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INVERSE COEFFICIENT PROBLEM FOR THE SEMI-LINEAR FRACTIONAL TELEGRAPH EQUATION

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Abstract. We establish the unique solvability for an inverse problem for semi-linear fractional telegraph equation

$$D_t^\alpha u + r(t) D_t^\beta u - \Delta u = F_0(x,t,u,D_t^\beta u), \quad (x,t) \in \Omega_0 \times (0,T]$$

with regularized fractional derivatives $D_t^\alpha u, D_t^\beta u$ of orders $\alpha \in (1,2)$, $\beta \in (0,1)$ with respect to time on bounded cylindrical domain. This problem consists in the determination of a pair of functions: a classical solution u of the first boundary-value problem for such equation, and an unknown continuous coefficient $r(t)$ under the over-determination condition

$$\int_{\Omega_0} u(x,t)\varphi(x)dx = F(t), \quad t \in [0,T]$$

with given functions φ and F.

1. Introduction

The existence and uniqueness theorems for fractional Cauchy problems were proved in [2, 4, 5, 7, 11, 12, 13, 14, 15, 16, 24] and other works. The conditions of classical solvability of the first boundary-value problem for equation

$$D_t^\beta u(x,t) - A(x,D)u(x,t) = F_0(x,t)$$

with regularized fractional derivative (see, for example, [4]) and some elliptic differential second order operator $A(x,D)$ were obtained in [18] and [19].

Equations with fractional derivatives are applied in studying of anomalous diffusion and various processes in physics, mechanics, chemistry and engineering. The telegraph fractional equations in theory of thermal stresses is considered, for example, in [21]. Inverse problems to such equations arise in many branches of science and engineering. Some inverse boundary-value problems to diffusion-wave equation with different unknown functions or parameters were investigated, for example, in [1, 3, 6, 9, 17, 20, 22, 25]. In particular, the article [1] was devoted to determination of a source term for a time fractional diffusion equation with an integral type over-determination condition.

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In this note we prove the existence and uniqueness of a classical solution (u, r) of the inverse boundary-value problem

$$D_t^\alpha u + r(t)D_t^\beta u - \Delta u = F_0(x, t, u, D_t^\beta u), \quad (x, t) \in \Omega_0 \times (0, T], \tag{1.1}$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega_0 \times [0, T], \tag{1.2}$$

$$u(x, 0) = F_1(x), \quad x \in \bar{\Omega}_0, \tag{1.3}$$

$$u_t(x, 0) = F_2(x), \quad x \in \bar{\Omega}_0, \tag{1.4}$$

$$\int_{\Omega_0} u(x, t)\varphi_0(x)dx = F(t), \quad t \in [0, T] \tag{1.5}$$

for regularized telegraph equation, where $\alpha \in (1, 2)$, $\beta \in (0, 1)$, Ω_0 is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a boundary $\partial\Omega_0$ of class C^{1+s}, $s \in (0, 1)$, F_0, F_1, F_2, F, φ_0 – given functions. We shall use Green's functions to prove the solvability of this problem.

2. Definitions and auxiliary results

We shall use the notation: $\Omega_1 = \partial\Omega_0$, $Q_i = \Omega_i \times (0, T]$, $i = 0, 1$, $Q_2 = \Omega_0$, $\mathfrak{D}(R^m)$ is a space of indefinitely differentiable functions with compact supports in R^m, $m = 1, 2, \dots$, $\mathfrak{D}(\overline{Q}_0) = \{v \in C^\infty(\overline{Q}_0) : (\frac{\partial}{\partial t})^k v|_{t=T} = 0,\ k = 0, 1, \dots\}$,

$\mathfrak{D}'(R^m)$ and $\mathfrak{D}'(\overline{Q}_0)$ are spaces of linear continuous functionals (generalized functions [23, p. 13-15]) over $\mathfrak{D}(R^m)$ and $\mathfrak{D}(\overline{Q}_0)$, respectively, (f, φ) stands for the value of $f \in \mathfrak{D}'(R^m)$ on the test function $\varphi \in \mathfrak{D}(R^m)$ and also the value of $f \in \mathfrak{D}'(\overline{Q}_0)$ on $\varphi \in \mathfrak{D}(\overline{Q}_0)$.

We denote by $f * g$ the convolution of generalized functions f and g, use the function

$$f_\lambda(t) = \begin{cases} \frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)} & \text{for } \lambda > 0 \\ f'_{1+\lambda}(t) & \text{for } \lambda \leq 0, \end{cases}$$

where $\Gamma(z)$ is the Gamma-function, and $\theta(t)$ is the Heaviside function. Note that

$$f_\lambda * f_\mu = f_{\lambda+\mu}\,.$$

Also note that the Riemann-Liouville derivative $v_t^{(\alpha)}(x, t)$ of order $\alpha > 0$ is defined by

$$v_t^{(\alpha)}(x, t) = f_{-\alpha}(t) * v(x, t)$$

and

$$\begin{aligned} D_t^\alpha v(x, t) &= \frac{1}{\Gamma(1-\alpha)}\Big[\frac{\partial}{\partial t}\int_0^t \frac{v(x, \tau)}{(t-\tau)^\alpha}d\tau - \frac{v(x, 0)}{t^\alpha}\Big] \\ &= v_t^{(\alpha)}(x, t) - f_{1-\alpha}(t)v(x, 0), \quad \text{for } \alpha \in (0, 1), \end{aligned}$$

while

$$\begin{aligned} D_t^\alpha v(x, t) &= \frac{1}{\Gamma(2-\alpha)}\Big[\frac{\partial}{\partial t}\int_0^t \frac{v_\tau(x, \tau)}{(t-\tau)^{\alpha-1}}d\tau - \frac{v_t(x, 0)}{(t-\tau)^{\alpha-1}}\Big] \\ &= v_t^{(\alpha)}(x, t) - f_{1-\alpha}(t)v(x, 0) - f_{2-\alpha}(t)v_t(x, 0) \quad \text{for } \alpha \in (1, 2). \end{aligned}$$

We denote $D_t^1 v = \frac{\partial v}{\partial t}$.

