Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 152, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TRAVELING WAVEFRONTS IN NONLOCAL DIFFUSIVE PREDATOR-PREY SYSTEM WITH HOLLING TYPE II FUNCTIONAL RESPONSE

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Abstract. This article concerns the existence of traveling wavefronts for a nonlocal diffusive predator-prey system with functional response of Holling type II. We first establish the existence principle for the system with a general functional response by using a fixed point theorem and upper-lower solution technique. We apply this result to a predator-prey model with Holling type II functional response. We deduce the existence of traveling wavefronts that connect the zero equilibrium and the positive equilibrium.

1. Introduction

We consider the reaction system based on the predator-prey interaction model with nonlocal diffusion

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= d_1[(J_1 * u)(x,t) - u(x,t)] + h_1(u(x,t)) - f(u(x,t))v(x,t),\\
\frac{\partial v(x,t)}{\partial t} &= d_2[(J_2 * v)(x,t) - v(x,t)] + h_2(v(x,t)) + \rho f(u(x,t))v(x,t),
\end{aligned}
\tag{1.1}
$$

where u and v are the densities of the prey and the predator, respectively; $d_1 > 0$, $d_2 > 0$ are the diffusion coefficients; $J_i(x)(i = 1, 2)$ are the kernel functions describing the spatial migration probability of individuals and is given by

$$
(J_1 * u)(x,t) = \int_{\mathbb{R}} J_1(x-y)u(y,t)dy, \quad (J_2 * v)(x,t) = \int_{\mathbb{R}} J_2(x-y)v(y,t)dy;
$$

$f(u)$ is the predator response function; $h_1(u)$ is the growth function of prey which is a positive function within the maximal carrying capacity of the prey, and $h_2(v)$ is the growth function of the predator; $\rho \in (0, 1)$ is the transmission coefficient. If the predator only depends on the prey given in (1.1), then $h_2(v)$ is a negative function. Otherwise, it may be positive.

2010 *Mathematics Subject Classification.* 45K05, 35K57, 92D25.
Key words and phrases. Traveling wavefront; nonlocal diffusive predator-prey system; functional response of Holling type II; upper-lower solutions.

Submitted June 24, 2013. Published June 10, 2015.

A classical mathematical model for describing the spatial-temporal pattern for prey-predator (abbreviated as P-P) species is

$$\begin{aligned} u_t &= D_1 u_{xx} + \alpha u(\beta - u) - f(u)v, \\ v_t &= D_2 v_{xx} - dv + \rho f(u)v, \end{aligned} \tag{1.2}$$

where in (1.2), the predator has only the u-species as its food resource. Assuming that the predator has resource other than the u-species, and obeys the logistic dynamical growth rule without the u-species, then the P-P model is

$$\begin{aligned} u_t &= D_1 u_{xx} + \alpha u(\beta - u) - f(u)v, \\ v_t &= D_2 v_{xx} + \gamma v(\delta - v) + \rho f(u)v. \end{aligned} \tag{1.3}$$

Both (1.2) and (1.3) used the Laplacian operator $\Delta := \frac{\partial^2}{\partial x^2}$ to model the diffusion of species, which is a local operator which suggests that the population at the location x can only be influenced by the variation of the population near the location x. In (1.1), at time t, the total individuals of u-species and v-species moving from the whole space into the location x are expressed as $\int_{-\infty}^{\infty} J_1(x-y)[u(t,y)-u(t,x)]dy$ and $\int_{-\infty}^{\infty} J_2(x-y)[v(t,y)-v(t,x)]dy$, respectively. Generally speaking, one may call (1.1) a system with nonlocal diffusion, and correspondingly, call (1.2) and (1.3) as systems with local diffusion. In recent years, models with nonlocal diffusion have attracted much attention, e.g., see [1, 2, 3, 4, 5, 15, 18, 19, 23, 25]. Similar to the study of traveling wave solutions of reaction-reaction systems with local diffusion (e.g., see [20] and [22]), the traveling wave solutions for the nonlocal reaction-diffusion systems are important in describing the phenomena in physical process, biological process, and other fields (e.g., see [2, 18, 19, 24]).

Pioneering works on the existence of traveling wave solutions connecting two steady states (point-to-point orbit) for diffusive predators-prey systems (1.2) are found in Dunbar [6, 7, 8] with $f(u) = uv$ ($D_1 = 0$ and $D_1 \neq 0$) and $f(u) = \frac{u}{1+Eu}$ ($D_1 = 0$) by using the shooting method, which is based on a variant of Wazewski's theorem [6, 7, 8] and LaSalle's invariant principle. Following Dunbar's ideas, (1.2) with $f(u) = \frac{u}{1+Eu}$ ($D_1 \neq 0$) and $f(u) = \frac{u^2}{1+Eu^2}$ ($D_1 = 0$) for point-to-point orbits were proved by Huang, Lu & Ruan [12] and Li & Wu [14], respectively. Also Yang & Yang [11] considered a model with a more general form. Lin, Weng & Wu [16, 21] considered a P-P model with Sigmoidal response function and simplified Dunbar's method by constructing a bounded set $\mathbb{W}$ to replace the unbounded Wazewski set in [6, 7, 8]. See also Huang [13] for further development.

Combining fixed point theory with the method of upper-lower solutions is effective in obtaining the existence of solutions for mixed quasi-monotonic reaction-diffusion systems. For P-P system (1.3), $\alpha u(\beta - u) - f(u)v$ is non-increasing on v and $\gamma v(\delta - v) + \rho f(u)v$ is nondecreasing on u. Thus, (1.3) is a mixed quasi-monotonic system (same for (1.2)). In this article, we obtain traveling wavefronts for the P-P system (1.1) only by using the upper-lower solution technique and fixed point theorem.

This article is organized as follows. In Section 2, some preliminaries are done and an existence theorem of traveling wavefronts connecting two steady states for (1.1) is derived briefly using fixed point theorem. As applications of the existence theorem, we need to find a pair of upper-lower solutions for the wave profile system with given functions h_1, h_2 and f. The main contribution of this article is the construction and verification of upper-lower solutions for the wave profile systems with logistic growth and Holling type II functional response. These works are done in Section 3.

2. An existence theorem of traveling wavefronts

We adopt the usual notation for the standard ordering in $\mathbb{R}^2$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^2$ and $\|\cdot\|$ denote the supermum norm in space $C(\mathbb{R},\mathbb{R}^2)$. Let $\mathbf{0}=(0,0)$.

We make the following assumptions on $h_1(u)$, $h_2(v)$ and $f(u)$.

- (H1) There exist two positive numbers u_0,v_0 such that $h_1(u_0)-f(u_0)v_0=0$, $h_2(v_0)+\rho f(u_0)v_0=0$, and $f(0)=h_1(0)=h_2(0)=0$;
- (H2) f, h_1 and h_2 are Lipschitz continuous functions on any compact interval;
- (H3) f is nondecreasing on $[0,+\infty)$.

Remark 2.1. (H1) guarantees that the system (1.1) has a trivial steady state $(0,0)$ and a positive steady state (u_0,v_0). On the other hand, (H1) and (H3) imply that $f(u)\geq 0$ for $u\geq 0$.

We first consider systems of the form

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=d_1\int_{\mathbb{R}}J_1(x-y)[u(y,t)-u(x,t)]dy+f_1(u(x,t),v(x,t)),\\ \frac{\partial v(x,t)}{\partial t}&=d_2\int_{\mathbb{R}}J_2(x-y)[v(y,t)-v(x,t)]dy+f_2(u(x,t),v(x,t)).\end{aligned}\tag{2.1}$$

Here we assume that J_i and f_i $(i=1,2)$, satisfy

- (J1) $\int_{\mathbb{R}}J_i(x)dx=1$, $J_i(x)\geq 0$, $J_i(x)=J_i(-x)$ for $x\in\mathbb{R}$; for $\nu\in(0,+\infty]$, $\int_{\mathbb{R}}J_i(x)e^{\nu x}dx<\infty$;
- (F1) $f_i\in C(\mathbb{R},\mathbb{R})$ $(i=1,2)$, $f_1(0,0)=f_2(0,0)=0$, $f_1(u_0,v_0)=f_2(u_0,v_0)=0$; there exist two positive constants $L_1>0$, $L_2>0$ such that
$$\begin{aligned}|f_1(\varphi_1,\psi_1)-f_1(\varphi_2,\psi_2)|&\leq L_1\|\Phi-\Psi\|,\\ |f_2(\varphi_1,\psi_1)-f_2(\varphi_2,\psi_2)|&\leq L_2\|\Phi-\Psi\|,\end{aligned}$$
for any $\Phi=(\varphi_1,\psi_1)$, $\Psi=(\varphi_2,\psi_2)\in C(\mathbb{R},\mathbb{R}^2)$ with $0\leq\Phi(t),\Psi(t)\leq\mathbf{K}$, and $\mathbf{K}=(K_1,K_2)\geq(u_0,v_0)$ is some vector in $\mathbb{R}^2$ which will be given later;
- (F2) $f_1(u,v)$ is non-increasing on v, and $f_2(u,v)$ is non-decreasing on u, where $(u,v)\in[\mathbf{0},\mathbf{K}]$.

A traveling wave solution of (2.1) is a solution with the form $(u(t,x),w(t,x))=(\varphi(x+ct),\psi(x+ct))$, where $(\varphi,\psi)\in C^1(\mathbb{R},\mathbf{K})$ is the wave profile which propagates at a constant velocity $c>0$. Substituting $(u(t,x),v(t,x))=(\varphi(x+ct),\psi(x+ct))$ into (2.1) and replacing $x+ct$ by t, then we obtain

$$\begin{aligned}c\varphi'(t)&=d_1(J_1*\varphi)(t)-d_1\varphi(t)+f_1(\varphi(t),\psi(t)),\\ c\psi'(t)&=d_2(J_2*\psi)(t)-d_2\psi(t)+f_2(\varphi(t),\psi(t)),\end{aligned}\tag{2.2}$$

where $(J_1*\varphi)(t)=\int_{\mathbb{R}}J_1(t-s)\varphi(s)ds$ and $(J_2*\psi)(t)=\int_{\mathbb{R}}J_2(t-s)\psi(s)ds$. Recalling the physical and biological motivation (see e.g., [2, 9, 17]), we also require that the traveling solution (φ,ψ) satisfies the asymptotic boundary conditions

$$\lim_{t\to-\infty}(\varphi(t),\psi(t))=(0,0),\quad \lim_{t\to+\infty}(\varphi(t),\psi(t))=(u_0,v_0).\tag{2.3}$$

We call a solution of (2.2) satisfying (2.3) as a traveling wavefront of (2.1).

Let $\beta > 0$ be large. For any $(\varphi, \psi) \in C_{[0,\mathbf{K}]} := C(\mathbb{R}, [\mathbf{0}, \mathbf{K}])$, define the operator $H = (H_1, H_2) : C_{[0,\mathbf{K}]} \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$\begin{aligned} H_1(\varphi,\psi)(t) &= \beta_1\varphi(t) + d_1(J_1 * \varphi)(t) - d_1\varphi(t) + f_1(\varphi(t), \psi(t)), \\ H_2(\varphi,\psi)(t) &= \beta_2\psi(t) + d_2(J_2 * \psi)(t) - d_2\psi(t) + f_2(\varphi(t), \psi(t)). \end{aligned} \tag{2.4}$$

Then (2.2) can be written as

$$\begin{aligned} c\varphi'(t) &= -\beta_1\varphi(t) + H_1(\varphi,\psi)(t), \\ c\psi'(t) &= -\beta_2\psi(t) + H_2(\varphi,\psi)(t). \end{aligned} \tag{2.5}$$

By (2.5), we further define $F = (F_1, F_2) : C_{[0,\mathbf{K}]} \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$\begin{aligned} F_1(\varphi,\psi)(t) &= \frac{1}{c} e^{-\frac{\beta_1}{c}t} \int_{-\infty}^{t} e^{\frac{\beta_1}{c}s} H_1(\varphi,\psi)(s)ds, \\ F_2(\varphi,\psi)(t) &= \frac{1}{c} e^{-\frac{\beta_2}{c}t} \int_{-\infty}^{t} e^{\frac{\beta_2}{c}s} H_2(\varphi,\psi)(s)ds. \end{aligned} \tag{2.6}$$

Thus, a fixed point of F is a solution of (2.5), which is a traveling wavefront of (2.1), and vice verse. Therefore, in what follows, we shall search for the fixed point of F connecting $(0, 0)$ and (u_0, v_0).

