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FERMAT TYPE DIFFERENTIAL AND DIFFERENCE EQUATIONS

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Abstract. This article we explore the relationship between the number of differential and difference operators with the existence of meromorphic solutions of Fermat type differential and difference equations. Some Fermat differential and difference equations of certain types are also considered.

1. Introduction

Some classical results on the meromorphic solutions of Fermat type functional equation

$$f(z)^n + g(z)^n = 1 \tag{1.1}$$

can be stated as follows. Gross [2] proved that (1.1) has no transcendental meromorphic solutions when $n \geq 4$, and Montel [11] showed that (1.1) has no transcendental entire solutions when $n \geq 3$. Iyer [5] concluded that if $n = 2$, then (1.1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, no other solutions exist. We remark that (1.1) can be rewritten as

$$f(z)^n + [g_1(z) + \dots + g_m(z)]^n = 1, \tag{1.2}$$

it is easy to see that there is no relationship between the number of m with the existence of meromorphic solutions of (1.2), since that n and functional fields are the determinants of the existence of solutions. In this paper, we will explore the corresponding problem when $g_i(z), i = 1, 2, \dots, m$, are some differential or difference operators of $f(z)$. Our aim is to explore the relationship between the number of differential or difference operators with the existence of meromorphic solutions of Fermat type differential or difference equations.

Let us begin from a result given in [15]. Yang and Li considered the entire solutions of Fermat type differential equations. Here, we rewrite the original theorem as follows for the goal of this article

Theorem 1.1. *Let n be a positive integer, $b_0, b_1, \dots, b_{n-1}$ be constants, b_n be a nonzero constant and let $L(f) = \sum_{k=0}^{n} b_k f^{(k)}$. Then the transcendental meromorphic solution of the following equation*

$$f^2 + L(f)^2 = 1 \tag{1.3}$$

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must have the form $f(z) = \frac{1}{2}\left(Pe^{\lambda z} + \frac{1}{P}e^{-\lambda z}\right) = ch(\lambda z + A)$, *where* $e^A = P$, P *is a non-zero constant and* λ *satisfies the following equations:*

$$\sum_{k=0}^{n} b_k \lambda^k = \frac{1}{i}, \quad \sum_{k=0}^{n} b_k(-\lambda)^k = -\frac{1}{i}. \tag{1.4}$$

In fact, all the transcendental meromorphic solutions of (1.3) should be transcendental entire solutions. Let $L(f) = f^{(n)}$. From (1.4) of Theorem 1.1, we see easily that if n is an odd, then (1.3) has transcendental entire solutions. If n is an even, then (1.3) has no transcendental entire solutions. Some related results on $f^2 + R(z)(f^{(n)})^2 = 1$ also can be found in [13], where $R(z)$ is a rational function and n is an odd. Let $L(f) = b_n f^{(n)} + b_{n+1} f^{(n+1)}$. Yang and Li also obtained the following result.

Theorem 1.2 ([15, Theorem 2])**.** *Let* b_n *and* b_{n+1} *be non-zero constants. Then*

$$f^2 + [b_n f^{(n)} + b_{n+1} f^{(n+1)}]^2 = 1 \tag{1.5}$$

has no transcendental meromorphic solutions.

It is not difficult to see that $b_{n+1}f^{(n+1)}$ can be replaced by $b_{n+2k+1}f^{(n+2k+1)}$ in Theorem 1.2, where k is a non-negative integer. However, if $f^{(n+1)}$ is replaced by $f^{(n+2)}$ or $f^{(n+2k)}$, then (1.5) can admit some transcendental entire solutions from Example 1.3 below.

Example 1.3. The function $f(z) = \cos z$ satisfies

$$f^2 + [\frac{3}{2}f' + \frac{1}{2}f''']^2 = 1,$$

and $f(z) = \sin z$ solve

$$f^2 + [\frac{1}{2}f' - \frac{1}{2}f''']^2 = 1.$$

In fact, a necessary condition of existence of transcendental solutions of

$$f^2 + [b_n f^{(n)} + b_{n+1} f^{(n+2k)}]^2 = 1$$

is that n must be an odd. Furthermore, the necessary condition of existence of solutions of $f^2 + [b_n f^{(n)} + b_m f^{(m)}]^2 = 1$ is that m, n are odds, which can be obtained by (1.4) of Theorem 1.1.