Let $C(Q_0)$, $C(\overline{Q}_0)$, $C[0, T]$ be spaces of continuous functions on Q_0, $\overline{Q}_0$ and $[0, T]$, respectively, $C^\gamma(\Omega_0)$ $(C^\gamma(\bar{\Omega}_0))$ be a space of bounded continuous functions on Ω_0 $(\bar{\Omega}_0)$ satisfying Hölder continuity condition, $C^\gamma(Q_i)$ $(C^\gamma(\bar{Q}_i))$, $i = 0, 1$, be a space of bounded continuous functions on Q_i ($\bar{Q}_i$) which for all $t \in (0, T]$ satisfies Hölder continuity condition with respect to space variables, $C^\gamma(Q_0 \times \mathbb{R}^2)$ be a space of bounded continuous functions $f(x, t, v, z)$ on $Q_0 \times \mathbb{R}^2$ which for all $t \in (0, T]$, $v, z \in \mathbb{R}$ satisfies Hölder continuity condition with respect to space variables $x \in \Omega_0$,

$$\begin{gathered}
C_\beta(Q_0) = \{v \in C(Q_0) : D_t^\beta v \in C(Q_0)\}, \\
C_\beta^\gamma(Q_0) = \{v \in C^\gamma(Q_0) : D_t^\beta v \in C^\gamma(Q_0)\}, \\
C_{2,\alpha}(Q_0) = \{v \in C(Q_0) : \Delta v, D_t^\alpha v \in C(Q_0)\}, \\
C_{2,\alpha}(\bar{Q}_0) = \{v \in C_{2,\alpha}(Q_0) : v, v_t \in C(\bar{Q}_0)\}.
\end{gathered}$$

We use the following assumptions:

- (A1) $F_0 \in C^\gamma(Q_0 \times \mathbb{R}^2)$, $\gamma \in (0, 1)$, $|F_0(x, t, v, w)| \le A_0 + B_0[|v|^q + |w|^p]$,
$$|F_0(x, t, v_1, w_1) - F_0(x, t, v_2, w_2)| \le D_0[|v_1 - v_2|^q + |w_1 - w_2|^p]$$
for all $v, w, v_1, v_2, w_1, w_2 \in \mathbb{R}$ where $p, q \in [0, 2]$, A_0, B_0, D_0 are some positive constants,
- (A2) $F_1 \in C^\gamma(\bar{\Omega}_0)$, $F_1|_{\Omega_1} = 0$,
- (A3) $F_2 \in C^\gamma(\bar{\Omega}_0)$,
- (A4) $F, D^\beta F, D^\alpha F \in C[0, T]$, there exists $f := \inf_{t \in [0,T]} |D^\beta F(t)| > 0$,
- (A5) $\varphi_0 \in C^2(\bar{\Omega}_0)$, $\varphi_0|_{\Omega_1} = 0$.

Definition 2.1. A pair of functions $(u, r) \in C_{2,\alpha}(\bar{Q}_0) \times C[0, T]$ satisfying (1.1) on Q_0 and the conditions (1.2)-(1.5) is called a solution of the problem (1.1)-(1.5).

From (1.3), (1.4) and (1.5), it follows the necessary agreement conditions

$$\int_{\Omega_0} F_1(x)\varphi_0(x)dx = F(0), \quad \int_{\Omega_0} F_2(x)\varphi_0(x)dx = F'(0). \tag{2.1}$$

We introduce the operators

$$\begin{gathered}
(Lv)(x, t) \equiv v_t^{(\alpha)}(x, t) - \Delta v(x, t), \quad (x, t) \in \overline{Q}_0, \quad v \in \mathfrak{D}'(\overline{Q}_0), \\
(L^{\mathrm{reg}} v)(x, t) \equiv D_t^\alpha v(x, t) - \Delta v(x, t), \quad (x, t) \in \overline{Q}_0, \quad v \in C_{2,\alpha}(\bar{Q}_0).
\end{gathered}$$

Definition 2.2. A vector-function $(G_0(x, t, y, \tau), G_1(x, t, y), G_2(x, t, y))$ is called a Green's vector-function of the problem

$$(L^{\mathrm{reg}} u)(x, t) = g_0(x, t), \quad (x, t) \in Q_0, \tag{2.2}$$

$$u(x, t) = 0, \quad (x, t) \in \bar{Q}_1, \tag{2.3}$$

$$u(x, 0) = g_1(x), \quad u_t(x, 0) = g_2(x), \quad x \in \bar{\Omega}_0, \tag{2.4}$$

if for rather regular g_0, g_1, g_2 the function

$$u(x, t) = \int_0^t d\tau \int_{\Omega_0} G_0(x, t, y, \tau) g_0(y, \tau) dy + \sum_{j=1}^{2} \int_{\Omega_0} G_j(x, t, y) g_j(y) dy, \tag{2.5}$$

for $(x, t) \in \overline{Q}_0$, is the classical solution (in $C_{2,\alpha}(\bar{Q}_0)$) of the first boundary-value problem (2.2)–(2.4).

It follows from Definition 2.2 that

$$\begin{gathered}
(LG_0)(x, t, y, \tau) = \delta(x - y, t - \tau), \quad (x, t), (y, \tau) \in Q_0, \\
(L^{\mathrm{reg}} G_j)(x, t, y) = 0, \quad (x, t) \in Q_0, \ y \in \Omega_0, \ j = 1, 2,
\end{gathered}$$

$$G_1(x,0,y) = \delta(x-y), \quad \frac{\partial}{\partial t} G_1(x,0,y) = 0,$$
$$G_2(x,0,y) = 0, \quad \frac{\partial}{\partial t} G_2(x,0,y) = \delta(x-y), \quad x,y \in \Omega_0$$

where δ is Dirac delta-function.

Lemma 2.3 ([15]). *The following relations hold:*

$$G_j(x,t,y) = \int_0^t f_{j-\alpha}(\tau) G_0(x,t,y,\tau)d\tau, \quad (x,t) \in \bar{Q}_0,\ y \in \Omega_0,\ j=1,2.$$

Lemma 2.4. *A Green's vector-function of the first boundary-value problem* (2.2)–(2.4) *exists.*

The above lemma is proved following the strategy in [17, Lemma 2].