We introduce an exponential decay norm as follows. Let $\mu > 0$ such that $\mu < \min\{\frac{\beta_1}{c}, \frac{\beta_2}{c}\}$. Define

$$B_\mu(\mathbb{R}, \mathbb{R}^2) = \big\{u(t) \in C(\mathbb{R}, \mathbb{R}^2) : \sup_{t\in\mathbb{R}} |u(t)|e^{-\mu|t|} < \infty\big\},$$

and $|u(t)|_\mu = \sup_{t\in\mathbb{R}} |u(t)|e^{-\mu|t|}$ for $u \in B_\mu(\mathbb{R}, \mathbb{R}^2)$. Then $(B_\mu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\mu)$ is a Banach space.

We give a definition of upper and lower solutions of (2.2) as follows.

Definition 2.2. A pair of continuous functions $\overline{\Phi}(t) = (\overline{\varphi}(t), \overline{\psi}(t))$ and $\underline{\Phi}(t) = (\underline{\varphi}(t), \underline{\psi}(t)) \in C_{[0,\mathbf{K}]}$ is called a pair of upper-lower solutions of (2.2), respectively, if $\overline{\Phi}'(t)$ and $\underline{\Phi}'(t)$ exist for $t \in \mathbb{R} \setminus \mathbb{T}$, which are bounded and satisfy

$$\begin{aligned} c\overline{\varphi}'(t) &\geq d_1(J_1 * \overline{\varphi})(t) - d_1\overline{\varphi}(t) + f_1(\overline{\varphi}(t), \underline{\psi}(t)), \quad t \in \mathbb{R}\setminus\mathbb{T}, \\ c\overline{\psi}'(t) &\geq d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) + f_2(\overline{\varphi}(t), \overline{\psi}(t)), \quad t \in \mathbb{R}\setminus\mathbb{T}; \end{aligned}$$

$$\begin{aligned} c\underline{\varphi}'(t) &\leq d_1(J_1 * \underline{\varphi})(t) - d_1\underline{\varphi}(t) + f_1(\underline{\varphi}(t), \overline{\psi}(t)), \quad t \in \mathbb{R}\setminus\mathbb{T}, \\ c\underline{\psi}'(t) &\leq d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) + f_2(\underline{\varphi}(t), \underline{\psi}(t)), \quad t \in \mathbb{R}\setminus\mathbb{T}. \end{aligned}$$

Here $\mathbb{T}$=$(t_1, t_2, \dots, t_n)$ is a set of finite points with $t_1 < t_2 < \dots < t_n$.

In this article, we assume that a pair of upper-lower solutions $\overline{\Phi}(t) = (\overline{\varphi}(t), \overline{\psi}(t))$ and $\underline{\Phi}(t) = (\underline{\varphi}(t), \underline{\psi}(t))$ of (2.2) satisfies

- (P1) $(0, 0) \leq (\underline{\varphi}, \underline{\psi})(t) \leq (\overline{\varphi}, \overline{\psi})(t) \leq (K_1, K_2)$, $t \in \mathbb{R}$;
- (P2) $\lim_{t\to-\infty} (\overline{\varphi}, \overline{\psi})(t) = (0, 0)$, $\lim_{t\to+\infty} (\underline{\varphi}, \underline{\psi})(t) = \lim_{t\to+\infty} (\overline{\varphi}, \overline{\psi})(t) = (u_0, v_0)$.

Now we state an existence theorem for traveling wave solution of (2.1). A similar proof can be found in [10, 23, 25], and we omit its proof here.

Theorem 2.3. *Assume that* (J1), (F1), (F2) *hold. If* (2.2) *has a pair of upper-lower solutions* $\overline{\Psi} = (\overline{\varphi}, \overline{\psi})$ *and* $\underline{\Psi} = (\underline{\varphi}, \underline{\psi})$ *satisfying* (P1)–(P2). *Then* (2.2) *has a solution satisfying* (2.3). *That is,* (2.1) *has a traveling wavefront satisfying* (2.3).

Remark 2.4. Let $f_1(u,v) = h_1(u) - f(u)v$, $f_2(u,v) = h_2(v) + \rho f(u)v$, where h_i $(i = 1,2)$ and f satisfy (H1)–(H3). It is obvious that f_1 and f_2 satisfy (F1) and (F2). Therefore, the conclusion in Theorem 2.3 holds for system (1.1) if J_i $(i = 1,2)$ satisfy (J1).

3. Prey-predator system with Holling type II response

We consider the prey-predator system with Holling type II response,

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= d_1(J_1 * u)(x,t) - d_1 u(x,t) + \alpha u(x,t)(\beta - u(x,t)) - \frac{u(x,t)v(x,t)}{1+u(x,t)},\\ \frac{\partial v(x,t)}{\partial t} &= d_2(J_2 * v)(x,t) - d_2 v(x,t) + \gamma v(x,t)(\delta - v(x,t)) + \rho\frac{u(x,t)v(x,t)}{1+u(x,t)}.\end{aligned} \tag{3.1}$$

in which $x \in \mathbb{R}$, $t \geq 0$, and all parameters are positive, and J_i $(i = 1,2)$ satisfy (J1).

Obviously, (3.1) has three trivial equilibria as follows:

$$E_0 = (0,0), \quad E_1 = (\beta,0), \quad E_2 = (0,\delta).$$

Furthermore, we assume that (3.1) has a unique positive equilibrium $E_3 = (u_0, v_0)$ satisfying

$$\alpha\beta - \alpha u_0 - \frac{v_0}{1+u_0} = 0, \quad \gamma\delta - \gamma v_0 + \frac{\rho u_0}{1+u_0} = 0. \tag{3.2}$$

It is clear that $u_0 < \beta$, $v_0 > \delta$ and $\gamma v_0 > \frac{\rho u_0}{1+u_0}$.

Let us consider the algebraic system

$$\alpha\beta - \alpha u - \frac{v}{1+u} = 0, \quad \gamma\delta - \gamma v + \frac{\rho u}{1+u} = 0, \tag{3.3}$$

which is equivalent to

$$(\alpha\beta - \alpha u)(1+u) = v, \quad (\gamma\delta - \gamma v)(1+u) + \rho u = 0.$$

Substituting the first equation into the second equation, we obtain

$$[\gamma\delta(1+u) - \gamma(\alpha\beta - \alpha u)(1+u)^2] + \rho u = 0.$$

That is,

$$Q(u) := \alpha u^3 + (2\alpha - \alpha\beta)u^2 - (2\alpha\beta - \alpha - \delta - \frac{\rho}{\gamma})u + (\delta - \alpha\beta) = 0.$$

Note $Q(0) = \delta - \alpha\beta$, $Q(-\infty) = -\infty$ and $Q(\infty) = \infty$, and $Q(u) = 0$ is a cubic algebraic equation. If $\delta - \alpha\beta < 0$, then $Q(u) = 0$ has a positive real root $u = u_0$. Since

$$Q(\beta) = \alpha\beta^3 + 2\alpha\beta^2 - \alpha\beta^3 - 2\alpha\beta^2 + \alpha\beta + \delta\beta + \frac{\rho}{\gamma}\beta + \delta - \alpha\beta = \delta\beta + \frac{\rho}{\gamma}\beta + \delta > 0,$$

we have $u_0 < \beta$. Let $v_0 = (\alpha\beta - \alpha u_0)(1+u_0)$, then $v_0 > 0$ and (u_0, v_0) is a positive solution of (3.3). Therefore, the assumption (3.2) is feasible.

Assume $c > 0$. Let $u(x,t) = \varphi(x+ct)$, $v(x,t) = \psi(x+ct)$, and denote the traveling wave coordinate $x + ct$ still by t, then the corresponding wave profile system is

$$\begin{aligned}c\varphi'(t) &= d_1(J_1 * \varphi)(t) - d_1\varphi(t)\alpha\varphi(t)(\beta - \varphi(t)) - \frac{\varphi(t)\psi(t)}{1+\varphi(t)},\\ c\psi'(t) &= d_2(J_2 * \psi)(t) - d_2\psi(t) + \gamma\psi(t)(\delta - \psi(t)) + \rho\frac{\varphi(t)\psi(t)}{1+\varphi(t)}.\end{aligned} \tag{3.4}$$

We first do some preliminaries. For $\lambda \in \mathbb{R}$ and $c > 0$, define

$$\Delta_1(\lambda, c) = d_1 \int_{\mathbb{R}} J_1(y)(e^{\lambda y} - 1)dy - c\lambda + \alpha\beta,$$
$$\Delta_2(\lambda, c) = d_2 \int_{\mathbb{R}} J_2(y)(e^{\lambda y} - 1)dy - c\lambda + \delta_2.$$

Here δ_2 is some constant which will be given in the following subsections. By (J1) and some direct calculations, we have the following lemma.

Lemma 3.1. *The following conclusions hold.*

(i) There is a $c_1^* > 0$, such that for any given $c > c_1^*$, $\Delta_1(\lambda, c)$ has two distinct real roots $\lambda_1(c)$ and $\lambda_2(c)$ satisfying $0 < \lambda_1(c) < \lambda_2(c)$ and

$$\Delta_1(\lambda, c) \begin{cases} > 0, & \text{either } 0 < \lambda < \lambda_1(c) \text{ or } \lambda > \lambda_2(c), \\ < 0, & \lambda_1(c) < \lambda < \lambda_2(c). \end{cases}$$

(ii) There is a $c_2^* \geq 0$, such that for any given $c > c_2^*$, $\Delta_2(\lambda, c)$ has two distinct real roots $\lambda_3(c)$ and $\lambda_4(c)(> 0)$ satisfying $\lambda_3(c) < \lambda_4(c)$ and

$$\Delta_2(\lambda, c) \begin{cases} > 0, & \text{either } \lambda < \lambda_3(c) \text{ or } \lambda > \lambda_4(c), \\ < 0, & \lambda_3(c) < \lambda < \lambda_4(c). \end{cases}$$

Remark 3.2. If $\delta_2 > 0$, then $c_2^* > 0$ and $\lambda_3(c) > 0$. If $\delta_2 = 0$, then $c_2^* = 0$ and $\lambda_3(c) = 0$. If $\delta_2 < 0$, then $c_2^* = 0$ and $\lambda_3(c) < 0$.

3.1. **Traveling waves connecting E_0 and E_3.** In this subsection, we are interested in the solution of (3.4) with asymptotic boundary conditions

$$\lim_{t \to -\infty} (\varphi(t), \psi(t)) = (0, 0), \quad \lim_{t \to +\infty} (\varphi(t), \psi(t)) = (u_0, v_0). \tag{3.5}$$

Here, we choose $K_1 = \beta$, $K_2 = \delta + \frac{\rho}{\gamma}\beta$, then we have $K_1 > u_0$ and $K_2 > v_0$.

