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PERIODIC SOLUTIONS FOR LIÉNARD DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Abstract. In this article, we study the second-order forced Liénard equation $x'' + f(x)x' + g(x) = e(t)$. By using the topological degree theory, we prove that the equation has at least one positive periodic solution when g admits a repulsive singularity near the origin and satisfies some semilinear growth conditions near infinity. Recent results in the literature are generalized and complemented.

1. Introduction

In this work, we are concerned with the existence of positive T-periodic solutions for the Liénard equation

$$x'' + f(x)x' + g(x) = e(t), \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous functions, $g : (0, \infty) \to \mathbb{R}$ is continuous and admits a repulsive singularity near the origin, e is continuous and T-periodic.

As we know, the Liénard equation appears in a number of physical models, for example, it is used to describe fluid mechanical and nonlinear elastic mechanical phenomena. During the last few decades, the Liénard equation has attracted many researchers. One important related topic is to look for periodic solutions under different conditions. We refer the reader to [20, 22, 23, 25, 29] and the references therein. Here we mention the following results: Fonda et al [12] used the Poincare-Birkhoff theorem to obtain the existence of positive periodic solutions, including all subharmonics, for the following special case of (1.1)

$$x'' + g(x) = e(t), \tag{1.2}$$

where $e \in C(\mathbb{R}, \mathbb{R})$ is T-periodic and $g \in C(\mathbb{R}^+, \mathbb{R})$ satisfies the following strong force condition at $x = 0$:

$$\lim_{x\to 0^+} g(x) = -\infty, \quad \lim_{x\to 0^+} G(x) = +\infty, \quad \Big(G(x) = \int_1^x g(s)ds\Big) \tag{1.3}$$

and g is superlinear at $x = +\infty$:

$$\lim_{x\to +\infty} g(x) = +\infty, \tag{1.4}$$

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Later, Wang [30] used the phase-plane analysis methods proved the existence of at least one positive T-periodic solution of (1.2) if g satisfies (1.3) near $x = 0$, and g satisfies semilinear condition at $x = +\infty$: there is an integer $k \geq 0$ and a small constant $\varepsilon > 0$ such that

$$(\frac{k\pi}{T})^2 + \varepsilon \leq \frac{g(x)}{x} \leq (\frac{(k+1)\pi}{T})^2 - \varepsilon \tag{1.5}$$

for all t and all $x \gg 1$. We note that the conditions (1.5) are the standard uniform nonresonance conditions with respect to the Dirichlet boundary condition, not the periodic boundary condition.

When $f \neq 0$, equation (1.1) is a non-conservative system. Lefschetz [19] gave the first existence theorem for equation (1.1) under some dissipativity conditions. Many researchers tried to improve the results of [19]. We assume that there exists a constant $d > 0$ such that

$$g(x)\operatorname{sgn}(x) > 0, \quad |x| \geq d. \tag{1.6}$$

Mawhin [22] studied the existence of periodic solutions under assumption that g satisfies (1.6) and the sublinear condition

$$\lim_{|x|\to+\infty} \frac{g(x)}{x} = 0. \tag{1.7}$$

Later, Mawhin and Ward [23] improved such a condition, and used the following condition

$$\limsup_{x\to+\infty} \frac{g(x)}{x} < (\frac{\pi}{T})^2.$$

instead of (1.7).

This article is mainly motivated by the work mentioned above and the recent papers [30, 33, 35]. The result is obtained using topological degree theory, thanks to a priori estimates on the solutions of a suitable family of problems. Our main result reads as follows:

Theorem 1.1. *Assume that $f, e : \mathbb{R} \to \mathbb{R}$, $g : (0,\infty) \to \mathbb{R}$ are continuous functions and e is T-periodic. Suppose further that*

