1),

(W4) $W(t,-u) = W(t,u)$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$.

Theorem 1.17. *If* (L0), (L1), (W1)–(W4) *are satisfied, then* (1.1) *has an unbounded sequence of g-quasi-homoclinic solutions.*

Remark 1.18. Under conditions (L0), (W1)–(W3) or (W1)–(W4), the authors in [15] proved Theorems 1.16-1.17 for $c = 0$. Therefore our results are extensions of theirs.

Remark 1.19. In [8, 9], the authors studied the existence and multiplicity of homoclinic solutions of the damped problem
$$\ddot{u} + q(t)\dot{u} - L(t)u + W_u(t,u) = 0,$$
with the condition
$$\lim_{|t|\to+\infty} Q(t) = +\infty, \tag{1.5}$$
where $q \in C(\mathbb{R},\mathbb{R})$ and $Q(t) = \int_0^t q(s)ds$. Evidently, $q(t) \equiv c$ does not satisfy (1.5).

Lastly, motivated by [3], we consider the case where $W(t,u)$ is indefinite in sign; that is, W satisfies the following condition:

(W5) $W(t,u) = \frac{1}{\theta}a(t)|u|^\theta - \frac{1}{\vartheta}b(t)|u|^\vartheta$, where $2 < \vartheta < \theta \le r$ (r is defined in (L1)) and $a, b \in C(\mathbb{R},\mathbb{R}^+) \cap L^\infty(\mathbb{R},\mathbb{R}^+)$.

Theorem 1.20. *If* (L0), (L1), (W5) *are satisfied, then* (1.1) *has at least one nontrivial g-quasi-homoclinic solution.*

The rest of this article is organized as follows. In section 2, we introduce some lemmas and preliminary results. In sections 3, 4 and 5, the main results are proved.

2. Preliminary results

It is well-known that (1.1) has a variational functional I defined on E
$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{ct}(|\dot{u}|^2 + \langle L(t)u,u\rangle) - \int_{\mathbb{R}} e^{ct}W(t,u) \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{ct}W(t,u) \\ &= I_1(u) - I_2(u), \end{aligned}$$

where $I_1(u) = \frac{1}{2}\|u\|^2$ and $I_2(u) = \int_{\mathbb{R}} e^{ct}W(t,u)$.

We give some lemmas which will be used later. Using the similar idea of [15, Lemma 1], we can obtain the following compact embedding lemma.

Lemma 2.1. *Suppose that $L(t)$ satisfies* (L0), *then $E \hookrightarrow\hookrightarrow L^2(e^{ct})$ for $c > 0$.*

Lemma 2.2 ([11, 15])**.** *If $c = 0$ and suppose that $L(t)$ satisfies (L0), then $E_0 \hookrightarrow\hookrightarrow L^p(\mathbb{R}, \mathbb{R}^N)$ for $p \in [2, \infty]$.*

Throughout this article, the letter C denotes positive (possibly different) constants. The next lemma is a technical result.

Lemma 2.3. *If $L(t)$ satisfies* (L0), *then $E \hookrightarrow\hookrightarrow L^\infty(e^{ct/2})$.*

Proof. For any $R > 0$, denote $\beta(R) = \inf_{|t|\geq R} l(t)$. We have $\beta(R) \to \infty$ as $R \to \infty$ from (L0). Let $K \subset E$ be a bounded set such that $\|u\| \leq M$, $\forall\ u \in K$. We shall show that K is precompact in $L^\infty(e^{ct/2})$.

For any $n \in \mathbb{N}$, $u \in E$ and $\tau \in \mathbb{R}$, writing

$$
\begin{aligned}
e^{ct/2}u(t) = &\int_\tau^t (e^{cs/2}u(s))' \frac{(s-\tau)^{n+1}}{(t-\tau)^n} + e^{cs/2}u(s)\frac{(n+1)(s-\tau)^n}{(t-\tau)^n} ds \\
&- \int_t^{\tau+1} (e^{cs/2}u(s))' \frac{(\tau+1-s)^{n+1}}{(\tau+1-t)^n} + e^{cs/2}u(s)\frac{(n+1)(\tau+1-s)^n}{(\tau+1-t)^n} ds
\end{aligned}
$$

for all $\tau \leq t \leq \tau + 1$, we have

$$
e^{ct/2}|u(t)| \leq \frac{2}{\sqrt{2n+3}}\Big(\int_\tau^{\tau+1} e^{cs}|\dot{u}(s)|^2\Big)^{1/2} + \frac{2n+2+c}{\sqrt{2n+1}}\Big(\int_\tau^{\tau+1} e^{cs}|u(s)|^2\Big)^{1/2} \tag{2.1}
$$

for all $\tau \leq t \leq \tau + 1$. Particularly, for any $R > 0$ and $u, v \in K$, we can derive the estimates as follows:

$$
\begin{aligned}
&e^{ct/2}|u(t) - v(t)| \\
&\leq \frac{2}{\sqrt{2n}}\Big(\int_{|s|\geq R} e^{cs}|\dot{u} - \dot{v}|^2\Big)^{1/2} + \frac{4n+c}{\sqrt{2n}}\Big(\int_{|s|\geq R} e^{cs}|u - v|^2\Big)^{1/2} \\
&\leq \frac{4(\|u\| + \|v\|)}{\sqrt{2n}} + \frac{4n+c}{\sqrt{2n}}\Big(\int_{|s|\geq R} \frac{e^{cs}\langle L(s)(u-v), (u-v)\rangle}{l(s)}\Big)^{1/2} \\
&\leq \frac{8M}{\sqrt{2n}} + \frac{4n+c}{\sqrt{2n}}\frac{2M}{\sqrt{\beta(R)}}, \quad \forall |t| \geq R.
\end{aligned} \tag{2.2}
$$