Theorem 2.5. *If $g_0 \in C^\gamma(Q_0)$, $\gamma \in (0,1)$, $g_j \in C^\gamma(\bar{\Omega}_0)$, $j = 1,2$, $g_1|_{\Omega_1} = 0$ then there exists a unique solution $u \in C_{2,\alpha}(\bar{Q}_0)$ of* (2.2)–(2.4)*. It is defined by*

$$u(x,t) = \big(\mathfrak{G}_0 g_0\big)(x,t) + \big(\mathfrak{G}_1 g_1\big)(x,t) + \big(\mathfrak{G}_2 g_2\big)(x,t), \quad (x,t) \in \bar{Q}_0 \tag{2.6}$$

where

$$(\mathfrak{G}_0 g_0)(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau) g_0(y,\tau)dy,$$
$$(\mathfrak{G}_j g_j)(x,t) = \int_{\Omega_0} G_j(x,t,y) g_j(y)dy, \quad j=1,2.$$

Proof. Taking Lemmas 2.3 and 2.4 into account, as in [7, 8, 10, 24] for the Cauchy problems, we show that the function (2.6) belongs to $C_{2,\alpha}(\bar{Q}_0)$ and satisfies the problem (2.2)–(2.4). We use the estimates founded in [7, 11, 15, 24]:

$$|G_0(x,t,y,\tau)| \le \frac{C_0^*}{(t-\tau)|x-y|^{N-2}}, \quad |x-y|^2 < 4(t-\tau)^\alpha,$$
$$|G_j(x,t,y)| \le \frac{C_j^* t^{j-1-\alpha}}{|x-y|^{N-2}}, \quad j=1,2,\ |x-y|^2 < 4t^\alpha,$$
$$\begin{aligned}
|G_0(x,t,y,\tau)| &\le \frac{\hat{C}_0(t-\tau)^{\alpha-1}}{|x-y|^N}\Big(\frac{|x-y|^2}{4(t-\tau)^\alpha}\Big)^{1+\frac{N}{2(2-\alpha)}} e^{-c(\frac{|x-y|^2}{4a_0(t-\tau)^\alpha})^{\frac{1}{2-\alpha}}} \\
&\le \frac{C_0(t-\tau)^{\alpha-1}}{|x-y|^N}, \quad |x-y|^2 > 4(t-\tau)^\alpha
\end{aligned}$$
$$\begin{aligned}
G_j(x,t,y) &\le \frac{\hat{C}_j t^{j-1}}{|x-y|^N}\Big(\frac{|x-y|^2}{4t^\alpha}\Big)^{\frac{N}{2(2-\alpha)}} e^{-c(\frac{|x-y|^2}{4t^\alpha})^{\frac{1}{2-\alpha}}} \\
&\le \frac{C_j t^{j-1}}{|x-y|^N}, \quad j=1,2,\ |x-y|^2 > 4t^\alpha
\end{aligned}$$

where $c, C_j^*, C_j, \hat{C}_j \quad (j=0,1,2)$ are positive constants;

$$|G_j(x+\Delta x, t+\Delta t, y,\tau) - G_j(x,t,y,\tau)| \le A_j(x,t,y,\tau)[|\Delta x| + |\Delta t|^{\alpha/2}]^\gamma,$$
$$\begin{aligned}
&|D_t^\beta G_j(x+\Delta x, t+\Delta t, y,\tau) - D_t^\beta G_j(x,t,y,\tau)| \\
&\le A_{\beta,j}(x,t,y,\tau)[|\Delta x| + |\Delta t|^{\alpha/2}]^\gamma \\
&\forall (x,t), (x+\Delta x, t+\Delta t) \in \bar{Q}_0, (y,\tau) \in \bar{Q}_j,\ j=0,1,2
\end{aligned}$$

with some $0 < \gamma < 1$ where non-negative functions $A_j(x,t,y,\tau)$, $A_{\beta,j}(x,t,y,\tau)$ have the same kind of estimates as $G_j(x,t,y,\tau)$, $D_t^\beta G_j(x,t,y,\tau)$, $j = 0,1,2$, have, respectively, and $G_j(x,t,y,\tau) = G_j(x,t,y)$, $A_j(x,t,y,\tau) = A_j(x,t,y)$, $A_{\beta,j}(x,t,y,\tau) = A_{\beta,j}(x,t,y)$ for $j = 1,2$. Note that for the general boundary-value problem to a parabolic equation with partial derivatives the last properties of a Green's vector-function were obtained in [10]. $\square$

3. Existence and uniqueness theorems for the inverse problem

Now we prove the existence of a solution for the inverse problem (1.1)–(1.5). It follows from the theorem 2.5 that under assumptions (A1), (A2), (A3) for a given $r \in C[0,T]$ the solution $u \in C_{2,\alpha}(\bar{Q}_0)$ of the first boundary-value problem (1.1)–(1.4) satisfies

$$\begin{aligned} u(x,t) = &-\int_0^t r(\tau)d\tau \int_{\Omega_0} G_0(x,t,y,\tau) D_\tau^\beta u(y,\tau)dy \\ &+ h_0(x,t,u,D_t^\beta u) + h(x,t), \quad (x,t) \in \Omega_0 \end{aligned} \tag{3.1}$$

where

$$\begin{gathered} h_0(x,t,u,D_t^\beta u) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau) F_0(y,\tau,u(y,\tau),D_\tau^\beta u(y,\tau))dy, \\ h(x,t) = \sum_{j=1}^2 \int_{\Omega_0} G_j(x,t,y) F_j(y)dy, \quad (x,t) \in \Omega_0. \end{gathered} \tag{3.2}$$

Conversely, any solution $u \in C_\beta^\gamma(Q_0)$ of (3.1) belongs to $C_{2,\alpha}(\bar{Q}_0)$ and is the solution of (1.1)–(1.5).