To construct lower-upper solutions of (3.4), we need the following lemma.

Lemma 3.3. *Suppose that*

$$\alpha u_0 \geq \frac{(3 + 2\sqrt{2})v_0}{1 + u_0}, \quad \gamma v_0 \geq \frac{4(\sqrt{2} - 1)\rho u_0}{1 + u_0} \tag{3.6}$$

hold, then there exist $\varepsilon_1 \in (0, (\sqrt{2} - 1)u_0]$ and $\varepsilon_2 \in (0, \frac{v_0}{2}]$ such that

$$\begin{gathered} -\alpha\varepsilon_1^2 + (2\sqrt{2} - 2)\alpha u_0 \varepsilon_1 + \frac{u_0 v_0}{1 + u_0} - \frac{2(u_0 - \varepsilon_1)v_0}{1 + (u_0 - \varepsilon_1)} > \varepsilon_0, \\ -\gamma\varepsilon_2^2 + \gamma v_0 \varepsilon_2 - \frac{\rho v_0 \varepsilon_1}{(1 + u_0)(1 + u_0 - \varepsilon_1)} > \varepsilon_0, \end{gathered} \tag{3.7}$$

where $\varepsilon_0 > 0$ is some constant. Particularly, one can choose $\varepsilon_1 = (\sqrt{2} - 1)u_0$ and $\varepsilon_2 = v_0/2$.

Proof. The proof of the first inequality is similar to that of [10, Lemma 4.2]. We only prove the second inequality. Let

$$g_1(\varepsilon_2) = -\gamma\varepsilon_2^2 + \gamma v_0 \varepsilon_2, \quad g_2(\varepsilon_1) = \frac{\rho v_0 \varepsilon_1}{(1 + u_0)(1 + u_0 - \varepsilon_1)}.$$

Note that $g_1(0) = 0$, $\max g_1(\varepsilon_2) = g_1(\frac{v_0}{2}) = \frac{1}{4}\gamma v_0^2$. If $\varepsilon_1 \in (0, (\sqrt{2} - 1)u_0]$, then

$$g_2(\varepsilon_1) = \frac{\rho v_0 \varepsilon_1}{(1 + u_0)(1 + u_0 - \varepsilon_1)} \leq \frac{(\sqrt{2} - 1)\rho v_0 u_0}{(1 + u_0)[1 + (2 - \sqrt{2})u_0]} < \frac{(\sqrt{2} - 1)\rho v_0 u_0}{1 + u_0}.$$

If $\gamma v_0 \geq \frac{4(\sqrt{2}-1)\rho u_0}{1+u_0}$, then for any $\varepsilon_1 \in (0, (\sqrt{2}-1)u_0]$, there exist $\varepsilon_2^* \in (0, v_0/2]$ such that $g_1(\varepsilon_2) > g_2(\varepsilon_1)$ for $\varepsilon_2^* < \varepsilon_2 \leq v_0/2$. The proof is complete. □

Remark 3.4. If we assume that

$$\alpha\beta > \frac{(4+2\sqrt{2})v_0}{1+u_0}, \quad \gamma\delta > \frac{(4\sqrt{2}-5)\rho u_0}{1+u_0}, \tag{3.8}$$

then from (3.8) we derive that

$$\begin{aligned}
\alpha u_0 &= \alpha\beta - \frac{v_0}{1+u_0} > \frac{(3+2\sqrt{2})v_0}{1+u_0}, \\
\alpha u_0 &= \alpha\beta - \frac{v_0}{1+u_0} > \alpha\beta - \frac{\alpha\beta}{4+2\sqrt{2}} = \frac{(2+\sqrt{2})\alpha\beta}{4}, \\
\gamma v_0 &= \gamma\delta + \frac{\rho u_0}{1+u_0} \geq \frac{4(\sqrt{2}-1)\rho u_0}{1+u_0}, \\
\gamma\delta &= \gamma v_0 - \frac{\rho u_0}{1+u_0} \geq \gamma v_0 - \frac{\gamma v_0}{4(\sqrt{2}-1)} > \frac{1}{3}\gamma v_0.
\end{aligned} \tag{3.9}$$

That is, (3.6) holds.

In this subsection, let $\delta_2 = \gamma\delta$, then we have from Remark 3.2 that $c_2^* > 0$ and $\lambda_3(c) > 0$. We denote $\lambda_i(c)$ by λ_i in what follows, where $i = 1, 2, 3, 4$. Let $c > c^* := \max\{c_1^*, c_2^*\}$, $q > 1$ and

$$\eta \in \big(1, \min\{\frac{\lambda_2}{\lambda_1}, \frac{\lambda_4}{\lambda_3}, \frac{\lambda_1+\lambda_3}{\lambda_1}, \frac{\lambda_1+\lambda_3}{\lambda_3}, 2\}\big). \tag{3.10}$$

Define two continuous functions

$$L_1(t) := e^{\lambda_1 t} - q e^{\eta\lambda_1 t}, \quad L_2(t) := e^{\lambda_3 t} - q e^{\eta\lambda_3 t}.$$

Let $L_i(t) = 0, i = 1, 2$, we obtain

$$\bar{t}_2 = -\frac{1}{(\eta-1)\lambda_1}\ln q < 0, \quad \bar{t}_4 = -\frac{1}{(\eta-1)\lambda_3}\ln q < 0.$$

Let $\varepsilon_1, \varepsilon_2$ be defined as in Lemma 3.3. For any given $q > 1$, we can choose small $\lambda > 0$ such that $t_2 < \bar{t}_2, t_4 < \bar{t}_4$ and

$$u_0 - \varepsilon_1 e^{-\lambda t_2} = L_1(t_2), \quad v_0 - \varepsilon_2 e^{-\lambda t_4} = L_2(t_4).$$

Using the above constants, define the continuous vector functions:

$$\overline{\varphi}(t) = \begin{cases} e^{\lambda_1 t}, & t \leq t_1, \\ \min\{K_1, u_0 + u_0 e^{-\lambda t}\}, & t \geq t_1, \end{cases} \quad \underline{\varphi}(t) = \begin{cases} e^{\lambda_1 t} - q e^{\eta\lambda_1 t}, & t \leq t_2, \\ u_0 - \varepsilon_1 e^{-\lambda t}, & t \geq t_2, \end{cases}$$

$$\overline{\psi}(t) = \begin{cases} e^{\lambda_3 t} + q e^{\eta\lambda_3 t}, & t \leq t_3, \\ \min\{K_2, v_0 + v_0 e^{-\lambda t}\}, & t \geq t_3, \end{cases} \quad \underline{\psi}(t) = \begin{cases} e^{\lambda_3 t} - q e^{\eta\lambda_3 t}, & t \leq t_4, \\ v_0 - \varepsilon_2 e^{-\lambda t}, & t \geq t_4, \end{cases}$$

where $\lambda > 0$ is small and will be determined later. We can see $\overline{\varphi}(t), \overline{\psi}(t), \underline{\varphi}(t), \underline{\psi}(t)$ satisfy (P1)–(P2) in Section 2. Moreover, if $q > 1$ is large enough, then it is obvious that

$$t_3 < 0 \quad \text{and} \quad t_1 \geq \max\{t_2, t_3, t_4\}.$$

Lemma 3.5. *Assume* (3.8) *holds. If $q > 1$ is large and $\lambda > 0$ is small, then* $(\overline{\varphi}(t), \overline{\psi}(t))$ *and* $(\underline{\varphi}(t), \underline{\psi}(t))$ *is a pair of upper solution and lower solution of* (3.4).

Proof. Firstly, we note the following facts: for $t \in \mathbb{R}$,

$$\begin{aligned} &\overline{\varphi}(t) \le e^{\lambda_1 t}, \quad \overline{\varphi}(t) \le u_0 + u_0 e^{-\lambda t}, \quad \overline{\varphi}(t) \le K_1, \\ &\overline{\psi}(t) \le e^{\lambda_3 t} + q e^{\eta\lambda_3 t} \quad \overline{\psi}(t) \le v_0 + v_0 e^{-\lambda t}, \quad \overline{\psi}(t) \le K_2, \\ &\underline{\varphi}(t) \ge e^{\lambda_1 t} - q e^{\eta\lambda_1 t}, \quad \underline{\varphi}(t) \ge u_0 - \varepsilon_1 e^{-\lambda t}, \quad \underline{\varphi}(t) \ge 0, \\ &\underline{\psi}(t) \ge e^{\lambda_3 t} - q e^{\eta\lambda_3 t}, \quad \underline{\psi}(t) \ge v_0 - \varepsilon_2 e^{-\lambda t}, \quad \underline{\psi}(t) \ge 0. \end{aligned} \tag{3.11}$$

These inequalities will be used in the following arguments without extra explanations.

Now we consider $\overline{\varphi}(t)$. If $t < t_1$, $\overline{\varphi}(t) = e^{\lambda_1 t}$ and $\underline{\psi}(t) \ge 0$, then

$$\begin{aligned} &d_1(J_1 * \overline{\varphi})(t) - d_1\overline{\varphi}(t) - c\overline{\varphi'}(t) + \alpha\overline{\varphi}(t)(\beta - \overline{\varphi}(t)) - \frac{\overline{\varphi}(t)\underline{\psi}(t)}{1+\overline{\varphi}(t)} \\ &\le e^{\lambda_1 t}\Delta_1(\lambda_1, c) = 0. \end{aligned}$$

If $t \ge t_1$, we have $\overline{\varphi}(t) = K_1 = \beta$, then the result is clear. Otherwise, $\overline{\varphi}(t) = u_0 + u_0 e^{-\lambda t}$, $\underline{\psi}(t) \ge v_0 - \varepsilon_2 e^{-\lambda t}$ implies that

$$\begin{aligned} &d_1(J_1 * \overline{\varphi})(t) - d_1\overline{\varphi}(t) - c\overline{\varphi'}(t) + \alpha\overline{\varphi}(t)(\beta - \overline{\varphi}(t)) - \frac{\overline{\varphi}(t)\underline{\psi}(t)}{1+\overline{\varphi}(t)} \\ &\le d_1\int_R J_1(y-t)(u_0 + u_0 e^{-\lambda y})dy - d_1(u_0 + u_0 e^{-\lambda t}) + c\lambda u_0 e^{-\lambda t} \\ &\quad + \alpha(u_0 + u_0 e^{-\lambda t})[\beta - (u_0 + u_0 e^{-\lambda t})] - \frac{(u_0 + u_0 e^{-\lambda t})\underline{\psi}(t)}{1 + u_0 + u_0 e^{-\lambda t}} \\ &= u_0 e^{-\lambda t}\Delta_1(-\lambda, c) + \alpha\beta u_0 - \alpha u_0^2 - 2\alpha u_0^2 e^{-\lambda t} - \alpha u_0^2 e^{-2\lambda t} - \frac{(u_0 + u_0 e^{-\lambda t})\underline{\psi}(t)}{1 + u_0 + u_0 e^{-\lambda t}} \\ &= u_0 e^{-\lambda t}[\Delta_1(-\lambda, c) - 2(2-\sqrt{2})\alpha u_0] \\ &\quad - u_0[(2\sqrt{2}-2)\alpha u_0 e^{-\lambda t} + \alpha u_0 e^{-2\lambda t} + \frac{(1+e^{-\lambda t})\underline{\psi}(t)}{1 + u_0 + u_0 e^{-\lambda t}} - \frac{v_0}{1+u_0}]. \end{aligned}$$

Note from the second inequality in (3.9) that $\Delta_1(0, c) - 2(2-\sqrt{2})\alpha u_0 = \alpha\beta - 2(2-\sqrt{2})\alpha u_0 < 0$, and thus $\Delta_1(-\lambda, c) - 2(2-\sqrt{2})\alpha u_0 < 0$ if $\lambda > 0$ is small. Then there exists a constant λ_1^* such that $\Delta_1(-\lambda, c) - 2(2-\sqrt{2})\alpha u_0 < 0$ for any $\lambda \in (0, \lambda_1^*)$.