For any $\varepsilon > 0$, taking first n large enough such that

$$
\frac{8M}{\sqrt{2n}} < \frac{\varepsilon}{2},
$$

and R_0 large enough such that

$$
\frac{4n+c}{\sqrt{2n}}\frac{2M}{\sqrt{\beta(R_0)}} < \frac{\varepsilon}{2},
$$

we see from (2.2) that

$$
\max_{|t|\geq R_0} e^{ct/2}|u(t) - v(t)| < \varepsilon \tag{2.3}
$$

for all u, $v \in K$. Obviously, $E \hookrightarrow\hookrightarrow L^\infty_{\rm loc}(e^{ct/2})$. There exist $u_1, \dots, u_m \in K$ such that $\forall u \in K$, there exists $u_i (1 \le i \le m)$ satisfying

$$\max_{|t| \le R_0} e^{ct/2}|u(t) - u_i(t)| < \varepsilon,$$

which jointly with (2.3) implies $|e^{ct/2}(u - u_i)|_\infty < \varepsilon$. Hence, K is precompact in $L^\infty(e^{ct/2})$. ☐

Remark 2.4. If $c > 0$, by (2.1), we obtain that $e^{ct/2}|u(t)| \to 0$ as $|t| \to \infty$ for $u \in E$. Therefore, we have $u(t) \to 0$ as $t \to +\infty$ and $e^{ct/2}u(t) \to 0$ as $t \to -\infty$. Hence, by Definition 1.2, a solution of (1.1) with $u \in E$ is g-quasi-homoclinic.

The following embedding theorem is crucial for the investigation of the existence of homoclinic solutions of (1.1) with a super-quadratic condition.

Lemma 2.5. *Suppose that* (L0) *and* (L1) *hold. Then the embedding of E into $L^r(e^{ct})$ is compact, where r is defined in* (L1).

Proof. For $u \in E$, by (L1) and Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}} e^{ct}|u|^r &\le |e^{ct/2}u|_\infty^{r-1}\Big(\int_{\mathbb{R}} e^{ct} l(t)|u(t)|^2\Big)^{1/2}\Big(\int_{\mathbb{R}} \frac{1}{l(t)e^{(r-2)ct}}\Big)^{1/2}\\ &\le C|e^{ct/2}u|_\infty^{r-1}\|u\|,\end{aligned}$$

which means that E is compactly embedded in $L^r(e^{ct})$. ☐

Remark 2.6. As a consequence of Lemmas 2.1 and 2.5, we have $E \hookrightarrow\hookrightarrow L^q(e^{ct})$ for any $q \in [2, r]$. Hence, there exists a constant $C_q > 0$ such that

$$|u|_q \le C_q\|u\|, \ \forall q \in [2, r]. \tag{2.4}$$

To prove Theorems 1.10–1.17, we need some definitions and theorems introduced in [17] and [2, 5, 13].

Definition 2.7 ([17])**.** The functional $I \in C^1(E, \mathbb{R})$ satisfies the (PS)-condition if any sequence $\{u_k\} \subset E$ such that $\{I(u_k)\}$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence.

We restate a version of the mountain pass theorem due to [2] and [5]. It can also be found in [13]. We first recall the definition of genus. Let E be a Banach space and

$$\Gamma := \{A \subset E \setminus \{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

Define $\Gamma_k := \{A \in \Gamma : \gamma(A) \le k\}$, where

$$\gamma(A) := \inf\{n \in \mathbb{N} : \exists f \in C(A, \mathbb{R}^n \backslash \{0\}), -f(x) = f(-x)\}.$$

If there is no such mapping f for any $n \in \mathbb{N}$, we set $\gamma(A) = +\infty$.

Theorem 2.8 ([2, 5, 13])**.** *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ be even, $I(0) = 0$ and I satisfies the following two conditions:*

- (I1) *I is bounded from below and satisfies (PS)-condition;*
- (I2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that* $\sup_{u \in A_k} I(u) < 0$.

Then $c_n = \inf_{A \in \Gamma_n} \sup_{u \in A} I(u) < 0$ is a critical value of I for every $n \in \mathbb{N}$ and $c_n \to 0^-$ as $n \to \infty$.

Theorem 2.9 (Mountain pass theorem [17]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying (PS)-condition. Suppose $I(0) = 0$ and*

- (I3) *there are constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$;*
- (I4) *there is an $e \in E \backslash B_\rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : \ g(0) = 0, \ g(1) = e\}.$$

Theorem 2.10 ([17]). *Let E be an infinite dimensional real Banach space. Suppose $I \in C^1(E, \mathbb{R})$ be even, satisfies (PS)-condition and $I(0) = 0$. If $E = V \bigoplus X$, where V is finite dimensional, and I satisfies*

- (I5) *there are constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$;*
- (I6) *for each finite dimensional subspace $\tilde{E} \subset E$, there is an $R = R(\tilde{E})$ such that $I(e) \leq 0$ on $\tilde{E} \backslash B_{R(\tilde{E})}$,*

then I possesses an unbounded sequence of critical values.