It follows from the equation (1.1) and assumption (A5) that

$$\begin{aligned} &\int_{\Omega_0} D_t^\alpha u(x,t)\varphi_0(x)dx + r(t)\int_{\Omega_0} D_t^\beta u(x,t)\varphi_0(x)dx \\ &= \int_{\Omega_0} u(x,t)\Delta\varphi_0(x)dx + \int_{\Omega_0} F_0(x,t,u(x,t),D_t^\beta u(x,t))\varphi_0(x)dx, \quad t \in (0,T]. \end{aligned}$$

Using (1.5) we obtain

$$\begin{aligned} &D^\alpha F(t) + r(t)D^\beta F(t) \\ &= \int_{\Omega_0} u(x,t)\Delta\varphi_0(x)dx + \int_{\Omega_0} F_0(x,t,u(x,t),D_t^\beta u(x,t))\varphi_0(x)dx \end{aligned}$$

(here $D^\alpha F(t) = D_t^\alpha F(t)$ and so on); that is, by using (A4),

$$\begin{aligned} r(t) = \Big[&\int_{\Omega_0} u(x,t)\Delta\varphi_0(x)dx + \int_{\Omega_0} F_0(x,t,u(x,t),D_t^\beta u(x,t))\varphi_0(x)dx \\ &- D^\alpha F(t)\Big][D^\beta F(t)]^{-1}, \quad t \in (0,T]. \end{aligned} \tag{3.3}$$

Note that, for $u \in C^{\gamma}_{\beta}(Q_0)$, the function $r(t)$, defined by (3.3), belongs to $C[0,T]$. By substituting the right-hand side of (3.3) into (3.1) in place of $r(t)$ we obtain

$$
\begin{aligned}
u(x,t) = \int_0^t [D^\beta F(\tau)]^{-1} d\tau \int_{\Omega_0} & G_0(x,t,y,\tau) \\
& \times \Big[\int_{\Omega_0} F_0(z,\tau,u(z,\tau),D^\beta_\tau u(z,\tau))\varphi_0(z)dz \\
& + \int_{\Omega_0} u(z,\tau)\Delta\varphi_0(z)dz - D^\alpha F(\tau)\Big] D^\beta_\tau u(y,\tau)dy \\
& + h_0(x,t,u(x,t),D^\beta_t u(x,t)) + h(x,t), \quad (x,t)\in \bar{Q}_0,
\end{aligned} \tag{3.4}
$$

where the functions h_0, h are defined by (3.2). We have reduced the problem (1.1)–(1.5) to system (3.3), (3.4). Conversely, a pair $(u,r) \in C^{\gamma}_{\beta}(Q_0) \times C[0,T]$ satisfying (3.3) and (3.4) is a solution of the problem (1.1)-(1.5).

Thus, under assumptions (A1)–(A5) and (2.1), a pair of functions $(u,r) \in C_{2,\alpha}(\bar{Q}_0) \times C[0,T]$ is the solution of (1.1)-(1.5) if and only if the function $u \in C^{\gamma}_{\beta}(Q_0)$ is the solution of (3.4), with $r \in C[0,T]$ defined by (3.3).

Theorem 3.1. *Under the assumptions* (A1)–(A5) *and the condition* (2.1)*, there exists $T^* \in (0,T]$ ($Q_0^* = \Omega_0 \times (0,T^*]$) and a solution $(u,r) \in C_{2,\alpha}(\bar{Q}_0^*) \times C[0,T^*]$ of* (1.1)-(1.5)*. The function u is the solution of* (3.4)*, and $r(t)$ is defined by* (3.3)*.*

Proof. From the previous conclusion, it is sufficient to prove the solvability of the equation (3.4) in $C^{\gamma}_{\beta}(Q_0)$. We shall use the Schauder principle. Let $\|r\|_{C[0,T]} = \max_{t\in[0,T]} |r(t)|$, and

$$
\begin{aligned}
\|v\|_{C^{\gamma}_{\beta}(Q_0)} = \max\Big\{ & \sup_{(x,t)\in Q_0} |v(x,t)|, \sup_{(x,t)\in Q_0} |D^\beta_t v(x,t)|, \\
& \sup_{(x,t)\in Q_0, |\Delta x|<1} \frac{|v(x+\Delta x,t) - v(x,t)|}{|\Delta x|^\gamma}, \\
& \sup_{(x,t)\in Q_0, |\Delta x|<1} \frac{|D^\beta_t v(x+\Delta x,t) - D^\beta_t v(x,t)|}{|\Delta x|^\gamma} \Big\},
\end{aligned}
$$

Let R be some positive number, and

$$
M_R = M_R(Q_0) = \{v \in C^{\gamma}_{\beta}(Q_0) : \|v\|_{C^{\gamma}_{\beta}(Q_0)} \le R\}.
$$

On M_R we consider the operator

$$
\begin{aligned}
(Pv)(x,t) := -\int_0^t [D^\beta F(\tau)]^{-1} d\tau \int_{\Omega_0} G_0(x,t,y,\tau) \Big[\int_{\Omega_0} v(z,\tau)\Delta\varphi_0(z)dz \\
+ F_0(z,\tau,v(z,\tau),D^\beta_t v(z,\tau))\varphi_0(z)dz - D^\alpha_t F(\tau)\Big] D^\beta_\tau v(y,\tau)dy \\
+ h_0(x,t,v,D^\beta_t v) + h(x,t), \quad (x,t)\in\Omega_0,\ v\in M_R
\end{aligned}
$$

with the functions h_0, h defined by (3.2).

At the beginning we show the existence of $R>0$, $T^*\in(0,T]$ and therefore $M_R^* = M_R(Q_0^*)$ such that $P : M_R^* \to M_R^*$.

