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STURM-PICONE TYPE THEOREMS FOR NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract. In this article, we establish a Picone-type inequality for a pair of first-order nonlinear differential systems. By using this inequality, we give Sturm-Picone type comparison theorems for these systems and a special class of second-order half-linear equations with damping term.

1. Introduction

Let $\alpha > 0$ and define $\varphi_\alpha(s) = |s|^{\alpha-1}s$ if $s \neq 0$ and $\varphi_\alpha(0) = 0$. By comparing with the zeros of the first component of the solution of the system

$$\begin{aligned} x' &= a(t)x + b(t)\varphi_{1/\alpha}(y) \\ y' &= -c(t)\varphi_\alpha(x) - d(t)y \end{aligned} \tag{1.1}$$

we would like to obtain some information about the existence and distribution of zeros of the first component of the solution of the system

$$\begin{aligned} u' &= A(t)u + B(t)\varphi_{1/\alpha}(v) \\ v' &= -C(t)\varphi_\alpha(u) - D(t)v \end{aligned} \tag{1.2}$$

where a, A, b, B, c, C, d and D are continuous real-valued functions on a given interval I and $b(t) > 0$ and $B(t) > 0$ in I. The existence and uniqueness of the solution of the initial and boundary value problems for (1.1) (or (1.2)) were considered by Elbert [7] and Mirzov [16, 17].

We have the following special cases, considering, for example the second system:

If $A(t) \equiv D(t)$ in I, then (1.2) is the nonlinear Hamiltonian system

$$u' = \frac{\partial H}{\partial v}, \quad v' = -\frac{\partial H}{\partial u}$$

where

$$H(t; u, v) = \frac{1}{\alpha+1} C(t)|u|^{\alpha+1} + A(t)uv + \frac{\alpha}{\alpha+1} B(t)|v|^{1+\frac{1}{\alpha}}. \tag{1.3}$$

When $A(t) \equiv 0$ in I, the system (1.2) is equivalent to the scaler second-order half-linear equation

$$(P(t)\varphi_\alpha(u'))' + R(t)\varphi_\alpha(u') + Q(t)\varphi_\alpha(u) = 0 \tag{1.4}$$

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where the coefficient functions are

$$P(t) \equiv B(t)^{-\alpha}, \quad R(t) = D(t)B(t)^{-\alpha}, \quad Q(t) = C(t).$$

If $A(t) \equiv 0$ and $D(t) \equiv 0$ in I, then (1.4) reduced to the half-linear Sturm-Liouville equation

$$(P(t)\varphi_\alpha(u'))' + Q(t)\varphi_\alpha(u) = 0. \tag{1.5}$$

Moreover, if we take the transformation

$$\begin{aligned} u &= h(t)W \\ v &= \frac{1}{h(t)}z \end{aligned} \tag{1.6}$$

where $h'(t) = A(t)h(t)$, i.e $h(t) = \exp\big(\int_{t_0}^t A(s)ds\big)$ in system (1.2) with $A(t) \equiv D(t)$, is equivalent for any $r \in C^1(I)$

$$\Big(P_1(t)\varphi_\alpha(W')\Big)' + R_1(t)\varphi_\alpha(W) + Q_1(t)\varphi_\alpha(W) = 0 \tag{1.7}$$

where the coefficient function are

$$P_1(t) = r(t), \quad R_1(t) = (\alpha+1)r(t)A(t) - r'(t) - \alpha r(t)\frac{B'(t)}{B(t)},$$
$$Q_1(t) = r(t)C(t)B^\alpha(t).$$

It is not difficult to see that if we choose $r(t) = B^{-\alpha}(t)$ we get the scalar second-order half-linear equation

$$\Big(P(t)\varphi_\alpha(W')\Big)' + (\alpha+1)R(t)\varphi_\alpha(W') + Q(t)\varphi_\alpha(W) = 0 \tag{1.8}$$

where P, R and Q are defined as in (1.4).

Most of the classical results in oscillation theory are formulated for the solutions of the self-adjoint Sturm-Liouville equations of the form

$$-(p_1(x)u')' + p_0(x)u = 0, \tag{1.9}$$

$$-(P_1(x)v')' + P_0(x)v = 0, \tag{1.10}$$

where p_0, p_1, P_0, P_1 are real valued continuous functions and p_1 and P_1 are positive on an appropriate interval. The starting point for this theory is the well known comparison theorem for Sturm [20] discovered in 1836.