3. The sub-quadratic case

Proof of Theorem 1.10. When $c > 0$, similar to [31, Lemmas 2.6 and 3.1], we can prove that $I \in C^1(E, \mathbb{R})$ and I satisfies (PS)-condition. By Remark 2.4, any critical point of I in E is a g-quasi-homoclinic solution of (1.1). It follows from (W0) that $I(0) = 0$ and I is even. Moreover, by (W0), it is easy to prove that I is bounded from below. Hence, to obtain Theorem 1.10, it is sufficient to show that $I(u)$ satisfies (I2) of Theorem 2.8. By (W0), there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that $\alpha(t) \geq \alpha_0 > 0$ for all $t \in [t_0 - \delta_0, t_0 + \delta_0]$. Therefore, we have

$$\lim_{u \to 0} \min_{|t - t_0| \leq \delta_0} \frac{e^{ct} W(t, u)}{|u|^2} \geq \min_{|t - t_0| \leq \delta_0} e^{ct} \alpha(t) \lim_{u \to 0} \frac{1}{|u|^{2 - \varsigma}} = +\infty, \tag{3.1}$$

where $1 < \varsigma < 2$. For simplicity, we assume that $t_0 = 0$ in (W0). For $r > 0$, let $J_r = [0, r]$. Fix $r > 0$ small enough such that $J_r \subset [-\delta_0, \delta_0]$. By (3.1), there exist two sequences $\{\varepsilon_m\}$, $\{M_m\}$ and constants $\varepsilon > 0$, $C > 0$ such that $\varepsilon_m > 0$, $M_m > 0$ and

$$\lim_{m \to \infty} \varepsilon_m = 0, \quad \lim_{m \to \infty} M_m = \infty,$$
$$\frac{e^{ct} W(t, \varepsilon_m)}{\varepsilon_m^2} \geq M_m \quad \text{for } t \in J_r, \tag{3.2}$$
$$\frac{e^{ct} W(t, u)}{u^2} \geq -C \quad \text{for } t \in J_r, \ u \in \mathbb{R}^N. \tag{3.3}$$

For any fixed $k \in \mathbb{N}$, we construct an $A_k \in \Gamma_k$ satisfying $\sup_{u \in A_k} I(u) < 0$. We divide J_r equally into k small closed subintervals and denote them by J_i with $1 \leq i \leq k$. Then the length of each J_i is $a = r/k$. We make an interval $E_i \subset J_i (i = 1, 2, \dots, k)$ such that E_i has the same center as that of J_i and the length of E_i is $a/2$. Define a function $\psi \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\psi(t) = 1$ for $t \in [a/4, 3a/4]$, $\psi(t) = 0$ for $t \in (-\infty, 0] \cup [a, +\infty)$. Then $\operatorname{supp} \psi \subset [0, a]$. Now for each $i \in \{1, 2, \dots, k\}$, we define $\psi_i(t)$ by a parallel translation $\psi(t - y_i)$ with a suitable $y_i \in \mathbb{R}$ such that

$$\operatorname{supp} \psi_i \subset D_i, \quad \operatorname{supp} \psi_i \cap \operatorname{supp} \psi_j = \phi \ (i \neq j) \tag{3.4}$$

$$\psi_i(t) = 1 \ (t \in E_i), \ 0 \le \psi_i(t) \le 1 \ (t \in \mathbb{R}). \tag{3.5}$$

Set

$$V_k = \{(t_1, t_2, \dots, t_k) \in \mathbb{R}^k : \max_{1\le i\le k} |t_i| = 1\}, \tag{3.6}$$

$$W_k = \{\sum_{i=1}^{k} t_i\psi_i(t) : (t_1, t_2, \dots, t_k) \in V_k\}. \tag{3.7}$$

Evidently, V_k is homeomorphic to the sphere S^{k-1} by an odd mapping. Therefore, $\gamma(V_k) = \gamma(S^{k-1}) = k$. Moreover, $\gamma(W_k) = \gamma(V_k) = k$ because the mapping $(t_1, \dots, t_k) \mapsto \sum t_i\psi_i$ is odd and homeomorphic. It is clear that W_k is compact. So there is a constant $C_k > 0$ such that

$$\|u\| \le C_k, \quad \forall u \in W_k. \tag{3.8}$$

For $s > 0$ and $u = \sum_{i=1}^{k} t_i\psi_i(t) \in W_k$, by (3.4) and (3.8), we obtain

$$I(su) \le \frac{s^2}{2}C_k^2 - \sum_{i=1}^{k}\int_{D_i} e^{ct}W(t, st_i\psi_i(t))dt. \tag{3.9}$$

By (3.6), there exists $j \in \{1, 2, \dots, k\}$ such that $|t_j| = 1$ and $|t_i| \le 1$ for the others i. It follows that

$$\begin{aligned}
&\sum_{i=1}^{k}\int_{D_i} e^{ct}W(t, st_i\psi_i(t))dt \\
&= \int_{E_j} e^{ct}W(t, st_i\psi_i(t))dt + \int_{D_j\setminus E_j} e^{ct}W(t, st_i\psi_i(t))dt \\
&\quad + \sum_{i\ne j}\int_{D_i} e^{ct}W(t, st_i\psi_i(t))dt.
\end{aligned} \tag{3.10}$$

Since $|t_j| = 1$, $\psi_j(t) \equiv 1$ on E_j and $W(t, u) = W(t, |u|)$, so we have

$$\int_{E_j} e^{ct}W(t, st_i\psi_i(t))dt = \int_{E_j} e^{ct}W(t, s)dt. \tag{3.11}$$