- (H1) $\lim_{x\to0^+} g(x) = -\infty$ *and* $\lim_{x\to0^+} \int_1^x g(r)dr = +\infty$*;*
- (H2) *There exist T-periodic continuous functions a, b such that*

$$a(t) \leq \liminf_{x\to+\infty} \frac{g(x)}{x} \leq \limsup_{x\to+\infty} \frac{g(x)}{x} \leq b(t). \tag{1.8}$$

Also,

$$\bar{a} > 0 \quad \text{and} \quad \underline{\lambda}_1(b) > 0, \tag{1.9}$$

here $\bar{a} = \frac{1}{T}\int_0^T a(t)dt$ and $\{\underline{\lambda}_1(q)\}$ denotes the first anti-periodic eigenvalues of

$$x'' + (\lambda + q(t))x = 0. \tag{1.10}$$

Then (1.1) has at least one positive T-periodic solution.

The main novelty in the present paper is represented by the conditions at infinity, which remind of a situation between the first and the second eigenvalue, but are more general since the comparison involves the mean and the weighted eigenvalue associated with the functions a, b controlling the $g(x)/x$.

During the previous few decades, singular differential equations or singular dynamical systems have been attracted the attention of many researchers [2, 9, 11, 14, 15, 18, 26, 33, 34]. It is well-known that electrostatic or gravitational forces are the most important applications of singular interactions. It was also found recently that one special singular differential equation which is called "Ermakov-Pinney equation" plays an important role in the studying of the stability of periodic solutions of conservative systems with degree of lower freedom (see [7] and the references therein). Usually, the proof is based on either variational approach [1, 2] or topological methods. The proof of the main results in this paper is based on topological methods, which started with the pioneering paper of Lazer and Solimini [18]. From then on, some fixed point theorems in cones for completely continuous operators[13, 28], the method of upper and lower solutions [15, 24], Schauder's fixed point theorem [27], degree theory [11, 31] and a nonlinear alternative principle of Leray-Schauder type [6, 8, 16] have been widely applied.

The rest of this article is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by the use of topological degree theory, we will state and prove the main results. To illustrate the new results, some applications are also given.

2. Preliminaries

In this section, we present some results which will be applied in Sections 3. Let us first introduce some known results on eigenvalues. Let q be a T-periodic potential such that $q \in L^1(\mathbb{R})$. Consider the eigenvalue problems of (1.10) with the T-periodic boundary condition :

$$x(0) - x(T) = x'(0) - x'(T) = 0, \tag{2.1}$$

or, with the anti-T-periodic boundary condition :

$$x(0) + x(T) = x'(0) + x'(T) = 0. \tag{2.2}$$

We use $\lambda_1^D(q) < \lambda_2^D(q) < \dots < \lambda_n^D(q) < \dots$ to denote all eigenvalues of (1.10) with the Dirichlet boundary condition:

$$x(0) = x(T) = 0. \tag{2.3}$$

The following are the standard results for eigenvalues. See, e.g. Reference [21].

(E1) there exist two sequences $\{\underline{\lambda}_n(q) : n \in \mathbb{N}\}$ and $\{\bar{\lambda}_n(q) : n \in \mathbb{Z}^+\}$ such that

$$-\infty < \overline{\lambda}_0(q) < \underline{\lambda}_1(q) \le \overline{\lambda}_1(q) < \underline{\lambda}_2(q) \le \overline{\lambda}_2(q) < \dots < \underline{\lambda}_n(q) \le \overline{\lambda}_n(q) < \dots$$

where $\underline{\lambda}_n(q) \to +\infty, \overline{\lambda}_n(q) \to +\infty$ as $n \to +\infty$. Moreover, λ is an eigenvalue of (1.10)-(2.1) if and only if $\lambda = \underline{\lambda}_n(q)$ or $\overline{\lambda}_n(q)$ with n is even; and λ is an eigenvalue of $(1.10) - (2.2)$ if and only if $\lambda = \underline{\lambda}_n(q)$ or $\overline{\lambda}_n(q)$ with n is odd.