On the other hand, let

$$\begin{aligned} I_1(\lambda, t) &:= (2\sqrt{2}-2)\alpha u_0 e^{-\lambda t} + \alpha u_0 e^{-2\lambda t} + \frac{(1+e^{-\lambda t})\underline{\psi}(t)}{1 + u_0 + u_0 e^{-\lambda t}} - \frac{v_0}{1+u_0} \\ &\ge (2\sqrt{2}-2)\alpha u_0 e^{-\lambda t} + \alpha u_0 e^{-2\lambda t} + \frac{(1+e^{-\lambda t})(v_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 + u_0 e^{-\lambda t}} - \frac{v_0}{1+u_0}. \end{aligned}$$

Let $t \ge t_1$ and $x := e^{-\lambda t} \in (0, \infty)$. Then

$$\begin{aligned} I_1(\lambda, t) &\ge (2\sqrt{2}-2)\alpha u_0 x + \alpha u_0 x^2 + \frac{(1+x)(v_0 - \varepsilon_2 x)}{1 + u_0 + u_0 x} - \frac{v_0}{1+u_0} \\ &= \Big((2\sqrt{2}-2)\alpha u_0 x(1 + u_0 + u_0 x) + \alpha u_0 x^2(1 + u_0 + u_0 x) \\ &\quad + (1+x)(v_0 - \varepsilon_2 x) - \frac{v_0}{1+u_0}(1 + u_0 + u_0 x)\Big)/\big(1 + u_0 + u_0 x\big) \\ &= \frac{x\hat{g}(x)}{1 + u_0 + u_0 x}. \end{aligned}$$

Here

$$\hat{g}(x) = \hat{a}x^2 + \hat{b}x + \hat{c}, \quad \hat{a} := \alpha u_0,$$
$$\hat{b} := \alpha u_0(1+u_0) - \varepsilon_2 + (2\sqrt{2}-2)\alpha u_0^2,$$
$$\hat{c} := \frac{v_0}{1+u_0} - \varepsilon_2 + (2\sqrt{2}-2)\alpha u_0(1+u_0).$$

Let $\varepsilon_2 = v_0/2$. Note that we have from (3.8),

$$\begin{aligned}
\hat{b} &= \alpha u_0(1+u_0) - \varepsilon_2 + (2\sqrt{2}-2)\alpha u_0^2 \\
&\geq \alpha u_0(1+u_0) - \frac{v_0}{2} \\
&= [\alpha\beta - \frac{v_0}{1+u_0}](1+u_0) - \frac{v_0}{2} \\
&= \alpha\beta(1+u_0) - \frac{3v_0}{2} \\
&> (4+\sqrt{2})v_0 - \frac{3v_0}{2} \geq 0, \\
\hat{g}(0) &= \frac{v_0}{1+u_0} - \varepsilon_2 + (2\sqrt{2}-2)\alpha u_0(1+u_0), \\
&\geq \frac{v_0}{1+u_0} - \frac{v_0}{2} + (2\sqrt{2}-2)(3+2\sqrt{2})v_0 > 0, \\
\hat{g}'(x) &= 2\hat{a}x + \hat{b} > 0 \text{ for } x > 0.
\end{aligned}$$

Therefore, $\hat{f}(x) := x\hat{g}(x)$ is increasing on $x \in (0,\infty)$. Since $\hat{f}(0) = 0$, we have $\hat{f}(x) > 0$ for $x \in (0,\infty)$. That is $I_1(\lambda, t) > 0$ for $t \geq t_1$. Thus $\overline{\varphi}(t)$ satisfies the definition of upper solution.

We now consider $\overline{\psi}(t)$. If $t \leq t_3$, then $\overline{\psi}(t) = e^{\lambda_3 t} + qe^{\eta\lambda_3 t}$, $\overline{\varphi}(t) \leq e^{\lambda_1 t}$. Noting $\eta\lambda_3 \leq \lambda_1 + \lambda_3$ and $t_3 < 0$, we have

$$\begin{aligned}
&d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi'}(t) + \gamma\overline{\psi}(t)(\delta - \overline{\psi}(t)) + \frac{\rho\overline{\varphi}(t)\overline{\psi}(t)}{1+\overline{\varphi}(t)} \\
&\leq d_2 \int_R J_2(y-t)(e^{\lambda_3 y} + qe^{\eta\lambda_3 y})dy - d_2(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}) - c(\lambda_3 e^{\lambda_3 t} + q\eta\lambda_3 e^{\eta\lambda_3 t}) \\
&\quad + \gamma(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})(\delta - e^{\lambda_3 t} - qe^{\eta\lambda_3 t}) + \frac{\rho\overline{\varphi}(t)(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1+\overline{\varphi}(t)} \\
&= e^{\lambda_3 t}\Delta_2(\lambda_3, c) + qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) - \gamma(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})^2 + \frac{\rho\overline{\varphi}(t)(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1+\overline{\varphi}(t)} \\
&\leq qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) - \gamma(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})^2 + \frac{\rho e^{\lambda_1 t}(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1+e^{\lambda_1 t}} \\
&\leq qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) + \rho e^{(\lambda_1+\lambda_3)t} + q\rho e^{\lambda_1 t + \eta\lambda_3 t} \\
&\leq e^{\eta\lambda_3 t}(q\Delta_2(\eta\lambda_3, c) + \rho + q\rho e^{\lambda_1 t}).
\end{aligned} \tag{3.12}$$

Note $\Delta_2(\eta\lambda_3, c) < 0$. Let $q > 1$ be large enough, then $-t_3 > 0$ is also large enough such that

$$q\Delta_2(\eta\lambda_3, c) + \rho + q\rho e^{\lambda_1 t_3} = q[\Delta_2(\eta\lambda_3, c) + \rho e^{\lambda_1 t_3}] + \rho < 0,$$

which leads to $q\Delta_2(\eta\lambda_3, c) + \rho + q\rho e^{\lambda_1 t} < 0$ for $t \leq t_3$.

If $t \geq t_3$ and $\overline{\psi}(t) = K_2$, then from the definition of K_1, K_2 we have

$$\begin{aligned} &d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + \gamma\overline{\psi}(t)(\delta - \overline{\psi}(t)) + \frac{\rho\overline{\varphi}(t)\overline{\psi}(t)}{1+\overline{\varphi}(t)} \\ &\leq \gamma K_2(\delta - K_2) + \rho K_1 K_2 = 0. \end{aligned} \tag{3.13}$$

Otherwise, $\overline{\psi}(t) = v_0 + v_0 e^{-\lambda t}$, $\overline{\varphi}(t) \leq u_0 + u_0 e^{-\lambda t}$ implies that

$$\begin{aligned} &d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + \gamma\overline{\psi}(t)(\delta - \overline{\psi}(t)) + \frac{\rho\overline{\varphi}(t)\overline{\psi}(t)}{1+\overline{\varphi}(t)} \\ &\leq d_2 \int_R J_2(y-t)(v_0 + v_0 e^{-\lambda y})dy - d_2(v_0 + v_0 e^{-\lambda t}) + c\lambda v_0 e^{-\lambda t} \\ &\quad + \gamma(v_0 + v_0 e^{-\lambda t})(\delta - v_0 - v_0 e^{-\lambda t}) + \frac{\rho\overline{\varphi}(t)(v_0 + v_0 e^{-\lambda t})}{1+\overline{\varphi}(t)} \\ &\leq v_0 e^{-\lambda t}\Delta_2(-\lambda, c) + \gamma\delta v_0 - \gamma(v_0 + v_0 e^{-\lambda t})^2 + \frac{\rho(u_0 + u_0 e^{-\lambda t})(v_0 + v_0 e^{-\lambda t})}{1 + u_0 + u_0 e^{-\lambda t}} \\ &= v_0 e^{-\lambda t}\Delta_2(-\lambda, c) - 2\gamma v_0^2 e^{-\lambda t} - \gamma v_0^2 e^{-2\lambda t} - \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho u_0 v_0 (1+e^{-\lambda t})^2}{1 + u_0 + u_0 e^{-\lambda t}} \\ &= v_0 e^{-\lambda t}\big[\Delta_2(-\lambda, c) - \gamma v_0 + \frac{\frac{1}{2}\rho u_0}{1 + u_0 + u_0 e^{-\lambda t}}\big] \\ &\quad - v_0\big[\gamma v_0 e^{-\lambda t} + \gamma v_0 e^{-2\lambda t} + \frac{\rho u_0}{1+u_0} - \frac{\rho u_0(1 + \frac{3}{2}e^{-\lambda t} + e^{-2\lambda t})}{1 + u_0 + u_0 e^{-\lambda t}}\big]. \end{aligned}$$

Note that $\Delta_2(0,c) - \gamma v_0 + \frac{\frac{1}{2}\rho u_0}{1+u_0+u_0 e^{-\lambda t}} \leq \gamma\delta - \gamma v_0 + \frac{\frac{1}{2}\rho u_0}{1+u_0} < 0$. Thus there exists a constant λ_2^* such that $\Delta_2(-\lambda, c) - \gamma v_0 + \frac{\frac{1}{2}\rho u_0}{1+u_0+u_0 e^{-\lambda t}} < 0$ for any $\lambda \in (0, \lambda_2^*)$.

On the other hand, from (3.9) we have

$$\begin{aligned} I_2(\lambda, t) :&= \gamma v_0 e^{-\lambda t} + \gamma v_0 e^{-2\lambda t} + \frac{\rho u_0}{1+u_0} - \frac{\rho u_0(1 + \frac{3}{2}e^{-\lambda t} + e^{-2\lambda t})}{1 + u_0 + u_0 e^{-\lambda t}} \\ &\geq \gamma v_0 e^{-\lambda t} + \gamma v_0 e^{-2\lambda t} + \frac{\rho u_0}{1+u_0} - \frac{\rho u_0(1 + \frac{3}{2}e^{-\lambda t} + e^{-2\lambda t})}{1 + u_0} \\ &= \gamma v_0 e^{-\lambda t} + \gamma v_0 e^{-\lambda t} - \frac{\rho u_0}{1+u_0}(\frac{3}{2}e^{-\lambda t} + e^{-2\lambda t}) \\ &= e^{-\lambda t}\big[\gamma v_0 - \frac{3}{2}\frac{\rho u_0}{1+u_0} + (\gamma v_0 - \frac{\rho u_0}{1+u_0})e^{-\lambda t}\big] > 0, \end{aligned}$$

uniformly for $t > t_3$. Therefore, $\overline{\psi}(t)$ satisfies the definition of upper solution.