Let us consider the case of $L(f) = b_n f^{(n)} + b_{n+1} f^{(n+1)} + b_{n+2} f^{(n+2)}$. From Theorem 1.1, we see that if n is an even, then

$$f^2 + [b_n f^{(n)} + b_{n+1} f^{(n+1)} + b_{n+2} f^{(n+2)}]^2 = 1 \tag{1.6}$$

may admit some transcendental solutions. If n is odd, then (1.6) has no transcendental solutions. From the above statements, we conclude that the necessary conditions for (1.3) to admit transcendental solutions are as follows:

(i) k is an odd (it is also a sufficient condition), provided that $L(f) = f^{(k)}$,
(ii) $m + n$ is an even and m, n are odds, provided that $L(f) = b_n f^{(n)} + b_m f^{(m)}$,
(iii) $n + p + q$ is an odd, provided that $L(f) = b_n f^{(n)} + b_p f^{(p)} + b_q f^{(q)}$.

We give two notations for more convenient statements. Denoting $D(L(f))$ by the number of differential operators in $L(f)$, for example $D(L(f)) = n + 1$, if $L(f) = \sum_{k=0}^{n} b_k f^{(k)}$ and $b_k \neq 0, k = 0, 1, \dots, n$. Denoting $W(L(f))$ by the sum of order of all differential operators in $L(f)$, for example $W(L(f)) = \frac{n(n+1)}{2}$, if

$L(f) = \sum_{k=0}^{n} b_k f^{(k)}$ and $b_k \neq 0, k = 0, 1, \dots, n$. So far, we have obtained that if $D(L(f))$ is odd (even), the necessary condition of existence of transcendental solutions of (1.3) is that $W(L(f))$ is odd (even), provided that $L(f)$ has no more than three terms. A natural question happens as follows.

Question 1.4. Can we get the parity of $D(L(f))$ and $W(L(f))$ are same for any $L(f)$, provided that there exist transcendental solutions of (1.3)?

Unfortunately, Question 1.4 is not true for $L(f)$ with four terms by Example 1.5 below.

Example 1.5. The function $f(z) = \cos(-\sqrt{\frac{b_2}{b_4}}z + Ai)$ is an entire solution of

$$f^2 + [b_1 f' + b_2 f'' + b_3 f''' + b_4 f^{(4)}]^2 = 1,$$

where $b_1 b_4 + b_2 b_3 = b_4\sqrt{\frac{b_2}{b_4}}$. Here $D(L(f)) = 4$ and $W(L(f)) = 10$. The function $f(z) = \sin z$ solves the equation

$$f^2 + [f' + \frac{3}{2}f'' + 2f^{(4)} + \frac{1}{2}f^{(6)}]^2 = 1.$$

Here, $D(L(f)) = 4$ and $W(L(f)) = 13$.

If we add the condition that all $|b_i| = 1$, Question 1.4 is also not true, which can be seen by the Example 1.6.

Example 1.6. Assume that λ satisfies

$$b_5\lambda^5 + b_4\lambda^4 + b_1\lambda + b_0 = \frac{1}{i},$$
$$b_5[-\lambda]^5 + b_4[-\lambda]^4 + b_1[-\lambda] + b_0 = -\frac{1}{i}\,.$$

Then $f(z) = ch(\lambda z + A)$ solves $f^2 + [b_5 f^{(5)} + b_4 f^{(4)} + b_1 f' + b_0 f]^2 = 1$, where A is a constant. If t satisfies

$$b_5 t^5 + b_4 t^4 + b_1 t^2 + b_0 = \frac{1}{i},$$
$$b_5[-t]^5 + b_4[-t]^4 + b_2[-t]^2 + b_0 = -\frac{1}{i},$$

then $f(z) = ch(tz + B)$ solves $f^2 + [b_5 f^{(5)} + b_4 f^{(4)} + b_2 f'' + b_0 f]^2 = 1$, where B is a constant.