Theorem 1.1 (Sturm Comparison Theorem). *Suppose that $p_1(x) \equiv P_1(x)$ and $P_0(x) \le p_0(x)$ and $P_0(x) \ne p_0(x)$ for $x \in [x_1, x_2]$. If x_1 and x_2 are consecutive zeros of a nontrivial real solutions u of* (1.9)*, then every real solution of v of* (1.10) *has a zero in (x_1, x_2).*

In 1909, Picone [19] modified Sturm's theorem as follows.

Theorem 1.2 (Sturm-Picone Theorem). *Suppose that $0 < P_1(x) \le p_1(x)$ and $P_0(x) \le p_0(x)$ for $[x_1, x_2]$. If x_1 and x_2 are consecutive zeros of a nontrivial real solutions u of* (1.9)*, then every real solution of v of* (1.10) *has one of the following properties:*

(i) *$v(x)$ has a zero in (x_1, x_2)*
(ii) *$v(x)$ is a constant multiple of $u(x)$.*

Note that Theorem 1.2 is a special case of Leighton's theorem [15]. For a detailed study and earlier developments of this subject, we refer the reader to the books [14, 21].

The original proof by Picone was based on the identity

$$\frac{d}{dx}\Big[\frac{u}{v}(vp_1u' - uP_1v')\Big] = (p_0 - P_0)u^2 + (p_1 - P_1)u'^2 + P_1\big(u' - \frac{u}{v}v'\big)^2 \tag{1.11}$$

which holds for all real valued functions u and v defined on $[x_1, x_2]$ such that u, v, p_1u' and P_1v' are differentiable on $[x_1, x_2]$ and $v(x) \neq 0$ for $x \in [x_1, x_2]$.

The identity (1.11) has been a useful tool not only in comparing equations (1.9) and (1.10) but also in establishing Wirtinger type inequalities for the second-order linear ordinary differential equation and lower bounds for the eigenvalues of the associated eigenvalue problems and was generalized to high-order ordinary differential operators as well as the partial differential operators of the elliptic type [4, 6, 9, 11, 13, 28, 29].

Sturm-Picone theorem is extended in several directions, see, Ahmad and Lazer [1] and Ahmad [2] for linear systems, Müller-Pfeiffer [18] for non-selfadjoint differential equations, Tyagi and V. Raghavenda [26] for implicit differential equations, W. Allegretto [5] for degenerate elliptic equations, Zhang and Sun [30] for linear equations on time scales, Jaroš and Kusano [9] for half linear equations, [23, 27] for nonlinear equations. There is also a good amount of interest in the qualitative theory of partial differential equations to determine whether the given equation is oscillatory or not. In this direction, Sturm-Picone theorem plays an important role. [3, 5, 6, 8, 10, 11, 21, 22, 24, 28, 29].

In 1999, Jaros and Kusano [9] generalized Picone's identity (1.11) to the class of nonlinear second-order differential equations

$$\Big(p_1(x)\varphi(u')\Big)' + p_0(x)\varphi(u) = 0, \tag{1.12}$$

$$\Big(P_1(x)\varphi(v')\Big)' + P_0(x)\varphi(v) = 0, \tag{1.13}$$

where $\varphi(s) := |s|^{\alpha-1}s$, $\alpha > 0$, p_1, p_0, P_1, P_0 are defined as before. The above equations are also called half-linear or sometimes homogeneous of degree α. They established a suitable Picone-type identity as follows

$$\begin{aligned} &\frac{d}{dt}\{\frac{u}{\varphi(v)}\left(\varphi(v)p_1\varphi(u') - \varphi(u)P_1\varphi(v')\right)\} \\ &= (p_1 - P_1)\,|u'|^{\alpha+1} + (P_0 - p_0)\,|u|^{\alpha+1} \\ &+ P_1\big[|u'|^{\alpha+1} + \alpha|\frac{uv'}{v}|^{\alpha+1} - (\alpha+1)u'\varphi\Big(\frac{uv'}{v}\Big)\big]. \end{aligned} \tag{1.14}$$

Using the above identity, they obtained the following comparison results which is extension of Theorem 1.2 to the class of half linear equations (1.12) and (1.13).