Now we consider $\underline{\varphi}(t)$. If $t \le t_2$, then $\underline{\varphi}(t) = e^{\lambda_1 t} - qe^{\eta\lambda_1 t}$, $0 \le \overline{\psi}(t) \le e^{\lambda_3 t} + qe^{\eta\lambda_3 t}$. Thus we have

$$\begin{aligned}
&d_1(J_1 * \underline{\varphi})(t) - d_1\underline{\varphi}(t) - c\underline{\varphi}'(t) + \alpha\underline{\varphi}(t)(\beta - \underline{\varphi}(t)) - \frac{\underline{\varphi}(t)\overline{\psi}(t)}{1+\underline{\varphi}(t)} \\
&\ge d_1 \int_R J_1(y-t)(e^{\lambda_1 y} - qe^{\eta\lambda_1 y})dy - d_1(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}) \\
&\quad - c(\lambda_1 e^{\lambda_1 t} - q\eta\lambda_1 e^{\eta\lambda_1 t}) + \alpha(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})[\beta - e^{\lambda_1 t} + qe^{\eta\lambda_1 t}] \\
&\quad - \frac{(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})\overline{\psi}(t)}{1+e^{\lambda_1 t} - qe^{\eta\lambda_1 t}} \\
&= e^{\lambda_1 t}\Delta_1(\lambda_1, c) - qe^{\eta\lambda_1 t}\Delta_1(\eta\lambda_1, c) - \alpha(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})^2 \\
&\quad - \frac{(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})\overline{\psi}(t)}{1+e^{\lambda_1 t} - qe^{\eta\lambda_1 t}} \\
&\ge -qe^{\eta\lambda_1 t}\Delta_1(\eta\lambda_1, c) - \alpha(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})^2 - \frac{(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1+e^{\lambda_1 t} - qe^{\eta\lambda_1 t}} \\
&\ge -qe^{\eta\lambda_1 t}\Delta_1(\eta\lambda_1, c) - \alpha e^{2\lambda_1 t} - e^{\lambda_1 t}(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}) \\
&\ge -e^{\eta\lambda_1 t}[q\Delta_1(\eta\lambda_1, c) + \alpha + 1 + qe^{(\lambda_1 + \eta\lambda_3 - \eta\lambda_1)t}].
\end{aligned} \tag{3.14}$$

Note that $\Delta_1(\eta\lambda_1, c) < 0$ by Lemma 3.1, and from (3.10) that $\lambda_1 + \eta\lambda_3 - \eta\lambda_1 > 0$. Let $q > 1$ be large enough, then $-t_2 > 0$ is also enough such that

$$\begin{aligned}
&q\Delta_1(\eta\lambda_1, c) + \alpha + 1 + qe^{(\lambda_1 + \eta\lambda_3 - \eta\lambda_1)t_2} \\
&= q[\Delta_1(\eta\lambda_1, c) + e^{(\lambda_1 + \eta\lambda_3 - \eta\lambda_1)t_2}] + (\alpha + 1) < 0,
\end{aligned}$$

which leads to

$$-e^{\eta\lambda_1 t}[q\Delta_1(\eta\lambda_1, c) + \alpha + 1 + qe^{(\lambda_1 + \eta\lambda_3 - \eta\lambda_1)t}] \ge 0 \quad \text{for } t \le t_2.$$

If $t \ge t_2$, $\underline{\varphi}(t) = u_0 - \varepsilon_1 e^{-\lambda t}$, $0 \le \overline{\psi}(t) \le v_0 + v_0 e^{-\lambda t}$, then

$$\begin{aligned}
&d_1(J_1 * \underline{\varphi})(t) - d_1\underline{\varphi}(t) - c\underline{\varphi}'(t) + \alpha\underline{\varphi}(t)(\beta - \underline{\varphi}(t)) - \frac{\underline{\varphi}(t)\overline{\psi}(t)}{1+\underline{\varphi}(t)} \\
&\ge d_1 \int_R J_1(y-t)(u_0 - \varepsilon_1 e^{-\lambda y})dy - d_1(u_0 - \varepsilon_1 e^{-\lambda t}) - c\lambda\varepsilon_1 e^{-\lambda t} \\
&\quad + \alpha(u_0 - \varepsilon_1 e^{-\lambda t})[\beta - (u_0 - \varepsilon_1 e^{-\lambda t})] - \frac{(u_0 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1+u_0 - \varepsilon_1 e^{-\lambda t}} \\
&= -\varepsilon_1 e^{-\lambda t}\Delta_1(-\lambda, c) + \alpha\beta u_0 - \alpha(u_0 - \varepsilon_1 e^{-\lambda t})^2 - \frac{(u_0 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1+u_0 - \varepsilon_1 e^{-\lambda t}} \\
&= \varepsilon_1 e^{-\lambda t}[-\Delta_1(-\lambda, c) + (4 - 2\sqrt{2})\alpha u_0] + (2\sqrt{2} - 2)\alpha u_0 \varepsilon_1 e^{-\lambda t} - \alpha\varepsilon_1^2 e^{-2\lambda t} \\
&\quad + \frac{u_0 v_0}{1+u_0} - \frac{(u_0 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1+u_0 - \varepsilon_1 e^{-\lambda t}}.
\end{aligned}$$

Note that $-\Delta_1(0, c) + (4 - 2\sqrt{2})\alpha u_0 = -\alpha\beta + (4 - 2\sqrt{2})\alpha u_0 > 0$ by the second inequality in (3.9). We can choose $\lambda_3^* > 0$ such that $-\Delta_1(-\lambda, c) + (4 - 2\sqrt{2})\alpha u_0 > 0$ for $\lambda \in (0, \lambda_3^*)$.

Let

$$
\begin{aligned}
&I_3(\lambda,t)\\
&:= (2\sqrt{2}-2)\alpha u_0\varepsilon_1 e^{-\lambda t} - \alpha\varepsilon_1^2 e^{-2\lambda t} + \frac{u_0v_0}{1+u_0} - \frac{(u_0-\varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1+u_0-\varepsilon_1 e^{-\lambda t}}\\
&\geq (2\sqrt{2}-2)\alpha u_0\varepsilon_1 e^{-\lambda t} - \alpha\varepsilon_1^2 e^{-2\lambda t} + \frac{u_0v_0}{1+u_0} - \frac{(u_0-\varepsilon_1 e^{-\lambda t})(v_0+v_0e^{-\lambda t})}{1+u_0-\varepsilon_1 e^{-\lambda t}}.
\end{aligned}
$$

By Lemma 3.3, we have

$$
I_3(\lambda,0) \geq (2\sqrt{2}-2)\alpha u_0\varepsilon_1 - \alpha\varepsilon_1^2 + \frac{u_0v_0}{1+u_0} - \frac{2v_0(u_0-\varepsilon_1)}{1+u_0-\varepsilon_1} > \varepsilon_0 > 0.
$$

Choose $\xi_1 > 1$ satisfying

$$
(2\sqrt{2}-2)\alpha u_0\varepsilon_1\xi - \alpha(\varepsilon_1\xi)^2 + \frac{u_0v_0}{1+u_0} - \frac{(v_0+v_0\xi)(u_0-\varepsilon_1\xi)}{1+u_0-\varepsilon_1\xi} > \frac{\varepsilon_0}{2} > 0 \tag{3.15}
$$

for $\xi \in [1,\xi_1]$. Let $\lambda_3^* > 0$ be small enough, such that $e^{-\lambda_3^* t_2} \leq \xi_1$. For given $\lambda \in (0,\lambda_3^*)$, $t \in [t_2,0]$, the above relations leads to $e^{-\lambda t} \in [1,\xi_1]$. Therefore, $I_3(\lambda,t) > 0$ for $t \in [t_2,0]$.

Let $t > 0$ and $x := e^{-\lambda t} \in (0,1)$, $\bar{a} := (2\sqrt{2}-2)\alpha u_0$, $\bar{b} := \frac{u_0v_0}{1+u_0}$. Then

$$
\begin{aligned}
&(2\sqrt{2}-2)\alpha u_0\varepsilon_1 e^{-\lambda t} - \alpha\varepsilon_1^2 e^{-2\lambda t} + \frac{u_0v_0}{1+u_0} - \frac{(u_0-\varepsilon_1 e^{-\lambda t})(v_0+v_0e^{-\lambda t})}{1+u_0-\varepsilon_1 e^{-\lambda t}}\\
&= \bar{a}\varepsilon_1 x - \alpha\varepsilon_1^2x^2 + \bar{b} - \frac{(u_0-\varepsilon_1 x)(v_0+v_0x)}{1+u_0-\varepsilon_1 x}\\
&= \frac{1}{1+u_0-\varepsilon_1 x}[\bar{a}\varepsilon_1(1+u_0)x - \bar{a}\varepsilon_1^2x^2 - \alpha(1+u_0)\varepsilon_1^2x^2 + \alpha\varepsilon_1^3x^3 + \bar{b}(1+u_0)\\
&\quad - \bar{b}\varepsilon_1 x - u_0v_0 - u_0v_0x + \varepsilon_1v_0x + \varepsilon_1v_0x^2]\\
&= \Big(\bar{a}\varepsilon_1(1+u_0)x - \bar{a}\varepsilon_1^2x^2 - \alpha(1+u_0)\varepsilon_1^2x^2 + \alpha\varepsilon_1^3x^3 - \bar{b}\varepsilon_1 x - u_0v_0x + \varepsilon_1v_0x\\
&\quad + \varepsilon_1v_0x^2\Big)/(1+u_0-\varepsilon_1 x)\\
&= \frac{\tilde{f}(x)}{1+u_0-\varepsilon_1 x}.
\end{aligned}
$$

Here $\tilde{f}(x) := x\tilde{g}(x) = x\{\tilde{a}x^2 + \tilde{b}x + \tilde{c}\}$ and

$$
\begin{gathered}
\tilde{a} := \alpha\varepsilon_1^3, \quad \tilde{b} := \varepsilon_1v_0 - \bar{a}\varepsilon_1^2 - \alpha(1+u_0)\varepsilon_1^2 = \varepsilon_1[v_0 - \bar{a}\varepsilon_1 - \alpha(1+u_0)\varepsilon_1],\\
\tilde{c} := \bar{a}\varepsilon_1(1+u_0) - \bar{b}\varepsilon_1 - u_0v_0 + \varepsilon_1v_0.
\end{gathered}
$$

Let $\varepsilon_1 = (\sqrt{2}-1)u_0$, then we have from the assumption $\alpha u_0 \geq \frac{(3+2\sqrt{2})v_0}{1+u_0}$ that

$$
\begin{aligned}
\tilde{c} &= (2\sqrt{2}-2)(\sqrt{2}-1)\alpha u_0^2(1+u_0) - u_0v_0 + \frac{\varepsilon_1v_0}{1+u_0}\\
&\geq 2(3-2\sqrt{2})(3+2\sqrt{2})u_0v_0 - u_0v_0 + \frac{\varepsilon_1v_0}{1+u_0}\\
&= 2u_0v_0 - u_0v_0 + \frac{\varepsilon_1v_0}{1+u_0} = u_0v_0 + \frac{\varepsilon_1v_0}{1+u_0} > 0.
\end{aligned}
$$

There are three cases.

Case 1: If $\tilde{b} \geq 0$, then a similar discussion to $I_1(\lambda,t)$ yields to $\tilde{g}(x) \geq 0$, and then $I_3(\lambda,t) \geq 0$ for $\lambda \in\in (0,\lambda_3^*)$ and $t > 0$.

Case 2: If $\tilde{b} < 0$ and $\tilde{b}^2 - 4\tilde{a}\tilde{c} \le 0$, then the equation $\tilde{g}(x) = 0$ has either no real roots or a double real root. Therefore, $g(x) \ge 0$ for $x \in (0, \infty)$, which leads to $I_3(\lambda, t) \ge 0$ for $\lambda \in (0, \lambda_3^*)$ and $t > 0$.