We always considered that there is just one term $f(z)^2$ in the beginning place of (1.3), we proceed to consider the following equation

$$[f + f']^2 + [f + f'']^2 = 1, \tag{1.7}$$

and obtain the following result.

Theorem 1.7. *The equation* (1.7) *has no transcendental meromorphic solutions.*

Furthermore, we want to explore when the following equation

$$[af + bf']^2 + [cf + df'']^2 = 1, \tag{1.8}$$

can admit a transcendental meromorphic solution, where a, b, c, d are constants. We obtain the following result.

Theorem 1.8. *If* (1.8) *admits transcendental meromorphic solutions* $f(z)$*, then one of the following holds:*

(i) $a = 0, c = 0$, $b \neq 0$, $d \neq 0$ *and* $f(z) = \frac{id}{b^2} sh(\frac{bi}{d} z - B) + A$, *where* A *is a constant.*

(ii) a, b, c, d *are non-zero constants and* $\frac{c}{a} = \frac{1-e^{2p}}{1+e^{2p}} i$, *then* $f(z) = De^{-\frac{a}{b} z} + \frac{e^p + e^{-p}}{2a}$, *where* p *is a constant,* D *is a non-zero constant.*

(iii) $a = 0$, $b \neq 0$, $c \neq 0$ *and* $f(z) = \frac{bi+d}{bc} sh(e^{\frac{c}{bi+d} z - A}) + B$, *where* $d + \frac{dc}{b^2 i + bd} = 0$ *and* A, B *are constants.*

Recently, using the difference analogues of Nevanlinna theory, the meromorphic solutions of complex difference equations or differential-difference equations with certain types also be considered, such as [6, 7, 8, 9, 10, 12]. The first author and his colleagues considered Fermat type difference equations, such as Liu, Cao and Cao [8] investigated the finite order entire solutions of

$$f(z)^2 + f(z + c)^2 = 1. \tag{1.9}$$

Here and in the following, c is a non-zero constant, unless otherwise specified. The result can be stated as follows.

Theorem 1.9 ([8, Theorem 1.1])**.** *The transcendental entire solutions with finite order of* (1.9) *must satisfy* $f(z) = \sin(Az + B)$*, where* B *is a constant and* $A = \frac{(4k+1)\pi}{2c}$*, with* k *an integer.*

When $f(z + c)$ is replaced by $f(z + c) - f(z)$ in (1.9), that is

$$f(z)^2 + [f(z + c) - f(z)]^2 = 1, \tag{1.10}$$

Liu [6] proved the following result.

Theorem 1.10 ([6, Proposition 5.3])**.** *There is no transcendental entire solutions with finite order of* (1.10).

Let $a_0, a_1, \dots, a_n$ be non-zero constants. If

$$f(z)^2 + [a_0 f(z) + a_1 f(z + c_1) + \dots + a_n f(z + c_n)]^2 = 1 \tag{1.11}$$

admits transcendental entire solutions with finite order, we hope that n is an odd. However, it is not true again by Example 1.11 below.

Example 1.11. The function $f(z) = \sin z$ is a solution of

$$f(z)^2 + [2f(z) + f(z + \frac{\pi}{2}) + f(z + \pi)]^2 = 1$$

and also a solution of

$$f(z)^2 + [2f(z) + f(z + \frac{\pi}{2}) + \frac{1}{2} f(z + \pi) + \frac{3}{2} f(z + 3\pi)]^2 = 1.$$

If we put the additional condition that $|a_i| = 1$ for $i = 0, 1, 2, \dots, n$, then Question 1.4 is also not true; we will construct an example as follows.