For $v \in M_R$, $(x,t)\in\bar{Q}_0$ we find the estimates

$|h_0(x,t,v,D^\beta_t v)|$

$$
\begin{aligned}
&= \Big| \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau) F_0(y,\tau,v(y,\tau), D_t^\beta v(y,\tau)) dy \Big| \\
&\leq \int_0^t d\tau \Big[\int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|<2(t-\tau)^{\alpha/2}}} |G_0(x,t,y,\tau)|\, |F_0(y,\tau,v(y,\tau), D_t^\beta v(y,\tau))| dy \\
&\quad + \int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|>2(t-\tau)^{\alpha/2}}} |G_0(x,t,y,\tau)|\, |F_0(y,\tau,v(y,\tau), D_t^\beta v(y,\tau))| dy \Big] \\
&\leq \int_0^t d\tau \Big[\int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|<2(t-\tau)^{\alpha/2}}} \frac{C^*}{(t-\tau)|y-x|^{N-2}} \big[A_0 + |v(y,\tau|^q + |D_t^\beta v(y,\tau))|^p\big] dy \\
&\quad + \int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|>2(t-\tau)^{\alpha/2}}} \frac{C^*}{(t-\tau)^{1-\alpha}|y-x|^N} \big[A_0 + |v(y,\tau|^q + |D_t^\beta v(y,\tau))|^p\big] \Big] dy \\
&\leq C \int_0^t \Big[\frac{1}{(t-\tau)} \int_0^{2(t-\tau)^{\alpha/2}} r dr + \frac{1}{(t-\tau)^{1-\alpha}} \int_{2(t-\tau)^{\alpha/2}}^{\operatorname{diam}\Omega_0} \frac{dr}{r} \Big] d\tau \Big[A_0 + B_0(R^q + R^p)\Big] \\
&\leq \hat{C} \int_0^t \Big[(t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1} \ln \frac{\operatorname{diam}\Omega_0}{(t-\tau)^{\alpha/2}} \Big] d\tau \Big[A_0 + B_0(R^q + R^p)\Big] \\
&\leq k_0 t^{\alpha_1} \big[A_0 + B_0(R^q + R^p)\big]
\end{aligned}
$$

where $C^*, C, \hat{C}, \hat{k}, k_0$ are positive constants, and $\alpha_1 = \alpha - \varrho$, with ϱ an arbitrary number in $(0,1)$. Also we have

$$
\begin{aligned}
&\Big| \int_{\Omega_0} G_j(x,t,y) F_j(y) dy \Big| \\
&\leq \Big[\int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|<2t^{\alpha/2}}} G_j(x,t,y) dy + \int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|>2t^{\alpha/2}}} G_j(x,t,y) dy \Big] \|F_j\|_{C(\bar{\Omega}_0)} \\
&\leq c_0 \Big[\int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|<2t^{\alpha/2}}} \frac{t^{j-1-\alpha}}{|x-y|^{N-2}} dy \\
&\quad + \int_{\substack{(y,\tau)\in\Omega_0:\\ |y-x|>2t^{\alpha/2}}} \frac{t^{j-1}}{|y-x|^N} \big(\frac{|y-x|^2}{4t^\alpha} \big)^{\frac{N}{2(2-\alpha)}} e^{-c(\frac{|y-x|^2}{4t^\alpha})^{\frac{1}{2-\alpha}}} dy \Big] \|F_j\|_{C(\bar{\Omega}_0)} \\
&\leq \hat{k}_j t^{j-1} \Big[1 + \int_{2t^{\alpha/2}}^{\operatorname{diam}\Omega_0} r^{\frac{N}{2-\alpha}-1} t^{-\frac{\alpha N}{2(2-\alpha)}} e^{-\hat{c}\left(\frac{r^2}{t^\alpha}\right)^{\frac{1}{2-\alpha}}} dr \Big] \cdot \|F_j\|_{C(\bar{\Omega}_0)} \\
&\leq k_j t^{j-1} \|F_j\|_{C(\bar{\Omega}_0)}, \quad j = 1,2
\end{aligned}
$$

where $c_0, \hat{c}, \hat{k}_j, k_j$ $(j=1,2)$ are positive constants. Therefore,

$$
|h(x,t)| \leq \sum_{j=1,2} k_j t^{j-1} \|F_j\|_{C(\bar{\Omega}_0)}, \quad (x,t) \in \bar{Q}_0.
$$

Similarly,

$$
\begin{aligned}
&\Big| \int_{\Omega_0} F_0(y,\tau,v(y,\tau), D_t^\beta v(y,\tau)) \varphi_0(z) dy \Big| \\
&\leq \big[A_0 + B_0(R^q + R^p)\big] \int_{\Omega_0} |\varphi_0(z)| dz \quad \forall (y,\tau) \in \bar{Q}_0
\end{aligned}
$$

and therefore

$$\Big|\int_0^t [D^\beta F(\tau)]^{-1} d\tau \int_{\Omega_0} G_0(x,t,y,\tau) F_0(z,\tau,v(z,\tau),D_t^\beta v(z,\tau))\varphi_0(z)dz\Big| \\ \le \frac{k_0}{f} t^{\alpha_1}\big[A_0 + B_0(R^q + R^p)\big]\int_{\Omega_0} |\varphi_0(z)|dz \quad \forall (y,\tau)\in \bar{Q}_0.$$

Then, given $R \ge 1$, we obtain the estimate

$$|(Pv)(x,t)| \le \frac{k_0}{f} t^{\alpha_1}(c_1 R + c_2 R^2) + H_0(t) \le q_0 t^{\alpha_1} R^2 + H_0, \quad (x,t)\in \bar{Q}_0$$

where

$$c_1 = \int_{\Omega_0} dx \cdot \|D^\alpha F\|_{C[0,T]},$$

$$c_2 = 2B_0\Big(f + \int_{\Omega_0} |\varphi_0(z)|dz\Big) + \int_{\Omega_0} |\Delta\varphi_0(z)|dz,$$

$$H_0(t) = A_0 k_0 t^{\alpha_1}\Big(1 + \frac{1}{f}\int_{\Omega_0} |\varphi_0(z)|dz\Big) + k_1\|F_1\|_{C(\bar{\Omega}_0)} + k_2 t\|F_2\|_{C(\bar{\Omega}_0)} \le H_0,$$

$$q_0 = \frac{k_0(c_1+c_2)}{f}.$$

In the same way for $v \in M_R$, $(x,t)\in Q_0$, $|\Delta x| < 1$ we obtain

$$\frac{|(Pv)(x+\Delta x,t) - (Pv)(x,t)|}{|\Delta x|^\gamma} \le q_1 t^{\alpha_1} R^2 + H_1, \quad (x,t)\in Q_0.$$

Here and later q_j, c_j, H_j (j=1,2,...) are positive numbers.