Case 3: If $\tilde{b} < 0$ and $\tilde{b}^2 - 4\tilde{a}\tilde{c} > 0$, then the equation $\tilde{g}(x) = 0$ has two real roots: $x_{1,2} = \frac{1}{2\tilde{a}}(-\tilde{b} \pm \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}})$. Consider the small root $x_1 = \frac{1}{2\tilde{a}}(-\tilde{b} - \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}})$. We derive from $\alpha u_0 \ge \frac{(3+2\sqrt{2})v_0}{1+u_0}$ that

$$\begin{gathered}
\alpha u_0(1+u_0) \ge (3+2\sqrt{2})v_0 > (\sqrt{2}+1)v_0 = \frac{v_0}{\sqrt{2}-1},\\
2(\sqrt{2}-1)^2\alpha u_0 + (\sqrt{2}-1)\alpha u_0(1+u_0) - v_0 > 2(\sqrt{2}-1)^2\alpha u_0^2,\\
-\tilde{b} = \bar{a}\varepsilon_1 + \alpha(1+u_0)\varepsilon_1 - v_0 > 2\alpha\varepsilon_1^2 = 2\tilde{a},\\
-\tilde{b} - 2\tilde{a} > 0.
\end{gathered}$$

Let $x_1 < 1$, which is equivalent to

$$-\tilde{b} - 2\tilde{a} < \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}} \Leftrightarrow (\tilde{b} + 2\tilde{a})^2 < \tilde{b}^2 - 4\tilde{a}\tilde{c} \Leftrightarrow \tilde{a} + \tilde{b} + \tilde{c} < 0.$$

It is derived that

$$\begin{aligned}
\tilde{a} + \tilde{b} + \tilde{c} &= (\sqrt{2}-1)^3\alpha u_0^3 - [2(\sqrt{2}-1)\alpha u_0 + \alpha(1+u_0)](\sqrt{2}-1)^2 u_0^2\\
&\quad + [2(\sqrt{2}-1)\alpha u_0 - \frac{u_0 v_0}{1+u_0} + 2v_0](\sqrt{2}-1)u_0 - u_0 v_0\\
&= -(\sqrt{2}-1)^3\alpha u_0^3 - (\sqrt{2}-1)^2\alpha u_0^2(1+u_0) + 2(\sqrt{2}-1)^2\alpha u_0^2\\
&\quad - \frac{(\sqrt{2}-1)u_0^2 v_0}{1+u_0} + 2(\sqrt{2}-1)u_0 v_0 - u_0 v_0 < 0.
\end{aligned}$$

Therefore, $x_1 < 1$. Since we already known that $\tilde{f}(1) = \tilde{g}(1) > 0$ (see (3.15)) and $\tilde{g}(0) = \tilde{c} > 0$, by the graph of $\tilde{g}$ we know that $x_2 < 1$, which leads to $0 > \tilde{b} + 2\tilde{a} > \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}$. This is a contradiction, and thus Case 3 is impossible. Now, we conclude $\tilde{g}(x) > 0$ for $x \in (0,1)$. That is, $I_3(\lambda, t) > 0$ for $t > 0$. Summarizing the above discussion, we know that $\underline{\varphi}(t)$ satisfies the definition of lower solution.

Now we consider $\underline{\psi}(t)$. If $t \le t_4$, then $\underline{\psi}(t) = e^{\lambda_3 t} - qe^{\eta\lambda_3 t}$, $\underline{\varphi}(t) \ge 0$. Noting $\eta \le 2$, we have

$$\begin{aligned}
&d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) - c\underline{\psi}'(t) + \gamma\underline{\psi}(t)(\delta - \underline{\psi}(t)) + \frac{\rho\underline{\varphi}(t)\underline{\psi}(t)}{1+\underline{\varphi}(t)}\\
&\ge d_2\int_R J_2(y-t)(e^{\lambda_3 y} - qe^{\eta\lambda_3 y})dy - d_2(e^{\lambda_3 t} - qe^{\eta\lambda_3 t}) + c(\lambda_3 e^{\lambda_3 t} - \eta\lambda_3 qe^{\eta\lambda_3 t})\\
&\quad + \gamma(e^{\lambda_3 t} - qe^{\eta\lambda_3 t})(\delta - e^{\lambda_3 t} + qe^{\eta\lambda_3 t})\\
&= -qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) - \gamma(e^{\lambda_3 t} - qe^{\eta\lambda_3 t})^2\\
&\ge -qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) - \gamma e^{2\lambda_3 t}\\
&\ge e^{\eta\lambda_3 t}(-q\Delta_2(\eta\lambda_3, c) - \gamma) \ge 0
\end{aligned}$$

for $q > 1$ large enough.

Note that the function $y = \frac{x}{1+x}$ is nondecreasing for $x \in (-1, \infty)$. If $\underline{\varphi}(t) \geq u_0 - \varepsilon_1 e^{-\lambda t} > -1$ for $t \geq t_4$, then

$$\frac{\underline{\varphi}(t)}{1+\underline{\varphi}(t)} \geq \frac{u_0 - \varepsilon_1 e^{-\lambda t}}{1 + u_0 - \varepsilon_1 e^{-\lambda t}}.$$

In fact, if $\lambda > 0$ is small enough, one can have $(\sqrt{2}-1)e^{-\lambda t_4} < \frac{u_0+1}{u_0}$, which leads to

$$u_0 - (\sqrt{2}-1)u_0 e^{-\lambda t_4} > -1 \;\Rightarrow\; u_0 - \varepsilon_1 e^{-\lambda t} > -1 \quad \text{for } \varepsilon_1 < (\sqrt{2}-1)u_0, t \geq t_4.$$

Now, if $t \geq t_4$, then $\underline{\psi}(t) = v_0 - \varepsilon_2 e^{-\lambda t}$ and $\underline{\varphi}(t) \geq u_0 - \varepsilon_1 e^{-\lambda t} > -1$ (assuming $\lambda > 0$ small). Thus

$$\begin{aligned}
&d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) - c\underline{\psi}'(t) + \gamma\underline{\psi}(t)(\delta - \underline{\psi}(t)) + \frac{\rho\underline{\varphi}(t)\underline{\psi}(t)}{1+\underline{\varphi}(t)} \\
&\geq d_2 \int_R J_2(y-t)(v_0 - \varepsilon_2 e^{-\lambda y})dy - d_2(v_0 - \varepsilon_2 e^{-\lambda t}) - c\lambda\varepsilon_2 e^{-\lambda t} \\
&\quad + \gamma(v_0 - \varepsilon_2 e^{-\lambda t})(\delta - v_0 + \varepsilon_2 e^{-\lambda t}) + \frac{\rho\underline{\varphi}(t)(v_0 - \varepsilon_2 e^{-\lambda t})}{1+\underline{\varphi}(t)} \\
&\geq -\varepsilon_2 e^{-\lambda t}\Delta_2(-\lambda, c) + \gamma\delta v_0 - \gamma(v_0 - \varepsilon_2 e^{-\lambda t})^2 + \frac{\rho\underline{\varphi}(t)(v_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} \\
&= -\varepsilon_2 e^{-\lambda t}\Delta_2(-\lambda, c) + \gamma\delta v_0 - \gamma v_0^2 + 2\varepsilon_2\gamma v_0 e^{-\lambda t} - \gamma\varepsilon_2^2 e^{-2\lambda t} \\
&\quad + \frac{\rho\underline{\varphi}(t)(v_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} \\
&\geq \varepsilon_2 e^{-\lambda t}[\Delta_2(-\lambda, c) + \gamma v_0 - \frac{\frac{m-1}{m}\rho(u_0 - \varepsilon_1 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}}] + \varepsilon_2\gamma v_0 e^{-\lambda t} - \gamma\varepsilon_2^2 e^{-2\lambda t} \\
&\quad - \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho v_0(u_0 - \varepsilon_1 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} - \frac{\frac{1}{m}\rho\varepsilon_2 e^{-\lambda t}(u_0 - \varepsilon_1 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}}.
\end{aligned}$$

Here m is some positive integer. Note

$$\Delta_2(0, c) + \gamma v_0 - \frac{\frac{m-1}{m}\rho(u_0 - \varepsilon_1 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} \geq -\gamma\delta + \gamma v_0 - \frac{\frac{m-1}{m}\rho u_0}{1+u_0} > 0,$$

we can choose $\lambda_4^* > 0$ such that

$$\Delta_2(-\lambda, c) + \gamma v_0 - \frac{\frac{m-1}{m}\rho(u_0 - \varepsilon_1 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} > 0$$

for $\lambda \in (0, \lambda_4^*)$.

Let

$$\begin{aligned}
I_4(\lambda, t) &:= \varepsilon_2\gamma v_0 e^{-\lambda t} - \gamma\varepsilon_2^2 e^{-2\lambda t} - \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho v_0(u_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} \\
&\quad - \frac{\frac{1}{m}\rho\varepsilon_2 e^{-\lambda t}(u_0 - \varepsilon_1 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} \\
&= \varepsilon_2\gamma v_0 e^{-\lambda t} - \gamma\varepsilon_2^2 e^{-2\lambda t} - \frac{\rho v_0\varepsilon_1 e^{-\lambda t}}{(1+u_0)(1 + u_0 - \varepsilon_1 e^{-\lambda t})} \\
&\quad - \frac{\frac{1}{m}\rho\varepsilon_2 e^{-\lambda t}(u_0 - \varepsilon_1 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}}.
\end{aligned}$$

Obviously, we have $I_4(\lambda,\infty)=0$. By Lemma 3.3, we obtain

$$I_4(\lambda,0)=\varepsilon_2\gamma v_0-\gamma\varepsilon_2^2-\frac{\rho v_0\varepsilon_1}{(1+u_0)(1+u_0-\varepsilon_1)}-\frac{\frac{1}{m}\rho\varepsilon_2(u_0-\varepsilon_1)}{1+u_0-\varepsilon_1}>\frac{\varepsilon_0}{2}>0$$

if m is large. Similar to the argument for $I_3(\lambda,t)$, we can also obtain that $I_4(\lambda,t)>0$ uniformly for $t\geq t_4$. We omit the details here. Therefore, $\underline{\psi}(t)$ satisfies the definition of lower solution.

Taking $\lambda\in(0,\min_{1\leq i\leq 4}\{\lambda_i^*\})$, we see that $(\overline{\varphi}(t),\overline{\psi}(t))$ and $(\underline{\varphi}(t),\underline{\psi}(t))$ is a pair of upper-lower solutions of (3.4). □

From the above statements, the following result is obvious. Note that the following asymptotic behaviors of $(\varphi(t),\psi(t))$ can be obtained from the forms of lower solutions and sandwich method.

Theorem 3.6. *Assume* (3.8) *holds and* $c>c^*=\max\{c_1^*,c_2^*\}$*. Then* (3.1) *has a traveling wave solution* (φ,ψ) *with speed* c *connecting* $E_0=(0,0)$ *and* $E_3=(u_0,v_0)$*. Furthermore,*

$$\lim_{t\to-\infty}\varphi(t)e^{-\lambda_1 t}=\lim_{t\to-\infty}\psi(t)e^{-\lambda_3 t}=1. \tag{3.16}$$

3.2. **The particular case $\delta=0$.** If $\delta=0$, then (3.1) reduces to

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=d_1(J_1*u)(x,t)-d_1u(x,t)+\alpha u(x,t)(\beta-u(x,t))\\&\quad-\frac{u(x,t)v(x,t)}{1+u(x,t)},\\ \frac{\partial v(x,t)}{\partial t}&=d_2(J_2*v)(x,t)-d_2v(x,t)-\gamma v^2(x,t)+\rho\frac{u(x,t)v(x,t)}{1+u(x,t)}.\end{aligned} \tag{3.17}$$

This is a P-P system with a mortality rate function γv^2 for the predator, which depends on the prey as its unique resource. (3.17) has three equilibria

$$E_0=(0,0),\quad E_1=(\beta,0),\quad E_2=(u_0,v_0),$$

where the positive equilibrium $E_2=(u_0,v_0)$ satisfying

$$\alpha\beta-\alpha u_0-\frac{v_0}{1+u_0}=0,\quad \gamma v_0-\frac{\rho u_0}{1+u_0}=0. \tag{3.18}$$

It is clear that $u_0<\beta$.