By (3.3), the second and the third terms are estimated as

$$\int_{D_j\setminus E_j} e^{ct}W(t, st_i\psi_i(t))dt + \sum_{i\ne j}\int_{D_i} e^{ct}W(t, st_i\psi_i(t))dt \ge -Crs^2. \tag{3.12}$$

Combining (3.9)–(3.12), we have

$$I(su) \le \frac{C_k^2}{2}s^2 + Crs^2 - \int_{E_j} e^{ct}W(t, s)dt.$$

Substituting s by ε_m and using (3.2), we obtain

$$I(\varepsilon_m u) \le \varepsilon_m^2\Big(\frac{C_k^2}{2} + Cr - M_m\frac{a}{2}\Big). \tag{3.13}$$

Since $M_m \to \infty$ as $m \to \infty$, we choose m_0 large enough such that the right-hand side of (3.13) is negative. Take $A_k = \varepsilon_{m_0}W_k$. Then we have

$$\gamma(A_k) = \gamma(W_k) = k, \quad \sup\{I(u) : u \in A_k\} < 0.$$

Consequently, (I2) holds. Therefore, by Theorem 2.8, (1.1) has infinitely many g-quasi-homoclinic solutions $\{u_n\}$ satisfying $I(u_n) \to 0^-$ as $n \to \infty$.

When $c = 0$, by Lemma 2.2 and Remark 1.3, it is not difficult to obtain that that $I \in C^1(E, \mathbb{R})$, I satisfies (PS)-condition and any critical point of I on E is a homoclinic solution of (1.1). Clearly $I(0) = 0$ and I is even via (W0). Moreover, by (W0), we can easily prove that I is bounded from below. It suffices to verify that I satisfies (I2) of Theorem 2.8. By (W0), there exist $\delta_1 > 0$ and $\alpha_1 > 0$ such that $\alpha(t) \geq \alpha_1 > 0$ for all $t \in [t_0 - \delta_1, t_0 + \delta_1]$. Hence, we have

$$\lim_{u\to 0} \min_{|t-t_0|\leq\delta_1} \frac{W(t,u)}{|u|^2} \geq \min_{|t-t_0|\leq\delta_1} \alpha(t) \lim_{u\to 0} \frac{1}{|u|^{2-\varsigma}} = +\infty,$$

The rest of the proof is done by repeating the above argument; so we omit it. □

Proof of Theorem 1.11. When $c > 0$, similar to [31, Lemmas 2.7 and 3.2], we can prove that $I \in C^1(E, \mathbb{R})$ satisfying (PS)-condition and any critical point of I on E is a g-quasi-homoclinic solution of (1.1). By (W0') and $V(u) = V(-u)$ for all $u \in \mathbb{R}^N$, we obtain that $I(0) = 0$ and $I(u)$ is even. Moreover, by (W0'), it is easy to prove that I is coercive, i.e. $I(u) \to +\infty$ as $\|u\| \to +\infty$. Hence, I is bounded from below. It is sufficient to show that I satisfies (I2) of Theorem 2.8. By (W0'), there exist $\delta_0' > 0$ and $\beta_0 > 0$ such that $\beta(t) \geq \beta_0 > 0$ for all $t \in [t_0' - \delta_0', t_0' + \delta_0']$. Therefore, by (1.3), we have

$$\lim_{u\to 0} \min_{|t-t_0'|\leq\delta_0'} \frac{e^{ct}W(t,u)}{|u|^2} \geq \min_{|t-t_0'|\leq\delta_0'} e^{ct}\beta(t) \lim_{u\to 0} \frac{V(u)}{|u|^2} = +\infty.$$

The rest of the proof is similar to that of Theorem 1.10; we omit it. □

Proof of Theorem 1.13. When $c > 0$, similar to [31, Lemma 2.8], we obtain that $I \in C^1(E, \mathbb{R})$ and any critical point of I on E is a g-quasi-homoclinic solution of (1.1). By (W0), we obtain $I(0) = 0$ and I is even. It follows from [31, Lemma 3.3] that I is coercive. So I is bounded from below. To get Theorem 1.13, it is then sufficient to show that $I(u)$ satisfies the (PS)-condition and (I2) of Theorem 2.8.

However, under condition (L0'), it is not easy to show the (PS)-condition for I since we can not prove compactness of the embedding E into $L^2(e^{ct})$. To overcome this difficulty, We adopt some arguments similar to those in [6]. Firstly, we claim that if $u_n \rightharpoonup u$ in E, then $I_2'(u_n) \to I_2'(u)$ as $n \to \infty$. Actually, it suffices to verify that

$$\varphi(n,h) := \int_{\mathbb{R}} e^{ct}|W_u(t,u_n) - W_u(t,u)||h|dt \to 0 \quad \text{as } n \to \infty,$$

uniformly for $h \in E$ with $\|h\| = 1$. Let $R > 0$ and $J_R = [-R, R]$. We rewrite $\varphi(n,h)$ as follows

$$\begin{aligned} \varphi(n,h) &= \int_{J_R} e^{ct}|W_u(t,u_n) - W_u(t,u)||h| + \int_{\mathbb{R}\setminus J_R} e^{ct}|W_u(t,u_n) - W_u(t,u)||h| \\ &= \varphi_1(n,h,R) + \varphi_2(n,h,R). \end{aligned}$$