(E2) If $q_1 \le q_2$ then

$$\underline{\lambda}_n(q_1) \ge \underline{\lambda}_n(q_2), \overline{\lambda}_n(q_1) \ge \overline{\lambda}_n(q_2), \lambda_n^D(q_1) \ge \lambda_n^D(q_2)$$

for any $n \ge 1$.

(E3) For any $n \ge 1$,

$$\underline{\lambda}_n(q) = \min\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\}, \overline{\lambda}_n(q) = \max\{\lambda_n^D(q_{t_0}) : t_0 \in \mathbb{R}\}$$

where q_{t_0} denotes the translation of $q : q_{t_0}(t) \equiv q(t + t_0)$.

(E4) $\bar{\lambda}_n(q), \underline{\lambda}_n(q)$ and $\lambda_n^D(q)$ are continuous in q in the L^1-topology of $L^1(0, T)$.

(E5) It follows from a variational principle that the first eigenvalue λ_1^D can be found as
$$\lambda_1^D(q) = \inf_{x\in H_0^1(0,T)\ x\neq 0} \frac{\int_0^T \big(x'^2(t) - q(t)x^2(t)\big)dt}{\int_0^T x^2(t)dt}.$$
In particular,
$$\lambda_1^D(q) \le -\bar{q},$$
where $\bar{q}$ is the mean value. Moreover, the equality holds if and only if $q(t) = \bar{q}$ for a.e. t.

To prove our results, we need the following preliminary results, recall some notation and terminology from [5]. Define $L : \mathrm{dom}L \subset X \to Z,\ Lx = \dot{x}$, a Fredholm mapping of index zero, with $\mathrm{dom}\, L = \{x \in X : x(\cdot) \text{ is absolutely continuous}\}$, where the Banach spaces X, Z are
$$X = \{x \in C([0,T], \mathbb{R}^m) : x(0) = x(T)\}, \quad Z = L^1([0,T], \mathbb{R}^m)$$
with their usual norms. Then L is a Fredholm mapping of index zero [32]. Let M_0 be the Nemitzky operator from X to Z induced by the map $F : X \to \mathbb{R}^m$; that is, $M_0 : X \to Z, x(\cdot) \to F(x(\cdot))$. Consider the equation
$$Lx = M_0x, \quad x \in \mathrm{dom}\, L.$$

Lemma 2.1 ([5, Theorem 1])**.** *Let $\Omega \subset X$ be a bounded open subset and assume that there is no $x(\cdot) \in \partial_X\Omega$ such that $\dot{x} = f_0(x)$. Then*
$$\deg_L(L - M_0, \Omega) = (-1)^m \deg_B(f_0, \Omega \cap R^m, 0),$$
where $\deg_L$, $\deg_B$ denote the Schauder degree and the Brouwer degree, respectively.

We refer the reader to [32] for more details about degree theory.

3. Proof of Theorem 1.1

We will apply Lemma 2.1 to the singular problem (1.1). To this end, we deform (1.1) to a simpler singular autonomous equation
$$x'' + c_0x = \frac{1}{x},$$
where c_0 for some positive constant satisfy $0 < c_0 < (\pi/T)^2$. The choice of such a c_0 implies that the constant functions $a(t) = b(t) \equiv c_0$ satisfy (1.9). Consider the homotopy equation
$$x'' + \tau f(x)x' + g(t, x; \tau) = 0, \quad \tau \in [0,1], \tag{3.1}$$
where
$$g(t, x; \tau) = \tau(g(x) - e(t)) + (1-\tau)(c_0x - \frac{1}{x}).$$
We need to find a priori estimates for the possible positive T-periodic solutions of (3.1).

Note that $g(t, x; \tau)$ satisfies the conditions (H_1) uniformly with respect to $\tau \in [0,1]$. Moreover, for each $\tau \in [0,1]$, $g(t, x; \tau)$ satisfies (1.8) with
$$a = a_\tau = \tau a(t) + (1-\tau)c_0,$$
$$b = b_\tau = \tau b(t) + (1-\tau)c_0.$$
We will prove that a_τ and b_τ satisfy (1.9) uniformly in $\tau \in [0,1]$. This fact follows from the convexity of the first eigenvalues with respect to potentials.