Assume that $c>0$, let $u(x,t)=\varphi(x+ct)$, $v(x,t)=\psi(x+ct)$, and denote the traveling wave coordinate $x+ct$ still by t. Then we can obtain a wave profile system (3.4) with $\delta=0$. Similar to §3.1, consider the traveling waves connecting E_0 and E_2. Here, we can choose $K_1=\beta$, $K_2=\frac{\rho\beta}{\gamma}$.

Let us consider Lemma 3.3. Assume that

$$\alpha\beta>\frac{(4+2\sqrt{2})v_0}{1+u_0} \tag{3.19}$$

which yields to

$$\begin{gathered}\alpha u_0=\alpha\beta-\frac{v_0}{1+u_0}\geq\frac{(3+2\sqrt{2})v_0}{1+u_0},\\ \alpha u_0=\alpha\beta-\frac{v_0}{1+u_0}>\alpha\beta-\frac{\alpha\beta}{4+2\sqrt{2}}=\frac{(2+\sqrt{2})\alpha\beta}{4}.\end{gathered} \tag{3.20}$$

It is easy to see, that δ has no effect on the selection of ε_1, thus under the first inequality of (3.20), we still have $\varepsilon_1 \in (0, (\sqrt{2}-1)u_0]$ such that the first inequality in (3.7) holds. Since $\delta = 0$, we have $\gamma v_0 = \frac{\rho u_0}{1+u_0}$ (thus the second inequality of (3.6) is false), and we need to re-choose a $\varepsilon_2 \in (0, \frac{v_0}{2}]$ and to find an inequality similar to the second one of (3.7). Let

$$\begin{gathered} g_1(\varepsilon_2) = -\gamma\varepsilon_2^2 + 2\gamma v_0 \varepsilon_2, \\ g_2(\varepsilon_2) = \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho u_0 \varepsilon_2}{50(1+u_0)} - \frac{\rho(u_0-\varepsilon_1)(v_0-\varepsilon_2)}{1+u_0-\varepsilon_1}. \end{gathered}$$

Note that

$$\begin{gathered} \max\{g_1(\varepsilon_2)\} = g_1(v_0) = \gamma v_0^2, \quad g_1(\frac{v_0}{2}) = \frac{3}{4}\gamma v_0^2, \\ g_1(0) = 0, \quad g_2(0) = \frac{\rho u_0 v_0}{1+u_0} - \frac{\rho(u_0-\varepsilon_1)v_0}{1+u_0-\varepsilon_1} > 0. \end{gathered}$$

If $\varepsilon_1 \in (0, (\sqrt{2}-1)u_0]$, then

$$\begin{aligned} g_2(\frac{v_0}{2}) &= \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho u_0 v_0}{100(1+u_0)} - \frac{\frac{1}{2}\rho(u_0-\varepsilon_1)v_0}{1+u_0-\varepsilon_1} \\ &\le \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho u_0 v_0}{100(1+u_0)} - \frac{\frac{2-\sqrt{2}}{2}\rho u_0 v_0}{1+(2-\sqrt{2})u_0} \\ &\le \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho u_0 v_0}{100(1+u_0)} - \frac{\frac{2-\sqrt{2}}{2}\rho u_0 v_0}{1+u_0} \\ &= \frac{\sqrt{2}}{2}\frac{\rho u_0 v_0}{1+u_0} + \frac{\rho u_0 v_0}{100(1+u_0)} \\ &= (\frac{\sqrt{2}}{2} + \frac{1}{100})\gamma v_0^2 < g_1(\frac{v_0}{2}), \end{aligned}$$

then there exist $\varepsilon_2^* \in (0, v_0/2]$ such that $g_1(\varepsilon_2^*) = g_2(\varepsilon_2^*)$ and

$$g_1(\varepsilon_2) > g_2(\varepsilon_2) \quad \text{for } \varepsilon_2^* < \varepsilon_2 \le \frac{v_0}{2}. \tag{3.21}$$

In particular, one can choose $\varepsilon_2 = v_0/2$.

In this subsection, let $\delta_2 = 0$, and then $\lambda_3 = \lambda_3(c) = 0$ for $c > c_2^* = 0$. Correspondingly, $c > c^* := c_1^*$. We denote $\lambda_i(c)$ by λ_i, where $i = 1, 2, 3, 4$. Assume that $\lambda_4 > \lambda_1$. Choose $\zeta > 0$ with $\lambda_1 > \lambda_4 - \zeta > 0$, $q > 1$ and

$$\eta \in \big(1, \min\{\frac{\lambda_2}{\lambda_1}, \frac{\lambda_1+\lambda_4-\zeta}{\lambda_1}, 2\}\big). \tag{3.22}$$

Define continuous functions

$$\begin{gathered} \overline{\varphi}(t) = \begin{cases} e^{\lambda_1 t}, & t \le t_1, \\ \min\{K_1, u_0 + u_0 e^{-\lambda t}\}, & t \ge t_1, \end{cases} \quad \underline{\varphi}(t) = \begin{cases} e^{\lambda_1 t} - q e^{\eta\lambda_1 t}, & t \le t_2, \\ u_0 - \varepsilon_1 e^{-\lambda t}, & t \ge t_2, \end{cases} \\ \overline{\psi}(t) = \begin{cases} e^{\lambda_4 t} + q e^{(\lambda_4-\zeta)t}, & t \le t_3, \\ \min\{K_2, v_0 + v_0 e^{-\lambda t}\}, & t \ge t_3, \end{cases} \quad \underline{\psi}(t) = \begin{cases} 0, & t \le t_4, \\ v_0 - \varepsilon_2 e^{-\lambda t}, & t \ge t_4, \end{cases} \end{gathered}$$

where $q > 1$ is large enough and $\lambda > 0$ is small which will be defined later. We can see that $\overline{\varphi}(t)$, $\overline{\psi}(t)$, $\underline{\varphi}(t)$, $\underline{\psi}(t)$ satisfy (P1)–(P2) in Section 2. Moreover, if $q > 1$ is large enough, then it is clear that

$$t_2 < 0, \quad t_3 < 0, \quad t_4 < 0, \quad t_1 \geq \max\{t_2, t_3\}.$$

Lemma 3.7. *Assume that* (3.19) *holds and* $\lambda_4 > \lambda_1$, $q > 1$ *is large enough and* $\lambda > 0$ *is small. Then* $(\overline{\varphi}(t), \overline{\psi}(t))$ *is an upper solution and* $(\underline{\varphi}(t), \underline{\psi}(t))$ *is a lower solution of* (3.17).

Proof. We omit the argument for $\overline{\varphi}(t)$ since it is similar to the corresponding argument in Lemma 3.5.

We now consider $\overline{\psi}(t)$. If $t \leq t_3$, then $\overline{\psi}(t) = e^{\lambda_4 t} + qe^{(\lambda_4 - \zeta)t}$, $\overline{\varphi}(t) \leq e^{\lambda_1 t}$. Using λ_4 and $\lambda_4 - \zeta$ to replace λ_3 and $\eta\lambda_3$ in (3.12) respectively, we obtain

$$\begin{aligned}
&d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) - \gamma\overline{\psi}^2(t) + \frac{\rho\overline{\varphi}(t)\overline{\psi}(t)}{1 + \overline{\varphi}(t)} \\
&\leq qe^{(\lambda_4 - \zeta)t}\Delta_2(\lambda_4 - \zeta, c) - \gamma(e^{\lambda_4 t} + qe^{(\lambda_4 - \zeta)t})^2 + \frac{\rho e^{\lambda_1 t} \cdot (e^{\lambda_4 t} + qe^{(\lambda_4 - \zeta)t})}{1 + e^{\lambda_1 t}} \\
&\leq e^{(\lambda_4 - \zeta)t}[q\Delta_2(\lambda_4 - \zeta, c) - \gamma q^2 e^{(\lambda_4 - \zeta)t} + \rho + \rho q e^{\lambda_1 t}].
\end{aligned}$$

Note $\Delta_2(\lambda_4 - \zeta, c) < 0$. Let $q > 1$ be large enough, then $\rho - \gamma q < 0$, and thus we have from $\lambda_1 > \lambda_4 - \zeta > 0$ and $t_3 < 0$ that

$$\begin{aligned}
&q\Delta_2(\lambda_4 - \zeta, c) - \gamma q^2 e^{(\lambda_4 - \zeta)t_3} + \rho q e^{\lambda_1 t_3} + \rho \\
&\leq q[\Delta_2(\lambda_4 - \zeta, c) + e^{\lambda_1 t_3}(\rho - \gamma q)] + \rho < 0,
\end{aligned}$$

which leads to

$$e^{(\lambda_4 - \zeta)t}[q\Delta_2(\lambda_4 - \zeta, c) - \gamma q^2 e^{(\lambda_4 - \zeta)t} + \rho + \rho q e^{\lambda_1 t}] < 0 \quad \text{for } t \leq t_3.$$

If $t \geq t_3$ and $\overline{\psi}(t) = K_2$, then the proof is similar to (3.13). Otherwise, $\overline{\psi}(t) = v_0 + v_0 e^{-\lambda t}$, $\overline{\varphi}(t) \leq u_0 + u_0 e^{-\lambda t}$ implies

$$\begin{aligned}
&d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) - \gamma\overline{\psi}^2(t) + \frac{\rho\overline{\varphi}(t)\overline{\psi}(t)}{1 + \overline{\varphi}(t)} \\
&\leq v_0 e^{-\lambda t}\Delta_2(-\lambda, c) - \gamma(v_0 + v_0 e^{-\lambda t})^2 + \frac{\rho(u_0 + u_0 e^{-\lambda t})(v_0 + v_0 e^{-\lambda t})}{1 + u_0 + u_0 e^{-\lambda t}} \\
&= v_0 e^{-\lambda t}\Delta_2(-\lambda, c) - \frac{\rho u_0 v_0 (1 + e^{-\lambda t})^2}{1 + u_0} + \frac{\rho u_0 v_0 (1 + e^{-\lambda t})^2}{1 + u_0 + u_0 e^{-\lambda t}} \\
&= v_0 e^{-\lambda t}\Delta_2(-\lambda, c) - \frac{\rho u_0^2 v_0 e^{-\lambda t}(1 + e^{-\lambda t})^2}{(1 + u_0)(1 + u_0 + u_0 e^{-\lambda t})} \\
&\leq v_0 e^{-\lambda t}[\Delta_2(-\lambda, c) - \frac{\rho u_0^2}{(1 + u_0)(1 + u_0 + u_0 e^{-\lambda t})}].
\end{aligned}$$

Note that for $t \geq t_3$,

$$\begin{aligned}
\Delta_2(0, c) - \frac{\rho u_0^2}{(1 + u_0)(1 + u_0 + u_0 e^{-\lambda t})} &= -\frac{\rho u_0^2}{(1 + u_0)(1 + u_0 + u_0 e^{-\lambda t})} \\
&\leq -\frac{\rho u_0^2}{(1 + u_0)(1 + u_0 + u_0 e^{-\lambda t_3})} < 0,
\end{aligned}$$

and thus there exists a constant λ_2^* such that $\Delta_2(-\lambda, c) - \frac{\rho u_0^2}{(1+u_0)(1+u_0+u_0 e^{-\lambda t})} < 0$ for any $\lambda \in (0, \lambda_2^*)$. Therefore, $\overline{\psi}(t)$ satisfies the definition of upper solution.