The term $\varphi_2(n,h,R)$ can be estimated as

$$\begin{aligned} \varphi_2(n,h,R) &\leq \int_{\mathbb{R}\setminus J_R} e^{ct}\alpha(t)(|u_n|^{\varsigma-1} + |u|^{\varsigma-1})|h|dt \\ &\leq \Big(\int_{\mathbb{R}\setminus J_R} e^{ct}|\alpha(t)|^{\frac{2}{2-\varsigma}}dt\Big)^{\frac{2-\varsigma}{2}} (|u_n|_2^{\varsigma-1} + |u|_2^{\varsigma-1})|h|_2 \\ &\leq C\Big(\int_{\mathbb{R}\setminus J_R} e^{ct}|\alpha(t)|^{\frac{2}{2-\varsigma}}dt\Big)^{\frac{2-\varsigma}{2}} \end{aligned}$$

uniformly for $\|h\| = 1$. Note that this last expression can be made arbitrarily small by taking $R > 0$ large enough. For given $R > 0$, Sobolev's theorem implies that $u_n \to u$ on $L^p(J_R, \mathbb{R})$, $p \in [2, +\infty]$. Therefore, $u_n \to u$ uniformly on J_R. So

$$\varphi_1(n, h, R) = \int_{J_R} e^{ct}|W_u(t, u_n) - W_u(t, u)||h|dt \to 0 \quad \text{as } n \to \infty$$

uniformly for $\|h\| = 1$.

Next, we verify the (PS)-condition. Assume that $\{u_j\} \subset E$ is a sequence such that $\{I(u_j)\}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then, there exists a constant $C > 0$ such that

$$\frac{2-\varsigma}{2}\|u_j\|^2 = I'(u_j)u_j - \varsigma I(u_j) \le C\|u_j\| + C. \tag{3.14}$$

Since $1 < \varsigma < 2$, the inequality (3.14) shows that $\{u_j\}$ is bounded in E. There exists a subsequence of $\{u_j\}$, again denoted by $\{u_j\}$, and $u \in E$ such that $u_j \rightharpoonup u$ in E. By the claim, $I_2'(u_j) \to I_2'(u)$ as $j \to \infty$. Therefore,

$$\|u_j - u\|^2 = (I'(u_j) - I'(u), u_j - u) + (I_2'(u_j) - I_2'(u), u_j - u) \to 0, \quad \text{as } j \to \infty.$$

Consequently, $\{u_j\}$ converges strongly to u in E.

The rest of the proof is analogous to that in Theorem 1.10; so we omit it. □

Proof of Theorem 1.14. When $c > 0$, similar to [31, Lemma 2.8], we conclude that $I \in C^1(E, \mathbb{R})$ and any critical point of I on E is a g-quasi-homoclinic solution of (1.1). By (W0') and $V(u) = V(-u)$, we obtain $I(0) = 0$ and I is even. It follows from [31, Lemma 3.4] that I is coercive. So I is bounded from below. To prove Theorem 1.14, it is then sufficient to show that $I(u)$ satisfies the (PS)-condition and (I2) of Theorem 2.8.

Firstly, similar to the proof of Theorem 1.13, we claim that if $u_n \rightharpoonup u$ in E, then $I_2'(u_n) \to I_2'(u)$ as $n \to \infty$. It suffices to verify that $\phi(n, h) \to 0$ as $n \to \infty$ uniformly for $h \in E$ with $\|h\| = 1$. By (W0'), we can estimate $\phi_2(n, h, R)$ as follows

$$\phi_2(n, h, R) \le \int_{\mathbb{R}\setminus J_R} e^{ct}\beta(t)(|V_u(u_n)| + |V_u(u)|)|h|dt \le C\Big(\int_{\mathbb{R}\setminus J_R} e^{ct}|\beta(t)|^2 dt\Big)^{1/2}$$

uniformly for $\|h\| = 1$. Note that this last term can be made arbitrarily small by taking $R > 0$ large. For given $R > 0$, similar to Theorem 1.13, we can derive that $\phi_1(n, h, R) \to 0$ as $n \to \infty$ uniformly for $\|h\| = 1$. So $\phi(n, h) \to 0$ as $n \to \infty$ uniformly for $h \in E$ with $\|h\| = 1$ and the claim holds.

Next, we verify the (PS)-condition. Assume that $\{u_j\} \subset E$ is a sequence such that $\{I(u_j)\}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then, by (W0'), we obtain

$$\frac{1}{2}\|u_j\|^2 = I(u_j) + \int_{\mathbb{R}} e^{ct}\beta(t)|V(u_j)| \le C + C|\beta|_2|u_j|_2 \le C + C\|u_j\|,$$

which implies that $\{u_j\}$ is bounded in E.

The rest part of the proof is similar to those of Theorems 1.13 and 1.11; so we omit it. □

4. The super-quadratic case

Using a similar idea to one in [15, Lemma 2], we obtain the following result.

Lemma 4.1. *Suppose that* (L0), (L1), (W1), (W3) *are satisfied. If* $u_k \rightharpoonup u$ *(weakly) in* E*, then* $W_u(t, u_k) \to W_u(t, u)$ *in* $L^2(e^{ct})$.