Lemma 3.1. *Assume $q_0, q_1 \in L^1(0,T)$. Then, for all $\tau \in [0,1]$,*

$$\underline{\lambda}_1(\tau q_1 + (1-\tau)q_0) \geq \tau\underline{\lambda}_1(q_1) + (1-\tau)\underline{\lambda}_1(q_0). \tag{3.2}$$

Note that

$$\bar{a}_\tau = \tau\bar{a} + (1-\tau)c_0 \geq \min(\bar{a}, c_0) > 0.$$

Applying Lemma 3.1 to $q_1 = b$ and $q_0 = c_0$, we have

$$\underline{\lambda}_1(b_\tau) \geq \tau\underline{\lambda}_1(b) + (1-\tau)\underline{\lambda}_1(c_0) \geq \min(\underline{\lambda}_1(b), \underline{\lambda}_1(c_0)) > 0.$$

Thus a_τ and b_τ defined above satisfy (1.9) uniformly in $\tau \in [0,1]$.

For obtaining a priori estimates for all possible positive solutions to (3.1)–(2.1), we simply prove this for all possible positive solutions to (1.1)–(2.1), because a_τ, b_τ satisfy (1.8) and also (1.9) uniformly in $\tau \in [0,1]$.

Lemma 3.2. *Assume that $\underline{\lambda}_1(b) > 0$ of the equation*

$$y'' + (\lambda + b(t))y = 0\,.$$

Then

$$\|y'\|_2^2 \geq \int_0^T b(t+t_0)y^2(t)dt + \lambda_1^D(b_{t_0})\int_0^T y^2(t)dt.$$

Proof. By (E3), we have

$$\lambda_1^D(b_{t_0}) \geq \underline{\lambda}_1(b) > 0.$$

for all $t_0 \in \mathbb{R}$. Then, by the theory of second order linear differential operators [10], the eigenvalues of

$$y'' + (\lambda + b(t+t_0))y = 0$$

with Dirichlet boundary conditions form a sequence

$$\lambda_1^D(b_{t_0}) < \lambda_2^D(b_{t_0}) < \dots,$$

which tends $+\infty$, and the corresponding eigenfunctions $\psi_1, \psi_2, \dots$ are an orthonormal base of $L^2(0,T)$. Hence, given $c_i \in \mathbb{R}$ and $y \in H_0^1(0,T)$, we can write

$$y(t) = \sum_{i\geq 1} c_i\psi_i(t),$$

and

$$\begin{aligned}
\int_0^T ((y'(t))^2 - b(t+t_0)y^2(t))dt &= \sum_{i\geq 1} c_i^2 \int_0^T ((\psi_i'(t))^2 - b(t+t_0)\psi_i^2(t))dt \\
&= \sum_{i\geq 1} c_i^2 \lambda_i^D(b_{t_0}) \int_0^T \psi_i^2(t)dt \\
&\geq \lambda_1^D(b_{t_0}) \int_0^T y^2(t)dt.
\end{aligned}$$

This completes the proof. □

The usual L^p-norm is denoted by $\|\cdot\|_p$, and the supremum norm of $C[0,T]$ is denoted by $\|\cdot\|_\infty$.

Lemma 3.3. *Under the assumptions as in Theorem 1.1, there exist $B_1 > B_0 > 0$ such that any positive T-periodic solution $x(t)$ of* (1.1)-(2.1) *satisfies*

$$B_0 < x(t_0) < B_1, \tag{3.3}$$

for some $t_0 \in [0, T]$.

Proof. Let $x(t)$ be a positive T-periodic solution of (1.1)-(2.1). By (H1), there exists $B_0 > 0$ such that

$$g(s) - e(t) < 0 \quad \text{for all } 0 < s < B_0.$$

Integrate (1.1) from 0 to T, we obtain

$$\int_0^T (g(x(t)) - e(t))dt = -\int_0^T x''(t)dt - \int_0^T f(x)x'dt = 0.$$

Thus $\int_0^T (g(x(t)) - e(t))dt = 0$, there exists $t^* \in [0, T]$ such that $x(t^*) > B_0$.