For every $g \in C^\gamma(Q_0)$ we have

$$D_t^\beta\big(\mathfrak{G}_0 g\big)(x,t) = \int_0^t \frac{(t-\tau)^{-\beta} d\tau}{\Gamma(1-\beta)} \int_{\Omega_0} G_0(x,t,y,\tau) g(y,\tau)dy, \quad (x,t)\in Q_0$$

and, as previously, we find that

$$|D_t^\beta\big(\mathfrak{G}_0 g\big)(x,t)| \le \tilde{q}_0 t^{\alpha_1-\beta}\|g\|_{C(Q_0)}, \quad (x,t)\in Q_0, \ \ \tilde{q}_0 = const > 0.$$

Furthermore,

$$D_t^\beta\big(\mathfrak{G}_1 F_1\big)(x,t) = \frac{\partial}{\partial t}\int_0^t \frac{(t-\tau)^{-\beta} d\tau}{\Gamma(1-\beta)} \int_{\Omega_0} G_1(x,\tau,y)F_1(y)dy - f_{1-\beta}(t)F_1(x) \\ = \int_0^t f_{1-\beta}(t-\tau)\big(\mathfrak{G}_1 F_1\big)_\tau(x,\tau)d\tau,$$

$$D_t^\beta\big(\mathfrak{G}_2 F_2\big)(x,t) = \frac{\partial}{\partial t}\int_0^t \frac{(t-\tau)^{-\beta} d\tau}{\Gamma(1-\beta)} \int_{\Omega_0} G_2(x,\tau,y)F_2(y)dy - f_{2-\beta}(t)F_2(x) \\ = \int_0^t f_{1-\beta}(t-\tau)\big(\mathfrak{G}_2 F_2\big)_\tau(x,\tau)d\tau.$$

Since $\big(\mathfrak{G}_j F_j\big)_\tau$ $(j=1,2)$ are continuous functions on $\bar{Q}_0$, we have

$$\big|D_t^\beta\big(\mathfrak{G}_j F_j\big)(x,t)\big| \le c_{2+j} t^{1-\beta}, \quad (x,t)\in \bar{Q}_0, \ j=1,2.$$

So,

$$|D_t^\beta(Pv)(x,t)| \le q_2 t^{\alpha_2} R^2 + H_2, \quad (x,t)\in \bar{Q}_0$$

and similarly

$$\frac{|D_t^\beta(Pv)(x+\Delta x,t)-D_t^\beta(Pv)(x,t)|}{|\Delta x|^\gamma}\le q_3t^{\alpha_2}R^2+H_3,\quad (x,t)\in Q_0,\ |\Delta x|<1$$

for all $v\in M_R$ with $\alpha_2=\alpha_1-\beta$ and

$$\begin{aligned}H_2&=c_5+c_6\|F_1\|_{C(\bar\Omega_0)}+c_7t\|F_2\|_{C(\bar\Omega_0)},\\ H_3&=c_8+c_9\|F_1\|_{C(\bar\Omega_0)}+c_{10}\|F_2\|_{C(\bar\Omega_0)}.\end{aligned}$$

As a result we obtain

$$\begin{aligned}&\|Pv\|_{C^\gamma_\beta(Q_0)}\\ &\le \max_{t\in[0,T]}\max\{q_0t^{\alpha_1}R^2+H_0,q_1t^{\alpha_1}R^2+H_1,q_2t^{\alpha_2}R^2+H_2,q_3t^{\alpha_2}R^2+H_3\}\\ &\le \max_{t\in[0,T]}[qR^2t^{\alpha_2}+H]\quad \forall v\in M_R,\ R\ge1\end{aligned}$$

with $q=\max\{q_0T^\beta,q_1T^\beta,q_2,q_3\}$, $H=\max\{H_0,H_1,H_2,H_3\}$.

To show the inequality

$$\max\{qR^2t^{\alpha_2}+H,1\}\le R\quad \forall t\in[0,T^*],\ v\in M_R^* \tag{3.5}$$

consider the function $w(s)=qs^2t^{\alpha_2}-s$, $s\ge0$. We find $w'(s)=2qt^{\alpha_2}s-1$ and prove that $s_0=s_0(t)=[2qt^{\alpha_2}]^{-1}$ is the point of the minimum of $f(s)$. Then the inequality

$$\|Pv\|_{C^\gamma_\beta(Q_0^*)}\le R\quad \forall v\in M_R^*$$

is satisfied for some $R\ge\max\{1,H\}$, $T^*=\min\{t^*,T\}$ if $qt^{\alpha_2}s_0^2-s_0\le -H$ and $2qt^{\alpha_2}\max\{1,H\}\le1$ for all $t\in[0,t^*]$. We have $qt^{\alpha_2}s_0-s_0=-\frac{1}{4qt^{\alpha_2}}$. There exists $t^*>0$ such that $4qt^{\alpha_2}H\le1$ and $2qt^{\alpha_2}\max\{1,H\}\le1$ for all $t\in[0,t^*]$. Then

$$t^*=\min\{[\frac{1}{4qH}]^{1/\alpha_2},[\frac{1}{2q\max\{1,H\}}]^{1/\alpha_2}\}.$$

Note that (2.1) and (A4) imply $\|F_1\|_{C(\bar\Omega_0)}>0$. We have proven the existence of $R\ge1$, $T^*>0$ such that $P:M_R^*\to M_R^*$.

The operator P is continuous on

$$\tilde M_R^*=\{v\in C_\beta(Q_0^*):\|v\|_{C_\beta(Q_0^*)}=\max\{\|v\|_{C(Q_0^*)},\|D_t^\beta v\|_{C(Q_0^*)}\}\le R\};$$