Proof. For all $k \in \mathbb{N}$ and $t \in \mathbb{R}$, from $(W_1) - (W_2)$, we obtain

$$|W_u(t, u_k(t))| \le |u_k(t)| + C|u_k(t)|^{p-1}.$$

Recalling Lemma 2.1, Lemma 2.5 and Remark 2.6, we have $u_k \to u$ in $L^2(e^{ct})$ and $u_k \to u$ in $L^{2p-2}(e^{ct})$. So we can assume $u_k \to u$ a.e. $t \in \mathbb{R}$ up to a subsequence. Then by [22, Lemma A.1], there exists a function $h \in L^2(e^{ct}) \cap L^{2p-2}(e^{ct})$ such that

$$|u_k(t)|, \; |u(t)| \le h(t) \quad \text{a.e. } t \in \mathbb{R}.$$

Therefore,

$$e^{ct}|W_u(t, u_k(t)) - W_u(t, u(t))|^2 \le Ce^{ct}(|h(t)|^2 + |h(t)|^{2p-2}) \in L^1(\mathbb{R}, \mathbb{R}).$$

Hence, the lemma is proved by Lebesgue's dominated convergence theorem. □

The proof of the next lemma is standard and we omit it.

Lemma 4.2. *If* (L0), (L1), (W1), (W3) *are satisfied, then* $I \in C^1(E, \mathbb{R})$ *and*

$$I'(u)v = \int_{\mathbb{R}} e^{ct}(\langle \dot{u}(t), \dot{v}(t)\rangle + \langle L(t)u(t), v(t)\rangle - \langle W_u(t, u(t)), v(t)\rangle)dt,$$

for all $u, v \in E$. *Furthermore, any critical point of* I *on* E *is a g-quasi-homoclinic solution of* (1.1).

The following result ensures the compactness of the functional I.

Lemma 4.3. *Assume that* (L0), (L1), (W1)–(W3) *hold. Then* I *satisfies the (PS)-condition.*

Proof. Let $\{u_j\} \subset E$ be such that $\{I(u_j)\}$ is bounded and $I'(u_j) \to 0$ as $j \to \infty$. Then there exists a constant $M_3 > 0$ such that

$$M_3 + \|u_j\| \ge I(u_j) - \frac{1}{\mu}I^{'}(u_j)u_j \ge (\frac{1}{2} - \frac{1}{\mu})\|u_j\|^2.$$

The sequence $\{u_j\}$ is bounded in E since $\mu > 2$. Without loss of generality, we assume $u_j \rightharpoonup u$ for some $u \in E$ as $j \to \infty$. It follows from the definition of I that

$$\|u_j - u\|^2 = (I^{'}(u_j) - I^{'}(u), u_j - u) + \int_{\mathbb{R}} e^{ct}\langle W_u(t, u_j) - W_u(t, u), u_j - u\rangle.$$

By Lemma 4.1 and Hölder inequality, one immediately has $u_j \to u$ in E as $j \to \infty$. □

With the help of the preceding three lemmas we can now complete the proofs of Theorems 1.16 and 1.17.

Proof of Theorem 1.16. We use Theorem 2.9 to prove the existence of a nontrivial critical point of I.

Step 1. It is clear that $I(0) = 0$ via (W1)–(W2) and by Lemmas 4.2 and 4.3, $I \in C^1(E, \mathbb{R})$ and I satisfies the (PS)-condition.

Step 2. We now prove that there exist constants ρ, $\alpha > 0$ such that I satisfies condition (I3) of Theorem 2.9. By (W1) and (W3), given $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|W_u(t, u)| \le \varepsilon|u| + c_\varepsilon|u|^{p-1}.$$

Noticing (2.4), for $u \in E$, one obtains

$$\int_{\mathbb{R}} e^{ct}W(t, u(t)) \le \varepsilon|u|_2^2 + c_\varepsilon|u|_p^p \le C(\varepsilon\|u\|^2 + \|u\|^p).$$

Since ε is arbitrarily and $p \in (2, +\infty)$, it follows that

$$\int_{\mathbb{R}} e^{ct} W(t,u) = o(\|u\|^2), \quad \text{as } u \to 0 \text{ in } E.$$

Consequently, I satisfies (I3) of Theorem 2.9.

Step 3. We verify that there exists $e \in E \backslash B_\rho$ such that $I(e) \le 0$. Consider

$$I(su) = \frac{s^2}{2}\|u\|^2 - \int_{\mathbb{R}} e^{ct} W(t, su)$$

for $s \in \mathbb{R}\backslash\{0\}$ and $u \in E\backslash\{0\}$. By (W2), there exists a continuous function $\alpha_0(t) > 0$ such that for all $|u| \ge 1$,

$$W(t,u) \ge \alpha_0(t)|u|^\mu.$$

Set $\bar{u} \in E$ with $\|\bar{u}\| = 1$. Then there exist a constant $\delta_1 > 0$ and a positive measure subset Ω of $\mathbb{R}$ such that $|\bar{u}(t)| \ge \delta_1 > 0$ for all $t \in \Omega$. Choosing $s > 0$ such that $s|\bar{u}(t)| \ge 1$ for all $t \in \Omega$, we deduce that

$$I(s\bar{u}) \le \frac{s^2}{2}\|\bar{u}\|^2 - \int_{\mathbb{R}} e^{ct} W(t, s\bar{u}) \le \frac{s^2}{2} - s^\mu \int_\Omega e^{ct}\alpha_0(t)|\bar{u}(t)|^\mu. \tag{4.1}$$

Since $\mu > 2$ and $\alpha_0(t) > 0$, (4.1) implies that $I(s\bar{u}) < 0 = I(0)$ for some $s > 0$ such that $s|\bar{u}(t)| \ge 1$ for all $t \in \Omega$ and $\|s\bar{u}\| > \rho$, where ρ is defined in Step 2. □

Proof of Theorem 1.17. By (W4), I is even. Using the same argument as in the proof of Theorem 1.16, we can easily show that $I(0) = 0$, $I \in C^1(E, \mathbb{R})$ and satisfies the (PS)-condition. Likewise, by taking $V = 0$ and $X = E$, (I5) of Theorem 2.10 holds.