Next, noticing (1.9), we can take some constant $\varepsilon_0 \in \big(0, \min\{\overline{a}, \underline{\lambda}_1(b)\}\big)$. It follows from ($H_2$) that there exists a constants $B_1 (> B_0)$ large enough such that

$$a(t) - \varepsilon_0 \le \frac{g(s) - e(t)}{s} \le b(t) + \varepsilon_0. \tag{3.4}$$

for all t and all $s \ge B_1$. We assert that $x(t_*) < B_1$ for some t_*. Otherwise, assume that $x(t) \ge B_1$ for all t. Define

$$p(t) := \frac{g(x(t)) - e(t)}{x(t)}.$$

By (3.4),

$$a(t) - \varepsilon_0 \le p(t) \le b(t) + \varepsilon_0$$

for all t. Moreover, $x(t)$ satisfies the following differential equation

$$x'' + f(x)x' + p(t)x = 0.$$

Write $x = \tilde{x} + \bar{x}$, where $\bar{x} = \frac{1}{T}\int_0^T x(t)dt$, then $\tilde{x}$ satisfies

$$-\tilde{x}'' - f(\tilde{x} + \bar{x})\tilde{x}' = p(t)\tilde{x} + p(t)\bar{x}. \tag{3.5}$$

Integrating (3.5) from 0 to T, we have

$$\int_0^T p(t)\tilde{x}(t)dt = -\bar{x}\int_0^T p(t)dt. \tag{3.6}$$

Multiplying (3.5) by $\tilde{x}$ and using integration by parts, we obtain

$$\begin{aligned} \|\tilde{x}'\|_2^2 &= \int_0^T p(t)\tilde{x}^2(t)dt + \bar{x}\int_0^T p(t)\tilde{x}(t)dt \\ &= \int_0^T p(t)\tilde{x}^2(t)dt - \bar{x}^2(t)\int_0^T p(t)dt \\ &\le \int_0^T p(t)\tilde{x}^2(t)dt, \end{aligned} \tag{3.7}$$

where the fact $\int_0^T p(t)dt > T(\bar{a} - \varepsilon_0) > 0$ is used.

Note that $\tilde{x}(t_0) = 0$ for some t_0, $\tilde{x}(t_0 + T) = 0$, so $\tilde{x}(t) \in H_0^1(t_0, t_0 + T)$. We assert that $\tilde{x} \equiv 0$. On the contrary, assume that $\tilde{x} \not\equiv 0$. Now by (3.7), the first Dirichlet eigenvalue

$$\lambda_1^D(p|_{[t_0,t_0+T]}) = \inf_{\varphi \in H_0^1(t_0,t_0+T),\, \varphi \neq 0} \frac{\int_{t_0}^{t_0+T} \big(\varphi'^2(t) - p(t)\varphi^2(t)\big)dt}{\int_{t_0}^{t_0+T} \varphi^2(t)dt} \leq 0.$$

So,

$$\underline{\lambda}_1(p) = \min\{\lambda_1^D(p)\} \leq 0.$$

On the other hand, $p(t) < b(t) + \varepsilon_0$,

$$\underline{\lambda}_1(p) \geq \underline{\lambda}_1(b + \varepsilon_0) = \underline{\lambda}_1(b) - \varepsilon_0 > 0.$$

This is a contradiction.