thus, on M_R^*. Namely, for $v_1,v_2\in\tilde M_R^*$, $(x,t)\in Q_0^*$,

$$\begin{aligned}&|(Pv_1)(x,t)-(Pv_2)(x,t)|\\ &=\Big|\int_0^t[D^\beta F(\tau)]^{-1}d\tau\int_{\Omega_0}G_0(x,t,y,\tau)\Big[\int_{\Omega_0}v_2(z,\tau)\Delta\varphi_0(z)dz[D_\tau^\beta v_2(y,\tau)\\ &\quad-D_\tau^\beta v_1(z,\tau)]-\int_{\Omega_0}[v_1(z,\tau)-v_2(z,\tau)]\Delta\varphi_0(z)dz\cdot D_\tau^\beta v_1(y,\tau)\\ &\quad+\int_{\Omega_0}[F_0(z,\tau,v_1(z,\tau),D_\tau^\beta v_1(z,\tau)-F_0(z,\tau,v_2(z,\tau),D_\tau^\beta v_2(z,\tau)]\varphi_0(z)dz\Big]dy\\ &\quad+h_0(z,\tau,v_1(z,\tau),D_\tau^\beta v_1(z,\tau)-h_0(z,\tau,v_2(z,\tau),D_\tau^\beta v_2(z,\tau)\Big|\\ &\le c_{11}\sup_{(x,t)\in Q_0^*}\int_0^t[D^\beta F(\tau)]^{-1}d\tau\int_{\Omega_0}\big|G_0(x,t,y,\tau)\big|dy\end{aligned}$$

$$\begin{aligned}
&\times \Big[\|v_2\|_{C(Q_0^*)}\|D^\beta v_1 - D^\beta v_2\|_{C(Q_0^*)} + \|D^\beta v_1\|_{C(Q_0^*)}\|v_1 - v_2\|_{C(Q_0^*)} \\
&+ 2D_0\|v_1 - v_2\|^q_{C(Q_0^*)} + 2D_0\|D^\beta v_1 - D^\beta v_2\|^p_{C(Q_0^*)}\Big] \\
&+ 2D_0 \sup_{(x,t)\in Q_0^*}\int_0^t d\tau \int_{\Omega_0} \big|G_0(x,t,y,\tau)\big| dy \Big[\|v_1 - v_2\|^q_{C(Q_0^*)} \\
&+ 2D_0\|D^\beta v_1 - D^\beta v_2\|^p_{C(Q_0^*)}\Big] \\
&\le c_{12}(T^*)^{\alpha_1} R\|v_1 - v_2\|_{C^\gamma_\beta(Q_0^*)}.
\end{aligned}$$

Similarly, by using previous estimates,

$$|D^\beta_t(Pv_1)(x,t) - D^\beta_t(Pv_2)(x,t)| \le c_{13}(T^*)^{\alpha_2} R\|v_1 - v_2\|_{C^\gamma_\beta(Q_0^*)}$$

for all $v_1, v_2 \in \tilde{M}^*_R$, $(x,t) \in \bar{Q}^*{}_0$.

The operator P is compact on $\tilde{M}^*_R$ (and thus on M^*_R): it was established earlier the uniform boundedness of the set

$$P\tilde{M}^*_R := \{(Pv)(x,t),\, (x,t) \in Q_0^* : v \in \tilde{M}^*_R\}$$

in addition, it follows from the properties of Green's operators that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for all $(x,t) \in Q_0^*$, $|\Delta x| < \delta$, $|\Delta t| < \delta$ and for all $v \in \tilde{M}^*_R$

$$\sup_{(x,t)\in Q_0^*} |(Pv)(x+\Delta x, t+\Delta t) - (Pv)(x,t)| < \varepsilon,$$

$$\sup_{(x,t)\in Q_0^*} |D^\beta_t(Pv)(x+\Delta x, t+\Delta t) - D^\beta_t(Pv)(x,t)| < \varepsilon.$$

As a result, the operator P is equicontinuous on M^*_R. According to Schauder principle there exists a solution $u \in M^*_R$ of the equation (3.4). □

Theorem 3.2. *Assume that $F_0 \in C^1(Q_0 \times \mathbb{R}^2)$ and is bounded, $D^\beta F \in C[0,T]$ and $D^\beta F(t) \ne 0$, $t \in [0,T]$, φ satisfies the assumption (A5) and $\varphi(x) \ne 0$, $x \in \Omega_0$. Then a solution $(u,r) \in C_{2,\alpha}(\bar{Q}_0) \times C[0,T]$ of the problem* (1.1)-(1.5) *is unique.*

Proof. Take two solutions $(u_1,r_1), (u_2,r_2) \in C_{2,\alpha}(\bar{Q}_0) \times C[0,T]$ of (1.1)-(1.5) and substitute them into equation (1.1). For $u = u_1 - u_2$, $r = r_1 - r_2$ we obtain the equation

$$D^\alpha_t u = \Delta u - r_1(t)D^\beta_t u - r(t)D^\beta_t u_2 + F_0(x,t,u_1,D^\beta_t u_1) - F_0(x,t,u_2,D^\beta_t u_2).$$

By Hadamard lemma

$$F_0(x,t,u_1,D^\beta_t u_1) - F_0(x,t,u_2,D^\beta_t u_2) = F_{01}(x,t)u + F_{02}(x,t)D^\beta_t u$$

with some known functions F_{0j}, $j = 1,2$, which are continuous and bounded on Q_0, depend on u_1, u_2, $D^\beta_t u_1, D^\beta_t u_2$. Then the previous equation becomes

$$D^\alpha_t u + \big(r_1(t) - F_{02}(x,t)\big)D^\beta_t u = \Delta u + F_{01}(x,t)u - r(t)D^\beta_t u_2, \quad (x,t) \in Q_0.$$

It follows from the boundary condition that

$$u(x,t) = 0, \quad x \in \bar{\Omega}_1, \quad t \in [0,T],$$

and from the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in \bar{\Omega}_0.$$

Let $G_0^*(x,t,y,\tau)$ be the main Green's function of the first boundary-value problem for the equation

$$D_t^\alpha u + \big(r_1(t) - F_{02}(x,t)\big) D_t^\beta u = \Delta u + F_{01}(x,t)u.$$

Then the function $u(x,t)$ satisfies the equation

$$u(x,t) = -\int_0^t d\tau \int_{\Omega_0} G_0^*(x,t,y,\tau) D_\tau^\beta u_2(y,\tau) r(\tau) dy, \quad (x,t)\in \bar{Q}_0.$$