The theorem will be proved if we can show that I satisfies condition (I6). Let $\tilde{E} \subset E$ be a finite dimensional subspace. From Step 3 of Theorem 1.16, we know that, for any $\tilde{u} \in \tilde{E} \subset E$ with $\|\tilde{u}\| = 1$, there exists $s_{\tilde{u}} > 0$ such that

$$I(s\tilde{u}) < 0, \forall |s| \ge s_{\tilde{u}} > 0. \tag{4.2}$$

Then we can take $R = R(\tilde{E}) > 0$ such that

$$I(u) < 0, \ \forall u \in \tilde{E}\backslash B_R.$$

By Theorem 2.10, I possesses an unbounded sequence of critical values $\{c_j\}_{j\in\mathbb{N}}$ with $c_j \to +\infty$. □

5. The sign-changing case

In this section, we use the Nehari manifold method to prove Theorem 1.20. Under condition (W5), the functional associated to (1.1) is

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{\theta}\int_{\mathbb{R}} e^{ct} a(t)|u|^\theta + \frac{1}{\vartheta}\int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta,$$

which is differentiable on E, and

$$I'(u)v = (u,v) - \int_{\mathbb{R}} e^{ct} a(t)\langle |u|^{\theta-2}u, v\rangle dt + \int_{\mathbb{R}} e^{ct} b(t)\langle |u|^{\vartheta-2}u, v\rangle dt.$$

The Nehari manifold is

$$\begin{aligned}\mathcal{N} &= \{u \in E\backslash\{0\} : I'(u)u = 0\} \\ &= \big\{u \in E\backslash\{0\} : \|u\|^2 = \int_{\mathbb{R}} e^{ct} a(t)|u|^\theta - \int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta\big\}.\end{aligned}$$

Define

$$m = \inf_{u \in \mathcal{N}} I(u).$$

By a series of lemmas, we show that m is attained by some $u \in \mathcal{N}$, which is a critical point of I on E, and then a solution to (1.1).

Lemma 5.1. *The Nehari manifold $\mathcal{N}$ is not empty.*

Proof. Let $u \in E\backslash\{0\}$. We study the behavior of the function on $[0,+\infty)$

$$\gamma(\tau) = I'(\tau u)(\tau u) = \tau^2\|u\|^2 - \tau^\theta \int_{\mathbb{R}} e^{ct}a(t)|u|^\theta + \tau^\vartheta \int_{\mathbb{R}} e^{ct}b(t)|u|^\vartheta.$$

Obviously, γ is continuous. By (W5), it is easy to see that

$$\gamma(\tau) = \tau^2\|u\|^2 + o(\tau^2) \quad \text{as } \tau \to 0^+,$$
$$\gamma(\tau) \to -\infty \text{ as } \tau \to +\infty.$$

Therefore, there exists $\tau^* \in (0,+\infty)$ such that $\gamma(\tau^*) = 0$, i.e. $\tau^* u \in \mathcal{N}$. □

Lemma 5.2. *The functional I is coercive on $\mathcal{N}$ and $m > 0$.*

Proof. For every $u \in \mathcal{N}$, we have

$$I(u) = (\frac{1}{2} - \frac{1}{\theta})\|u\|^2 + (\frac{1}{\vartheta} - \frac{1}{\theta}) \int_{\mathbb{R}} e^{ct}b(t)|u|^\vartheta \geq (\frac{1}{2} - \frac{1}{\theta})\|u\|^2$$

which shows that I is coercive on $\mathcal{N}$. By (2.4), one has

$$\|u\|^2 \leq \|u\|^2 + \int_{\mathbb{R}} e^{ct}b(t)|u|^\vartheta = \int_{\mathbb{R}} e^{ct}a(t)|u|^\theta \leq |a|_\infty C_\theta^\theta \|u\|^\theta.$$

Consequently,

$$\|u\| \geq \left(\frac{1}{C_\theta^\theta |a|_\infty}\right)^{\frac{1}{\theta-2}} > 0.$$

It follows that

$$m = \inf_{u\in\mathcal{N}} I(u) \geq \inf_{u\in\mathcal{N}} (\frac{1}{2} - \frac{1}{\theta})\|u\|^2 \geq (\frac{1}{2} - \frac{1}{\theta}) \left(\frac{1}{C_\theta^\theta |a|_\infty}\right)^{\frac{2}{\theta-2}} > 0.$$

□

Lemma 5.3. *There exists $u \in \mathcal{N}$ such that $I(u) = m$.*

Proof. Let $\{u_n\} \subset \mathcal{N}$ be a minimizing sequence for I. Since I is coercive on $\mathcal{N}$, the sequence $\{u_n\}$ is bounded and then, up to subsequence, $u_n \rightharpoonup u$ in E. By Lemma 2.5, for all $\kappa \in [2,r]$, $u_n \to u$ in $L^\kappa(e^{ct})$ and $u_n \to u$ a.e. in $\mathbb{R}$. Hence, we obtain that

$$I(u) \leq \liminf_{n\to\infty} I(u_n) = m,$$
$$\|u\|^2 \leq \int_{\mathbb{R}} e^{ct}a(t)|u|^\theta - \int_{\mathbb{R}} e^{ct}b(t)|u|^\vartheta.$$

Since for all $n \in \mathbb{N}$,

$$\|u_n\| \geq \left(\frac{1}{C_\theta^\theta |a|_\infty}\right)^{\frac{1}{\theta-2}} > 0,$$

we obtain

$$\int_{\mathbb{R}} e^{ct}a(t)|u|^\theta - \int_{\mathbb{R}} e^{ct}b(t)|u|^\vartheta > 0,$$

which shows that $u \not\equiv 0$.