Now it follows from (3.6) that $\bar{x} = 0$ and $x \equiv 0$, a contradiction to the positiveness of $x(t)$. We have proved that $x(t^*) > B_0$ for some $t^* \in [0, T]$ and $x(t_*) < B_1$ for some $t_* \in [0, T]$. Thus the intermediate value theorem implies that (3.3) holds. □

Lemma 3.4. *There exist $B_2 > B_1 > 0$, $B_3 > 0$ such that any positive T-periodic solution $x(t)$ of* (1.1)-(2.1) *satisfies*

$$\|x\|_\infty < B_2, \quad \|x'\|_\infty < B_3.$$

Proof. From (H2) and (3.4), we know that there exists $h_0 > 0$ such that

$$g(s) - e(t) \leq (b(t) + \varepsilon_0)s + h_0 \tag{3.8}$$

for all t and $s > 0$. Multiplying (1.1) by x and then integrating over $[0, T]$, using the fact that

$$\int_0^T f(x(t))x'(t)x(t)dt = 0,$$

we obtain

$$\begin{aligned} \|x'\|_2^2 &= \int_0^T -(xx'' + xf(x)x')dt \\ &= \int_0^T (g(x(t)) - e(t))x(t)dt \\ &\leq \int_0^T ((b(t) + \varepsilon_0)x(t) + h_0)x(t)dt \\ &= \int_0^T b(t)x^2(t)dt + \varepsilon_0\|x\|_2^2 + h_0\|x\|_1. \end{aligned} \tag{3.9}$$

It follows from Lemma 3.3 that there exists t_0 satisfying $B_0 < x(t_0) < B_1$. Let $u(t) = x(t + t_0) - x(t_0)$, then $u \in H_0^1(0, T)$. Thus

$$\begin{aligned} \int_0^T b(t)x^2(t)dt &= \int_0^T b(t + t_0)x^2(t + t_0)dt \\ &= \int_0^T b(t + t_0)\big(x^2(t_0) + 2x(t_0)u(t) + u^2(t)\big)dt \\ &\leq B_1^2\|b\|_1 + 2B_1\|b\|_2\|u\|_2 + \int_0^T b(t + t_0)u^2(t)dt. \end{aligned}$$

The other terms in (3.9) by the Hölder inequality can be estimated as follows:

$$\varepsilon_0\|x\|_2^2 \leq \varepsilon_0(TB_1^2 + 2B_1T^{1/2}\|u\|_2 + \|u\|_2^2),$$

$$h_0\|x\|_1 \le h_0(TB_1 + T^{1/2}\|u\|_2).$$

Thus (3.9) reads

$$\|u'\|_2^2 \le A_0 + A_1\|u\|_2 + \varepsilon_0\|u\|_2^2 + \int_0^T b(t+t_0)u^2(t)dt, \tag{3.10}$$

where

$$A_0 = \varepsilon_0 TB_1^2 + h_0TB_1 + B_1^2\|b\|_1,$$
$$A_1 = 2\varepsilon_0 B_1 T^{1/2} + h_0T^{1/2} + 2B_1\|b\|_2$$

are positive constants.

On the other hand, using Lemma 3.2,

$$\underline{\lambda}_1(b(t))\|u\|_2^2 \le \lambda_1^D(b_{t_0})\|u\|_2^2 \le \int_0^T \big(u'^2(t) - b(t+t_0)u^2(t)\big)dt,$$

and we obtain from (3.10) that

$$(\underline{\lambda}_1(b(t)) - \varepsilon_0)\|u\|_2^2 \le A_1\|u\|_2 + A_0.$$

Consequently, $\|u\|_2 < A_2$ for some $A_2 > 0$. By (3.10), one has $\|x'\|_2 = \|u'\|_2 < A_3$ for some $A_3 > 0$. From these, for any $t \in [t_0, t_0 + T]$,

$$\begin{aligned} |x(t)| &\le |x(t_0)| + \Big|\int_{t_0}^t x'(t)dt\Big| \\ &\le B_1 + T^{1/2}\|x'\|_2 \\ &\le B_1 + T^{1/2}A_3 := B_2. \end{aligned}$$

Thus $\|x\|_\infty < B_2$ is obtained.