If $t \le t_2$, then $0 \le \underline{\varphi}(t) = e^{\lambda_1 t} - qe^{\eta\lambda_1 t} \le e^{\lambda_1 t}$, $0 \le \overline{\psi}(t) \le e^{\lambda_4 t} + qe^{(\lambda_4-\zeta)t}$. Using $\lambda_4 - \zeta$ to replace $\eta\lambda_3$ in (3.14), we have from (3.22) that

$$\begin{aligned}
&d_1(J_1 * \underline{\varphi})(t) - d_1\underline{\varphi}(t) - c\underline{\varphi}'(t) + \alpha\underline{\varphi}(t)(\beta - \underline{\varphi}(t)) - \frac{\underline{\varphi}(t)\overline{\psi}(t)}{1+\underline{\varphi}(t)} \\
&\ge -e^{\eta\lambda_1 t}[q\Delta_1(\eta\lambda_1, c) + (\alpha+1) + qe^{(\lambda_1+\lambda_4-\zeta-\eta\lambda_1)t}].
\end{aligned}$$

The coming argument for $t \le t_2$, and a further discussion for $t \ge t_2$ are the same as the corresponding argument in Lemma 3.5, so we omit it. Then we know that $\underline{\varphi}(t)$ satisfies the definition of lower solution.

If $t \le t_4$, $\underline{\psi}(t) = 0$, $\underline{\varphi}(t) \ge 0$, the following inequality is clear:

$$d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) - c\underline{\psi}'(t) - \gamma\underline{\psi}^2(t) + \frac{\rho\underline{\varphi}(t)\underline{\psi}(t)}{1+\underline{\varphi}(t)} \ge 0.$$

If $t \ge t_4$, then $\underline{\psi}(t) = v_0 - \varepsilon_2 e^{-\lambda t}$, $\underline{\varphi}(t) \ge u_0 - \varepsilon_1 e^{-\lambda t} > -1$ ($\lambda > 0$ small), and thus

$$\begin{aligned}
&d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) - c\underline{\psi}'(t) - \gamma\underline{\psi}^2(t) + \frac{\rho\underline{\varphi}(t)\underline{\psi}(t)}{1+\underline{\varphi}(t)} \\
&\ge -\varepsilon_2 e^{-\lambda t}\Delta_2(-\lambda, c) - \gamma(v_0 - \varepsilon_2 e^{-\lambda t})^2 + \frac{\rho(u_0 - \varepsilon_1 e^{-\lambda t})(v_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} \\
&= \varepsilon_2 e^{-\lambda t}[-\Delta_2(-\lambda, c) + \frac{\rho u_0}{50(1+u_0)}] + 2\varepsilon_2\gamma v_0 e^{-\lambda t} - \gamma\varepsilon_2^2 e^{-2\lambda t} - \frac{\rho u_0 v_0}{1+u_0} \\
&\quad - \frac{\rho u_0 \varepsilon_2 e^{-\lambda t}}{50(1+u_0)} + \frac{\rho(u_0 - \varepsilon_1 e^{-\lambda t})(v_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}}.
\end{aligned}$$

Note $-\Delta_2(0,c) + \frac{\rho u_0}{50(1+u_0)} = \frac{\rho u_0}{50(1+u_0)} > 0$, thus there exists a constant λ_4^* such that $-\Delta_2(-\lambda, c) + \frac{\rho u_0}{50(1+u_0)} > 0$ for any $\lambda \in (0, \lambda_4^*)$.

Let

$$\begin{aligned}
&I_4(\lambda, t) \\
&:= 2\varepsilon_2\gamma v_0 e^{-\lambda t} - \gamma\varepsilon_2^2 e^{-2\lambda t} - \frac{\rho u_0 v_0}{1+u_0} - \frac{\rho u_0\varepsilon_2 e^{-\lambda t}}{50(1+u_0)} + \frac{\rho(u_0 - \varepsilon_1 e^{-\lambda t})(v_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}} \\
&= \frac{99}{50}\varepsilon_2\gamma v_0 e^{-\lambda t} - \gamma\varepsilon_2^2 e^{-2\lambda t} - \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho(u_0 - \varepsilon_1 e^{-\lambda t})(v_0 - \varepsilon_2 e^{-\lambda t})}{1 + u_0 - \varepsilon_1 e^{-\lambda t}}.
\end{aligned}$$

Obviously, we have $I_4(\lambda, \infty) = 0$ and

$$\begin{aligned}
I_4(\lambda, 0) &= g_1(\varepsilon_2) + g_2(\varepsilon_2) \\
&= 2\varepsilon_2\gamma v_0 - \gamma\varepsilon_2^2 - \frac{\rho u_0 v_0}{1+u_0} - \frac{\rho u_0 \varepsilon_2}{50(1+u_0)} + \frac{\rho(u_0 - \varepsilon_1)(v_0 - \varepsilon_2)}{1 + u_0 - \varepsilon_1}.
\end{aligned}$$

In view of (3.21), there exists $\varepsilon_2 \in (0, \frac{v_0}{2}]$ such that $I_4(\lambda, 0) > \varepsilon_0 > 0$ where $\varepsilon_0 > 0$ is a constant. Similar to the argument in (3.15), we can also obtain $I_4(\lambda, t) > 0$ uniformly for $t \in [t_4, 0]$ and $\lambda \in (0, \lambda_4^*)$.

Now we show that $I_4(\lambda, t) > 0$ uniformly for $t > 0$. Let $x = e^{-\lambda t}$ and $\kappa = \frac{99}{50}$. Then

$$\begin{aligned}
I_4(\lambda, t) &= \kappa\varepsilon_2\gamma v_0 x - \gamma\varepsilon_2^2 x^2 - \frac{\rho u_0 v_0}{1+u_0} + \frac{\rho(u_0 - \varepsilon_1 x)(v_0 - \varepsilon_2 x)}{1 + u_0 - \varepsilon_1 x} \\
&= \frac{1}{1+u_0-\varepsilon_1 x}\{[\kappa\varepsilon_2\gamma v_0 x - \gamma\varepsilon_2^2 x^2 - \frac{\rho u_0 v_0}{1+u_0}][1 + u_0 - \varepsilon_1 x]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \rho(u_0-\varepsilon_1 x)(v_0-\varepsilon_2 x)\}\\
&= \frac{1}{1+u_0-\varepsilon_1 x}\{x[\breve{a}x^2+\breve{b}x+\breve{c}]\}\\
&=: \frac{1}{1+u_0-\varepsilon_1 x}\{x\breve{g}(x)\},
\end{aligned}
$$

where

$$
\begin{gathered}
\breve{a} := \gamma\varepsilon_1\varepsilon_2^2, \quad \breve{b} := -\kappa\varepsilon_1\varepsilon_2\gamma v_0 - \varepsilon_2^2\gamma(1+u_0) + \rho\varepsilon_1\varepsilon_2,\\
\breve{c} := \kappa\varepsilon_2\gamma v_0(1+u_0) - \rho\varepsilon_1 v_0 - \rho\varepsilon_2 u_0 + \frac{\rho\varepsilon_1 u_0 v_0}{1+u_0}.
\end{gathered}
$$

Let $\varepsilon_1 = (\sqrt{2}-1)u_0$ and $\varepsilon_2 = \frac{v_0}{2}$. Then from $\frac{\rho u_0}{1+u_0} = \gamma v_0$ we have

$$
\begin{aligned}
\breve{c} &= \kappa\varepsilon_2\gamma v_0(1+u_0) - \rho\varepsilon_1 v_0 - \rho\varepsilon_2 u_0 + \frac{\rho\varepsilon_1 u_0 v_0}{1+u_0}\\
&= \gamma v_0^2[\frac{\kappa}{2} - (\sqrt{2}-1) - \frac{1}{2}] + \gamma u_0 v_0^2[\frac{\kappa}{2} - (\sqrt{2}-1) - \frac{1}{2} + (\sqrt{2}-1)] > 0,\\
\breve{b} &= -\kappa\varepsilon_1\varepsilon_2\gamma v_0 - \varepsilon_2^2\gamma(1+u_0) + \rho\varepsilon_1\varepsilon_2\\
&= \gamma v_0^2[-\frac{1}{4} + \frac{\sqrt{2}-1}{2}] + \gamma u_0 v_0^2[-\frac{\kappa(\sqrt{2}-1)}{2} - \frac{1}{4} + \frac{\sqrt{2}-1}{2}] < 0,
\end{aligned}
$$

$$
\begin{aligned}
&\breve{a} + \breve{b} + \breve{c}\\
&= \gamma v_0^2[-\frac{1}{4} + \frac{\sqrt{2}-1}{2} + \frac{\kappa}{2} - (\sqrt{2}-1) - \frac{1}{2}]\\
&\quad + \gamma u_0 v_0^2[\frac{\sqrt{2}-1}{4} - \frac{\kappa(\sqrt{2}-1)}{2} - \frac{1}{4} + \frac{\sqrt{2}-1}{2} + \frac{\kappa}{2} - \frac{1}{2}]\\
&= \gamma v_0^2[-\frac{3}{4} - \frac{\sqrt{2}-1}{2} + \frac{\kappa}{2}] + \gamma u_0 v_0^2[-\frac{3}{4} + \frac{3(\sqrt{2}-1)}{4} - \frac{\kappa(\sqrt{2}-1)}{2} + \frac{\kappa}{2}] > 0.
\end{aligned}
$$

Since $\breve{a} + \breve{b} + \breve{c} > 0$ is equivalent to $x_1 > 1$, where x_1 is the smallest real root of $\breve{g}(x) = 0$, similar to proving procedure of Lemma 3.5, we know that $I_4(\lambda, t) > 0$ uniformly for $t > 0$ and $\lambda \in (0, \lambda_4^*)$. Therefore, $\underline{\psi}(t)$ satisfies the definition of lower solution.

Let $\lambda^* = \min_{i=1,2,3,4}\{\lambda_i^*\}$. Summarizing the above discussion, we know that $(\overline{\varphi}(t), \overline{\psi}(t))$ and $(\underline{\varphi}(t), \underline{\psi}(t))$ satisfy the definition of upper-lower solutions. □

From the above statements, the following result is obvious.

Theorem 3.8. *Assume* (3.19) *holds,* $c > c^* := c_1^*$ *and* $\lambda_4 > \lambda_1$. *Then* (3.17) *has a traveling wave solution with speed* c *connecting* $E_0 = (0,0)$ *and* $E_2 = (u_0, v_0)$.

Concluding remarks. Using the upper-lower solution method, we proved the existence of traveling wavefronts connecting E_0 and E_3 for (1.1) with $h_1 = \alpha u(\beta - u), h_2 = \gamma v(\delta - v)$ and f of Holling type II, with either $\delta > 0$ or $\delta = 0$.

Assume that $\delta > 0$. Note that the predator in system (3.1) has logistic growth, which implies that it has resources other than the prey. Therefore, it is possible to consider the immigration of the prey into the residence area of predator, and further a co-existence steady state may be reached as $x + ct \to \infty$. Similar arguments show the existence of traveling waves connecting E_2 and E_3.

We did not discuss the existence of traveling wavefronts connecting E_1 and E_3, which raised in the case of mild invasions. Furthermore, we don't know if traveling wave solutions can be studied using the upper-lower solution technique under one of the following situations:

(i) $h_1 = \alpha u(\beta - u), h_2 = -dv$ and $f = \frac{u}{1+u}$ (Holling type II);
(ii) $h_1 = \alpha u(\beta - u), h_2 = \gamma v(\delta - v)$ and $f = u$ (Holling type I).

These problems are still open, and we leave it for future investigation.

Acknowledgments. Research is supported by the NSF of China (11171120), and by the Doctoral Program of Higher Education of China (20094407110001).

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