Theorem 1.3 ([9]). *Suppose that* $0 < P_1(x) \leq p_1(x)$ *and* $p_0(x) \leq P_0(x)$ *for* $x \in [x_1, x_2]$. *If* x_1, x_2 *are consecutive zeros of a nontrivial real solution* u *of* (1.12), *then every solution* v *of (1.8) has a zero in* (x_1, x_2) *except possibly it is a constant multiple of* u.

While qualitative theory of scalar cases are well-developed, only little is known about the general systems, particularly in the case where $a(t) \neq 0$, $A(t) \neq 0$ or $a(t) \neq d(t)$, $A(t) \neq D(t)$ in I (for some results concerning the case $\alpha = 1$ see

[13, 14]). Elbert [7], proved that if $b(t) > 0$ and $B(t) > 0$ on I and (x, y) is a solution of (1.2) such that the function $x(t)$ has consecutive zeros at $t_1, t_2 \in I$ and (1.1) is a Sturmian majorant for (1.2) in the sense that

$$[B(t) - b(t)]|\xi|^{\alpha+1} + \big[A(t) - a(t) - \frac{d(t) - D(t)}{\alpha}\big]\xi\varphi_\alpha(\eta) + \frac{C(t) - c(t)}{\alpha}|\eta|^{\alpha+1} \geq 0, \tag{1.15}$$

for all $\xi, \eta \in R$ and $t \in I$, then for any solution (u, v) of (1.1) the first component $u(t)$ has at least one zero in (t_1, t_2).

Note that the inequality (1.15) holds for $\xi, \eta \in R\backslash\{0\}$ if $B(t) > b(t)$, $C(t) > c(t)$, and

$$\big(B(t) - b(t)\big)\big(C(t) - c(t)\big)^{\alpha+1} \geq \big(\frac{\alpha}{\alpha+1}\big)^{\alpha+1}|\alpha\big(A(t) - a(t)\big) - (d(t) - D(t))\,|^{\alpha+1}.$$

Elbert proved his result by means of the generalized Prüfer transformation. In the particular case $a(t) \equiv A(t) \equiv D(t) \equiv 0$, Elbert's criterion reduces to the half-linear generalization of the classical Sturm-Picone comparison theorem due to Mirzov [16].

Recently, Jaroš studied the system (1.2) under suitable sufficient conditions. He established Picone-type identity for the nonlinear system of the form (1.2) and applied it to derive Wirtinger type inequalities. He also gave some results to obtain information about the existence and distribution of zeros of the first component of the solution of (1.2). Indeed the following result is interesting.

Theorem 1.4 ([12]). *If for some nontrivial C^1-function x defined on $[t_1, t_2]$ and satisfying $x(t_1) = x(t_2) = 0$, the condition*

$$J(x) = \int_{t_1}^{t_2} \Big[B(t)^{-\alpha}|x' - \frac{\alpha A(t) + D(t)}{\alpha + 1}x|^{\alpha+1} - c(t)|x|^{\alpha+1}\Big]dt \leq 0$$

holds, then for any solution (u, v) of (1.2) the first component $u(t)$ either has a zero in (t_1, t_2) or is a constant multiple of $x(t)\exp\big(\int_{t_0}^{t}\frac{A(s)-D(s)}{\alpha+1}ds\big)$ for some $t_0 \in I$

We would like to obtain some information about the existence and distribution of the zeros of the first component of the solution of (1.2) by comparing with the zeros of the first component of the solution of (1.1) and obtain sufficient conditions for the case including $B(t) \geq b(t)$ and $C(t) \geq c(t)$.

Note that our results, that are formulated in terms of the continuous function $a(t) = \frac{\alpha A(t)+D(t)}{(\alpha+1)}$ yield a variety of comparison results. Even if we reduce our consideration to the special cases of $a(t)$ mentioned above, our results seem to be new.

2. Picone-type inequality and Leightonian comparison theorems

Let

$$\Phi_\alpha(\xi, \eta) := \xi\varphi_\alpha(\xi) + \alpha\eta\varphi_\alpha(\eta) - (\alpha + 1)\xi\varphi_\alpha(\eta). \tag{2.1}$$

for $\varepsilon, \eta \in R$ and $\alpha > 0$. From the Young inequality, it follows that $\Phi_\alpha(\xi, \eta) \geq 0$ for all $\xi, \eta \in R$, and the equality holds if and only if $\xi = \eta$. The Picone-type inequality in the following lemma is of basic importance for our main results, it may be verified directly by differentiation.