To prove $\|x'\|_\infty < B_3$, we write (1.1) as

$$-x''(t) = H(t) := f(x(t))x'(t) + g(x(t)) - e(t).$$

As $\int_0^T H(t)dt = 0$, thus $\|H(t)\|_1 = 2\|H^+(t)\|_1$. From (3.8) we have

$$\begin{aligned} H^+(t) &= \max(H(t), 0) \\ &\le |f(x(t))| \cdot |x'(t)| + |b(t) + \varepsilon_0|x(t) + h_0 \\ &\le C_1|x'(t)| + C_2, \end{aligned}$$

where $C_1 = \max_{0\le y\le B_2}|f(y)|$. Since $x(0) = x(T)$, there exists $t_1 \in [0, T]$ such that $x'(t_1) = 0$. Therefore,

$$\begin{aligned} \|x'\|_\infty &= \max_{0\le t\le T}|x'(t)| \\ &= \max_{0\le t\le T}\Big|\int_{t_1}^t x''(s)ds\Big| \\ &\le \int_0^T |H(s)|\,ds \\ &= 2\int_0^T \big|H^+(s)\big|\,ds \\ &\le 2\big(C_3T^{1/2}\|x'\|_2 + TC_4\big) \\ &\le 2\big(A_3C_1T^{1/2} + TC_2\big) := B_3. \end{aligned}$$

We have proved is that the $W^{2,1}$ norms of x are bounded. □

Next, we obtain the positive lower estimates for $x(t)$ based on the condition (H1).

Lemma 3.5. *There exists a constant $B_4 \in (0, B_0)$ such that any positive solution $x(t)$ of (1.1)–(2.1) satisfies*

$$x(t) > B_4 \quad \text{for all } t.$$

Proof. From (H1), we fix some $A_4 \in (0, B_0)$ such that

$$g(s) - e(t) < -B_3C_1.$$

for all t and all $0 < s \le A_4$, where C_1 is the same as above. Assume now that

$$m = \min_{t\in[0,T]} x(t) = x(t_2) < A_4.$$

By Lemma 3.3, $\max_t x(t) > B_0$. Let $t_3 > t_2$ be the first time instant such that $x(t) = A_4$. Then for any $t \in [t_2, t_3]$, we have $x(t) \le A_4$ and $|-f(x(t))x'(t)| \le B_3C_1$. Hence, for $t \in [t_2, t_3]$,

$$x''(t) = -f(x(t))x'(t) - g(x(t)) - e(t) > B_3C_1 - f(x(t))x'(t) \ge 0.$$

As $x'(t_2) = 0, x'(t) > 0$ for $t \in (t_2, t_3]$. Therefore, the function $x : [t_2, t_3] \to \mathbb{R}$ has an inverse, denoted by ξ.

Now multiplying (1.1) by $x'(t)$ and integrating over $[t_2, t_3]$, we obtain

$$\begin{aligned}\int_m^{A_4} -g(\xi(x))dx &= \int_{t_2}^{t_3} -g(x(t))x'(t)dt\\ &= \int_{t_2}^{t_3} (x''(t)x'(t) + f(x(t)(x'(t))^2 + e(t)x'(t))dt \le A_5\end{aligned}$$

for some $A_5 > 0$, where Lemma 3.4 is used. By (H1),

$$\int_m^{A_4} -g(\xi(x))dx \to +\infty \tag{3.11}$$

if $m \to 0^+$. Thus we know from (3.11) that $m > B_4$ for some constant $B_4 > 0$. □

Now we give the proof of Theorem 1.1. Consider the homotopy equation (3.1), we can get a priori estimates as in Lemmas 3.3, 3.4 and 3.5. That is, any positive T-periodic solution of (3.1) satisfies

$$B_4' < x(t) < B_2', \quad \|x'\|_\infty < B_3'$$

for some positive constants B_4', B_2', B_3'.