It follows from the over-determination condition (1.5) that

$$\begin{aligned} r(t)D^\beta F(t) = \int_{\Omega_0} \Big[& u(x,t)\Delta\varphi_0(x) \\ & + \Big(F_{01}(x,t)u(x,t) + F_{02}(x,t)D_t^\beta u(x,t)\Big)\varphi_0(x)\Big] dx, \end{aligned} \tag{3.6}$$

and then $u(x,t)$ satisfies the equation

$$\begin{aligned} u(x,t) = \int_0^t \frac{d\tau}{D^\beta F(\tau)} \int_{\Omega_0} K^*(x,t,\tau) \Big[& \big(F_{01}(z,\tau)\varphi_0(z) + \Delta\varphi_0(z)\big) u(z,\tau) dz \\ & + F_{02}(z,\tau)\varphi_0(z) D_\tau^\beta u(z,\tau)\Big] dz, \quad (x,t)\in\bar{Q}_0 \end{aligned} \tag{3.7}$$

where

$$K^*(x,t,\tau) = \int_{\Omega_0} G_0^*(x,t,y,\tau) D_\tau^\beta u_2(y,\tau) dy$$

is continuous with respect to x and integrable in time function.

If $F_{01}(z,\tau) = F_{02}(z,\tau) = 0$, $(z,\tau)\in Q_0$ then by the uniqueness of the solution of this linear second type Volterra integral equation we obtain $u(x,t) = 0$, $(x,t)\in Q_0$. Then it follows from (3.6) that $r(t)D^\beta F(t) = 0$, $t\in[0,T]$. Since $D^\beta F(t) \neq 0$ on $[0,T]$ (under the assumptions of this theorem), it follows that $r(t)\equiv 0$ for $t\in[0,T]$.

Assume that

$$|F_{01}(z,\tau)| + |F_{02}(z,\tau)| \neq 0, \quad (z,\tau)\in Q_0. \tag{3.8}$$

If $F_{02}(z,\tau) = 0$, $(z,\tau)\in Q_0$ then by the uniqueness of the solution of the linear second type Volterra integral equation (3.7) with integrable kernel

$$K^*(x,t,\tau)\big(F_{01}(z,\tau)\varphi_0(z) + \Delta\varphi_0(z)\big)$$

we obtain, as in previous case, that $u(x,t) = 0$, $(x,t)\in Q_0$ and then, from (3.6), that $r(t)\equiv 0$, $t\in[0,T]$.

In the general case denote

$$V(x,t) = \big(F_{01}(x,t)\varphi_0(x) + \Delta\varphi_0(x)\big)u(x,t) + F_{02}(x,t)\varphi_0(x)D_t^\beta u(x,t).$$

Then (3.7) implies

$$\begin{aligned} V(x,t) = & \int_0^t \frac{d\tau}{D^\beta F(\tau)} \int_{\Omega_0} \big(F_{01}(x,t)\varphi_0(x) + \Delta\varphi_0(x)\big) K^*(x,t,\tau) V(z,\tau) dz \\ & + F_{02}(x,t)\varphi_0(x) D_t^\beta \int_0^t \frac{d\tau}{D^\beta F(\tau)} \int_{\Omega_0} K^*(x,t,\tau) V(z,\tau) dz; \end{aligned}$$

that is,

$$V(x,t) = \int_0^t d\tau \int_{\Omega_0} K(x,t,z,\tau) V(z,\tau) dz, \quad (x,t)\in\bar{Q}_0,$$

where

$$
\begin{aligned}
&K(x,t,z,\tau)\\
&= \frac{1}{D^\beta F(\tau)}\Big[F_{01}(x,t)\varphi_0(x) + \Delta\varphi_0(x) + \frac{F_{02}(x,t)\varphi_0(x)(t-\tau)^{-\beta}}{\Gamma(1-\beta)}\Big]K^*(x,t,\tau).
\end{aligned}
$$

By the uniqueness of the solution of this linear second type Volterra integral equation with integrable kernel $K(x,t,z,\tau)$ we obtain $V(x,t)=0$, $(x,t)\in Q_0$.

Note that (3.8) implies

$$
|F_{01}(x,t)\varphi_0(x) + \Delta\varphi_0(x)| + |F_{02}(x,t)\varphi_0(x)| \neq 0, \quad (x,t)\in Q_0.
$$

Note also that

$$
\begin{aligned}
D_t^\beta u(x,t) = 0 &\iff f_{-\beta}(t) * u(x,t) = 0\\
&\iff f_\beta(t) * f_{-\beta}(t) * u(x,t) = 0\\
&\iff u(x,t) = 0, \ \ (x,t)\in Q_0,
\end{aligned}
$$

for the function $u(x,t)$ satisfying zero initial conditions.

Then it follows from the previous results that $u(x,t)=0$, $(x,t)\in Q_0$ and from (3.6), by the assumptions of this theorem, we obtain $r(t)\equiv 0$, $t\in[0,T]$.

In separate case $F_{01}(x,t)\varphi_0(x) + \Delta\varphi_0(x) = 0$ for all $(x,t)\in Q_0$ we may put $V_1(x,t) = F_{02}(x,t)\varphi_0(x)D_t^\beta u(x,t)$ and, as before, obtain the linear second type Volterra integral equation

$$
V_1(x,t) = \int_0^t d\tau \int_{\Omega_0} K_1(x,t,z,\tau)V_1(z,\tau)dz, \quad (x,t)\in \bar{Q}_0
$$

with integrable kernel

$$
K_1(x,t,z,\tau) = F_{02}(x,t)\varphi_0(x)K^*(x,t,\tau)(t-\tau)^{-\beta}/\Gamma(1-\beta)D^\beta F(\tau).
$$

As before, from here we obtain $V_1(x,t)=0$, $(x,t)\in Q_0$ and, as in the general case, since $F_{02}(x,t)\varphi_0(x)\neq 0$ on Q_0, conclude that $u(x,t)=0$, $(x,t)\in Q_0$ and $r(t)\equiv 0$, $t\in[0,T]$. □

The similar result holds for the inverse problem on determination of a pair of functions (u,b): a solution u of the first (or second) boundary-value problem for the equation

$$
D_t^\alpha u = \Delta u + b(t)u = F_0(x,t), \quad (x,t)\in Q_0
$$

and an unknown coefficient $b(t)$ under the same over-determinating condition (1.2). We may study the cases $N=1,2$ in the same way.

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