Example 1.12. The function $f(z) = \sin z$ is a solution of

$$f(z)^2 + [if(z) + f(z + c_1) + f(z + c_2) + f(z + c_3)]^2 = 1,$$

where $e^{c_1} = A$, $e^{c_2} = B$, $e^{c_3} = C$ and

$$\begin{gathered} A + B + C = 0, \\ \frac{1}{A} + \frac{1}{B} + \frac{1}{C} = -2i, \end{gathered} \tag{1.12}$$

it is also a solution of

$$f(z)^2 + [if(z) + f(z+t_1) + f(z+t_2) + f(z+t_3) + f(z+t_4)]^2 = 1,$$

where $e^{t_1} = A_1$, $e^{t_2} = B_1$, $e^{t_3} = C_1$, $e^{t_4} = D_1$ and

$$\begin{gathered} A_1 + B_1 + C_1 + D_1 = 0, \\ \frac{1}{A_1} + \frac{1}{B_1} + \frac{1}{C_1} + \frac{1}{D_1} = -2i. \end{gathered} \tag{1.13}$$

As a generalization of (1.10), we want to consider when the following equation

$$D^2 f(z)^2 + [Af(z+c) + Bf(z)]^2 = 1 \tag{1.14}$$

admits transcendental entire solutions, we get the next result.

Theorem 1.13. *Let A, B be constants. If there are transcendental entire solutions with finite order of* (1.14)*, then $A^2 = B^2 + D^2$.*

Example 1.14. We see that $f(z) = \sin(az + b)$ is an entire solution of

$$f(z)^2 + [-\sqrt{2}f(z+c) + f(z)]^2 = 1,$$

here $D = 1$, $B = 1$, $A = -\sqrt{2}$ and $c = \frac{\pi}{4a}$.

Finally, we also consider a difference equation similar to (1.7), and get the following result.

Theorem 1.15. *Let a_1, a_2, a_3, a_4 be non-zero constants. If*

$$[a_1 f(z+c) + a_2 f(z)]^2 + [a_3 f(z+c) + a_4 f(z)]^2 = 1 \tag{1.15}$$

admits transcendental entire solutions with finite order, then $a_1^2 + a_3^2 = a_2^2 + a_4^2$ and $f(z) = \frac{a_3 \cos(aiz+bi) + a_1 \sin(aiz+bi)}{a_2 a_3 - a_1 a_4}$, where a is non-zero constant and b is a constant.

Example 1.16. It is easy to see that

$$[f(z+c) + f(z)]^2 + [f(z+c) - f(z)]^2 = 1 \tag{1.16}$$

can admit a transcendental entire solution of $f(z) = \frac{\sqrt{2}}{2}\sin(z + \frac{\pi}{4}) = \frac{\cos z + \sin z}{2}$, which implies that $a = -i$ and $b = 0$ in Theorem 1.15.

2. Proof of Theorem 1.7

For the proof of Theorem 1.7, we need the following lemma.

Lemma 2.1 ([16, Theorem 1.46])**.** *Suppose that $f(z)$ is a transcendental meromorphic function and $h(z)$ is a non-constant entire function. Then*

$$\lim_{r\to\infty} \frac{T(r, f(h))}{T(r, h)} = \infty.$$

Now, we give the proof of Theorem 1.7. Using the classical results of Fermat equations in the beginning of the introduction, we have

$$\begin{gathered} f + f' = \sin(h(z)), \\ f + f'' = \cos(h(z)). \end{gathered} \tag{2.1}$$

It is a system of differential equations, however we do not know what is $h(z)$. Hence, it is not convenient to slove using the basic theory of differential equations. Here, we use the following method. Firstly, from (2.1), we get

$$f' - f'' = \sin(h(z)) - \cos(h(z)). \tag{2.2}$$

Then taking the first order derivative of first equation of (2.1), we have

$$f' + f'' = h'(z)\cos(h(z)). \tag{2.3}$$

Thus, combining the second equation of (2.1) and (2.3), we have

$$f - f' = \cos(h(z)) - h'(z)\cos(h(z)), \tag{2.4}$$

then taking derivative of the above equation, we have

$$f' - f'' = -h'(z)\sin(h(z)) - h''(z)\cos(h(z)) + [h'(z)]^2\sin(h(z)). \tag{2.5}$$

Combining (2.2) and (2.5), we have

$$\tan(h(z)) = \frac{h''(z) - 1}{(h'(z))^2 - h'(z) - 1}, \tag{2.6}$$

which implies that $h(z)$ should be a constant from Lemma 2.1. By a simple computation, we have $f(z)$ should be a constant. So, there is no transcendental entire solutions.