Lemma 2.1 (Picone-type inequality). *Suppose that (u, v) is a solution of (1.2) such that $u(t) \neq 0$ in I. If there exists a solution (x, y) of (1.1), then*

$$
\begin{aligned}
\frac{d}{dt}\Big[&\frac{x}{\varphi_\alpha(u)} \\
\Big(\varphi_\alpha(u)y - \varphi_\alpha(x)v\Big)\Big] &\geq \Big[C(t) - c(t) - \frac{1}{\alpha+1}|a(t)-d(t)|\Big]x\varphi_\alpha(x) \\
&\quad + \Big[b(t) - \frac{b^{\alpha+1}(t)}{B^\alpha(t)} - \frac{\alpha}{\alpha+1}|a(t)-d(t)|\Big]y\varphi_{1/\alpha}(y) \\
&\quad + B^{-\alpha}(t)\Phi_\alpha(b(t)\varphi_{1/\alpha}(y), B(t)\frac{x}{u}\varphi_{1/\alpha}(v)) \\
&\quad - \Big[(\alpha+1)a(t) - \alpha A(t) - D(t)\Big]x\varphi_\alpha\Big(\frac{x}{u}\Big)v.
\end{aligned}
\tag{2.2}
$$

We begin with the following functionals $V_{\sigma\tau}$ and $M_{\sigma\tau}$ defined for $t_1 < \sigma < \tau < t_2$ and solutions (x, y) of (1.1) and (u, v) of (1.2) with $u(t) \neq 0$ in I by

$$
\begin{aligned}
&V_{\sigma\tau}(x) \\
&= \int_\sigma^\tau \{\Big[C(t) - c(t) - \frac{1}{\alpha+1}|a(t)-d(t)|\Big]|x|^{\alpha+1} \\
&\quad + \Big[b(t) - \frac{b^{\alpha+1}(t)}{B^\alpha(t)} - \frac{\alpha}{\alpha+1}|a(t)-d(t)|\Big]b^{-(\alpha+1)}(t)|x' - a(t)x|^{(\alpha+1)}\}dt
\end{aligned}
\tag{2.3}
$$

and

$$
M_{\sigma\tau}[x; u, v] = \int_\sigma^\tau B^{-\alpha}(t)\Big(\Phi_\alpha(x' - ax, B(t)\frac{x}{u}\varphi_{1/\alpha}(v)\Big)dt.
$$

From Lemma 2.1, by using the definition of $V_{\sigma\tau}(x)$ we have the following lemma.

Lemma 2.2. *Let (x, y) and (u, v) be solutions of (1.1) and (1.2) respectively such that $u(t) \neq 0$ in I and let $[\sigma, \tau] \subset I$. Then for the first component $x(t)$ of the solution of (1.1), the following inequality holds:*

$$
\begin{aligned}
&\big[\frac{x}{\varphi_\alpha(u)}(\varphi_\alpha(u)y - \varphi_\alpha(x)v)\big]\big|_\sigma^\tau \\
&\geq V_{\sigma\tau}(x) - \int_\sigma^\tau \big[(\alpha+1)a(t) - \alpha A(t) - D(t)\big]x\varphi_\alpha\Big(\frac{x}{u}\Big)vdt.
\end{aligned}
\tag{2.4}
$$

Moreover, the inequality holds in (2.4) if and only if

$$
x' = \Big(a(t) + B(t)\frac{\varphi_{1/\alpha}(v)}{u}\Big)x. \tag{2.5}
$$

Proof. Integrating (2.2) from σ to τ and using positive semidefiniteness of the form Φ_α, we obtain

$$
\begin{aligned}
&\big[\frac{x}{\varphi_\alpha(u)}(\varphi_\alpha(u)y - \varphi_\alpha(x)v)\big]\big|_\sigma^\tau \\
&\geq V_{\sigma\tau}(x) + M_{\sigma\tau}(x; u, v) - \int_\sigma^\tau \big[(\alpha+1)a(t) - \alpha A(t) - D(t)\big]x\varphi_\alpha\Big(\frac{x}{u}\Big)vdt \\
&\geq V_{\sigma\tau}(x) - \int_\sigma^\tau \big[(\alpha+1)a(t) - \alpha A(t) - D(t)\big]x\varphi_\alpha\Big(\frac{x}{u}\Big)vdt
\end{aligned}
\tag{2.6}
$$

which gives (2.4). The equality obviously holds in (2.4) if and only if $\Phi_\alpha(x' - ax, B(t)\frac{x}{u}\varphi_{1/\alpha}(v)) = 0$ in $[\sigma, \tau]$ which is equivalent with the condition (2.5). □