Let $C = \max\{B_4', B_2', B_3'\}$ and let the open bounded in X be

$$\Omega = \{x \in X : \frac{1}{C} < x(t) < C \text{ and } |x'(t)| < C \text{ for all } t \in [0, T]\}.$$

Obviously, Ω contains the constant function $x(t) = r_0$, where $r_0 > 0$ is the solution of

$$c_0x - \frac{1}{x} = 0.$$

Let X be a Banach space of functions such that $C^1([0, T]) \subseteq X \subseteq C([0, T])$, with continuous immersions. Set $X_* = \{x \in X : \min_t x(t) > 0\}$.

Define the space:

$$D(L) = \{x \in W^{2,1}(0,T) : x(0) = x(T), x'(0) = x'(T)\},$$

and the following two operators:

$$L : D(L) \subset X \to L^1(0,T), \quad (Lx)(t) = -x''(t),$$

and

$$N_\tau : X_* \to L^1(0,T),$$
$$(N_\tau x)(t) = \tau f(x(t))x'(t) + \tau(g(x(t)) - e(t)) + (1-\tau)(c_0 x - \frac{1}{x}).$$

Taking $\sigma \in \mathbb{R}$ not belonging to the spectrum of L, the T-periodic for equation (3.1) is thus equivalent to the operator equation

$$Lx = N_\tau x,$$

which is also can be translated to

$$x - (L - \sigma I)^{-1}(N_\tau - \sigma I)x = 0,$$

since $L - \sigma I$ is invertible. By the homotopy invariance of degree and Lemma 2.1,

$$\begin{aligned} \deg(I - (L-\sigma I)^{-1}(N_1 - \sigma I), \Omega, 0) &= \deg(I - (L-\sigma I)^{-1}(N_0 - \sigma I), \Omega, 0) \\ &= \deg(c_0 x - \frac{1}{x}, \Omega \cap \mathbb{R}, 0) = +1. \end{aligned}$$

Thus (3.1), with $\tau = 1$, has at least one solution in Ω, which is a positive T-periodic solution of (1.1). The proof of Theorem 1.1 is thus complete.

Remark 3.6. It is known (see (E3), (E5)) that

$$\bar{\lambda}_0(a) \le -1/T \int_0^T a(t)dt < 0,$$

Therefore (1.9) implies

$$\bar{\lambda}_0(a) < 0 < \underline{\lambda}_1(b).$$

Remark 3.7. Some classes of potentials q for $\underline{\lambda}_1(q) > 0$ to hold have been found recently in [4]. To describe these, let $K(\alpha, T)$ denote the best Sobolev constant in the inequality

$$C\|u\|_\alpha^2 \le \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0,T).$$

The explicit formula for $K(\alpha, T)$ is

$$K(\alpha,T) = \begin{cases} \frac{2\pi}{\alpha T^{1+2/\alpha}} \left(\frac{2}{\alpha+2}\right)^{1-2/\alpha} \left(\frac{\Gamma(1/\alpha)}{\Gamma(1/2+1/\alpha)}\right)^2, & \text{for } 1 \le \alpha < \infty, \\ 4/T, & \text{for } \alpha = \infty, \end{cases}$$

and $\Gamma(\cdot)$ is the Euler's Gamma function.

Now [4, Theorem 2.1] reads as follows: let $q \in L^p(0,T)$ for some $1 \le p \le \infty$, $q_+ = \max\{q, 0\}$ is the positive part of q and $p^* = \frac{p}{p-1}$ the conjugate of p. If

$$\|q_+\|_p < K(2p^*, T),$$

then

$$\underline{\lambda}_1(q) \ge \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{\|q_+\|}{K(2p^*, T)}\right) > 0.$$

Example 3.8. Let $f, \varphi, h \in C(\mathbb{R}, \mathbb{R})$, $\varphi(t) \geq 0, \gamma \geq 1$. For some $1 \leq p \leq \infty$, if

$$\|\varphi_+\|_p < K(2p^*, T),$$

then the singular equation

$$x'' + f(x)x' + \varphi(t)x - \frac{1}{x^\gamma} = h(t)$$

has at least one positive T-periodic solution.

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