If $\|u\|^2 < \int_{\mathbb{R}} e^{ct} a(t)|u|^\theta - \int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta$, consider the function

$$\gamma(\tau) = \tau^2\|u\|^2 - \tau^\theta \int_{\mathbb{R}} e^{ct} a(t)|u|^\theta + \tau^\vartheta \int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta.$$

Since $\gamma(1) < 0$ and $\gamma(\tau) > 0$ in a right neighborhood of 0, there exists $\tau_0 \in (0,1)$ such that $\gamma(\tau_0) = 0$, which means that $\tau_0 u \in \mathcal{N}$. Consequently,

$$\begin{aligned}
m &\le I(\tau_0 u)\\
&= \frac{1}{2}\|\tau_0 u\|^2 - \frac{1}{\theta}\int_{\mathbb{R}} e^{ct} a(t)|\tau_0 u|^\theta + \frac{1}{\vartheta}\int_{\mathbb{R}} e^{ct} b(t)|\tau_0 u|^\vartheta\\
&= \big(\frac{1}{2} - \frac{2}{\theta+\vartheta}\big)\|\tau_0 u\|^2 + \big(\frac{2}{\theta+\vartheta} - \frac{1}{\theta}\big)\int_{\mathbb{R}} e^{ct} a(t)|\tau_0 u|^\theta\\
&\quad + \big(\frac{1}{\vartheta} - \frac{2}{\theta+\vartheta}\big)\int_{\mathbb{R}} e^{ct} b(t)|\tau_0 u|^\vartheta\\
&< \big(\frac{1}{2} - \frac{2}{\theta+\vartheta}\big)\|u\|^2 + \big(\frac{2}{\theta+\vartheta} - \frac{1}{\theta}\big)\int_{\mathbb{R}} e^{ct} a(t)|u|^\theta + \big(\frac{1}{\vartheta} - \frac{2}{\theta+\vartheta}\big)\int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta\\
&\le \liminf_{n\to\infty} I(u_n) = m.
\end{aligned}$$

This is a contradiction. Therefore,

$$\|u\|^2 = \int_{\mathbb{R}} e^{ct} a(t)|u|^\theta - \int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta.$$

Then $u \in \mathcal{N}$ and is the required minimum. The proof is complete. □

Lemma 5.4. *Let $u \in \mathcal{N}$ be such that $I(u) = m$. Then $I'(u) = 0$.*

Proof. Take $v \in E$ and let $\varepsilon > 0$ be small that $u + sv \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. We know that there exists $\tau(s) \in \mathbb{R}$ such that $\tau(s)(u + sv) \in \mathcal{N}$. Consider the function

$$F(s,\tau) = \tau^2\|u+sv\|^2 - \int_{\mathbb{R}} e^{ct} a(t)|\tau(u+sv)|^\theta dt + \int_{\mathbb{R}} e^{ct} b(t)|\tau(u+sv)|^\vartheta dt$$

for $(s,\tau) \in (-\varepsilon,\varepsilon) \times \mathbb{R}$. Since $u \in \mathcal{N}$, we have

$$F(0,1) = \|u\|^2 - \int_{\mathbb{R}} e^{ct} a(t)|u|^\theta dt + \int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta dt = 0.$$

On the other hand,

$$\begin{aligned}
\frac{\partial F}{\partial \tau}(0,1) &= 2\|u\|^2 - \theta\int_{\mathbb{R}} e^{ct} a(t)|u|^\theta + \vartheta\int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta\\
&\le 2\|u\|^2 - \frac{\theta+\vartheta}{2}\int_{\mathbb{R}} e^{ct} a(t)|u|^\theta dt + \frac{\theta+\vartheta}{2}\int_{\mathbb{R}} e^{ct} b(t)|u|^\vartheta dt\\
&= (2 - \frac{\theta+\vartheta}{2})\|u\|^2 < 0.
\end{aligned}$$

So, by the Implicit Function Theorem, for ε small enough, we can determine a function $\tau \in C^1(-\varepsilon,\varepsilon)$ such that $F(s,\tau(s)) = 0$, and $\tau(0) = 1$. This says that $\tau(s) \neq 0$ at least for ε small, and then $\tau(s)(u+sv) \in \mathcal{N}$.

Setting

$$\gamma(s) = I(\tau(s)(u+sv)),$$

we obtain that γ is differentiable and has a minimum point at $s = 0$, thus

$$0 = \gamma'(0) = I'(\tau(0)u)(\tau'(0)u + \tau(0)u) = \tau'(0)I'(u)u + I'(u)v = I'(u)v.$$

Since $v \in E$, is arbitrary, we conclude that $I'(u) = 0$. □

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