From Lemma 2.2, we easily obtain the variation $V(x)$ and $M(x;u,v)$ if we assume the existence of the limits

$$V(x) = \lim_{\sigma\to t_1^+,\tau\to t_2^-} V_{\sigma\tau}(x), \quad M(x;u,v) = \lim_{\sigma\to t_1^+,\tau\to t_2^-} M_{\sigma\tau}(x;u,v). \tag{2.7}$$

Now define the domains D_V and D_M of V and M respectively, to be sets of all real-valued solutions of (1.1) such that $V(x)$ and $M(x;u,v)$ exist. Also for the solution $x \in D_V \cap D_M$ of (1.1) and the solution (u,v) of (1.2) with $u(t) \neq 0$ in $I = (t_1,t_2)$, we denote

$$\begin{aligned} S_1(x;u,v) &= \lim_{t\to t_1^+} \Big[x\varphi_\alpha\Big(\frac{x'-a(t)x}{b(t)}\Big) - x\varphi_\alpha\Big(\frac{x}{u}\Big)v\Big] \\ S_2(x;u,v) &= \lim_{t\to t_2^-} \Big[x\varphi_\alpha\Big(\frac{x'-a(t)x}{b(t)}\Big) - x\varphi_\alpha\Big(\frac{x}{u}\Big)v\Big] \end{aligned} \tag{2.8}$$

whenever the limits in (2.8) exist.

Theorem 2.3. *Let (x,y) and (u,v) be solutions of* (1.1) *and* (1.2) *respectively with $u(t)\neq 0$ in I satisfying*

$$\Big[(\alpha+1)a(t) - \alpha A(t) - D(t)\Big]\frac{v}{\varphi_\alpha(u)} \le 0 \tag{2.9}$$

in I. Then the solution $x \in D_V \cap D_M$ of (1.1) *for which the limits in* (2.8) *exist, the inequality*

$$S_2(x;u,v) - S_1(x;u,v) \ge V(x) \tag{2.10}$$

holds. Furthermore if $[(\alpha+1)a(t) - \alpha A(t) - D(t)]\frac{v}{\varphi_\alpha(u)} = 0$ in I, then the equality in (2.10) *occurs if and only if $x(t)$ is a solution of* (2.5).

As an immediate consequence of the above theorem we have the following result.

Corollary 2.4. *Let (x,y) and (u,v) be solutions of* (1.1) *and* (1.2) *respectively with $u(t)\neq 0$ in I and*

$$\Big[(\alpha+1)a(t) - \alpha A(t) - D(t)\Big]\frac{v}{\varphi_\alpha(u)} = 0 \tag{2.11}$$

in I. Then for every solution $x \in D_V \cap D_M$ of (1.1) *for which both limits in* (2.8) *exists, the inequality* (2.10) *is valid. Moreover, the inequality holds in* (2.10) *if and only if*

$$x(t) = Ku(t)\exp\Big(\int_{t_0}^t (a(s) - A(s))ds\Big)$$

for some constants $K \neq 0$ and $t_0 \in I$.

In the case where $a(t) \equiv A(t) \equiv D(t)$ in I the condition (2.9) is trivially satisfied. Clearly, in this special case, the equality in (2.10) is satisfied if and only if $x(t)$ is a constant multiple of $u(t)$.

Another way, to guarantee the equality in (2.9) is to choose

$$a(t) = \frac{\alpha A(t) + D(t)}{\alpha+1}. \tag{2.12}$$

By choosing $a(t)$ this way, we have the following important results.

Corollary 2.5. *If (x,y) and (u,v) are solutions of* (1.1) *and* (1.2) *respectively with $u(t)\neq 0$ in I and $x\in D_V\cap D_M$ is such that the limits in* (2.8) *exist and satisfy $S_2(x;u,v)\geq 0$, $S_1(x;u,v)\leq 0$, then*

$$\begin{aligned}
V(x) = \int_{t_1}^{t_2} \Big\{ &\Big[C(t) - c(t) - \frac{1}{\alpha+1}\big|\frac{\alpha A(t)+D(t)}{\alpha+1} - d(t)\big|\Big]|x|^{\alpha+1} \\
&+ \Big[b(t) - \frac{b^{\alpha+1}(t)}{B^{\alpha}(t)} - \frac{\alpha}{\alpha+1}\big|\frac{\alpha A(t)+D(t)}{\alpha+1} - d(t)\big|\Big] \\
&\times B^{-(\alpha+1)}|x' - \frac{\alpha A(t)+D(t)}{\alpha+1}x|^{\alpha+1}\Big\}dt \leq 0
\end{aligned} \tag{2.13}$$

Furthermore, the equality in (2.13) *is satisfied if and only if*

$$x(t) = Ku(t)\exp\Big(-\int_{t_0}^{t}\frac{(A(s)-D(s))}{\alpha+1}ds\Big)$$

for some $t_0\in I$.

Corollary 2.6. *Let $V(x)$ be defined as in* (2.13). *If (x,y) is a solution of* (1.1) *satisfying $x(t_1)=x(t_2)=0$, the condition $V(x)\geq 0$ holds, then for any solution (u,v) of* (1.2) *the first component $u(t)$ has one of the following properties:*

(i) *u has a zero in (t_1,t_2) or,*
(ii) *u is a nonzero constant multiple of $x(t)\exp\big(\int_{t_0}^{t}\frac{(A(s)-D(s))}{\alpha+1}ds\big)$, for some $t_0\in I$.*

Remark 2.7. If the condition $V(x)\geq 0$ is strengthened to $V(x)>0$, conclusion (ii) of Corollary 2.6 does not hold.

From Corollary 2.6 we immediately have the following result which is an extension of Sturm-Picone Comparison Theorem of the systems (1.1) and (1.2).

Theorem 2.8. *Suppose there exists a nontrivial solution (x,y) of* (1.1) *in (t_1,t_2) such that $x(t_1)=x(t_2)=0$. If*

$$C(t) \geq c(t) + \frac{1}{\alpha+1}\Big|\frac{\alpha A(t)+D(t)}{\alpha+1} - d(t)\Big| \tag{2.14}$$

and

$$b(t) \geq \frac{b^{\alpha+1}(t)}{B^{\alpha}(t)} + \frac{\alpha}{\alpha+1}\Big|\frac{\alpha A(t)+D(t)}{\alpha+1} - d(t)\Big|$$

for every $t\in(t_1,t_2)$, then the first component $u(t)$ of every nontrivial solution (u,v) of (1.2) *has at least one zero in (t_1,t_2) unless u is a nonzero constant multiple of*

$$x(t)\exp\Big(\int_{t_0}^{t}\frac{A(s)-D(s)}{\alpha+1}ds\Big).$$

Remark 2.9. Note that when $a(t)\equiv d(t)\equiv A(t)\equiv D(t)$, the case (2.12) is already satisfied, hence we can obtain special cases of the above results given in Corollary 2.5-2.6 and Theorem 2.8.

Now we consider a class of second-order half-linear equations with damping term:

$$\Big(b^{-\alpha}(t)\varphi_\alpha(w')\Big)' + (\alpha+1)b^{-\alpha}(t)D(t)\varphi_\alpha(w') + c(t)\varphi_\alpha(w) = 0, \tag{2.15}$$

$$\Big(B^{-\alpha}(t)\varphi_\alpha(W')\Big)' + (\alpha+1)B^{-\alpha}(t)D(t)\varphi_\alpha(W') + C(t)\varphi_\alpha(W) = 0 \tag{2.16}$$

Note that Equation (2.16) is the same as Equation (1.8), which is obtained from (1.2), Equation (2.15) can be obtained from (1.1) using similar transformations. From Remark 2.9, we immediately have the following theorem which is straightforward Sturm-Picone comparison result for the above damped half-linear equations.

Theorem 2.10. *Suppose that there exists a nontrivial real solution w of* (2.15) *in (t_1, t_2) such that $w(t_1) = 0 = w(t_2)$. If $B(t) \geq b(t)$ and $C(t) \geq c(t)$, then every nontrivial solution W of* (2.16) *either has a zero in (t_1, t_2) or it is a nonzero constant multiple of w.*

Remark 2.11. Note that, Theorem 2.10 is a partial answer to the open problem given in [25].

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