3. Proof of Theorem 1.8

The method is a factorization. Here, we give the details. We easily get

$$af + bf' = \frac{e^{p(z)} + e^{-p(z)}}{2} \tag{3.1}$$

and

$$cf + df'' = \frac{e^{p(z)} - e^{-p(z)}}{2i}, \tag{3.2}$$

where $p(z)$ is a transcendental entire function. Taking the first derivative of (3.1), we have

$$af' + bf'' = \frac{p'(z)e^{p(z)} - p'(z)e^{-p(z)}}{2}. \tag{3.3}$$

Combining (3.2) with (3.3), we have

$$bcf - adf' = \frac{e^{p(z)} - e^{-p(z)}}{2}[-bi - p'(z)d]. \tag{3.4}$$

Combining (3.4) with (3.1), we have

$$[a^2d + b^2c]f = \frac{ade^{p(z)} + ade^{-p(z)}}{2} + \frac{e^{p(z)} - e^{-p(z)}}{2}[-b^2i - p'(z)bd] \tag{3.5}$$

and

$$[a^2d + b^2c]f' = \frac{bce^{p(z)} + bce^{-p(z)}}{2} - \frac{e^{p(z)} - e^{-p(z)}}{2}[-bai - p'(z)ad]. \tag{3.6}$$

Next, we talk about two cases.

Case 1: If $a^2d + b^2c = 0$, then from (3.5) we have

$$e^{2p(z)}[ad - b^2i - p'(z)bd] = -ad - b^2i - p'(z)bd. \tag{3.7}$$

Subcase 1.1. If $a = 0$, $d \neq 0$ and $b^2i + p'(z)bd \neq 0$, then $c = 0$ follows. Thus, it implies $e^{2p(z)} = 1$, that is $p(z)$ reduces to a constant, which is a contradiction with f is transcendental. If $a = 0$, $d \neq 0$ and $b^2i + p'(z)bd = 0$, then $c = 0$ and $b \neq 0$ (a, b are not zeros simultaneously) and $p(z) = -\frac{bi}{d}z + B$, where B is a constant, which implies that $f(z) = \frac{di}{b^2}sh(\frac{bi}{d}z - B) + A$, where A is a constant.

Subcase 1.2. If $a \neq 0$, $d = 0$ and $b^2i + p'(z)bd \neq 0$, then $b \neq 0$ and $c = 0$ follows, a contradiction. If $a \neq 0$, $d = 0$ and $b^2i + p'(z)bd = 0$, then $b = 0$, thus f should be

a constant from (1.8).
Subcase 1.3. If $ad \neq 0$, then $bc \neq 0$, thus $p(z)$ should be a constant for avoiding a contradiction. From a simple computation, we have $f(z) = De^{-\frac{a}{b}z} + \frac{e^p + e^{-p}}{2a}$ and $\frac{c}{a} = \frac{1-e^{2p}}{1+e^{2p}} i$.

Case 2: If $a^2d + b^2c \neq 0$, then taking derivative for (3.5) and combining with (3.6), we have

$$e^{2p(z)}[iba + p''(z)bd + \phi] = iba + p''(z)bd - \phi, \tag{3.8}$$

where $p'(z)b^2 i + [p'(z)]^2 bd + bc = \phi$, hence we have either $iba + p''(z)bd = 0$ and $\phi = 0$ or $p(z)$ should be a constant.
Subcase 2.1. If $iba + p''(z)bd = 0$ and $\phi = 0$, we assume that $b = 0$, form (3.1) and (3.2), we have

$$e^{2p(z)}[\frac{dp''(z)}{a} + \frac{d}{a}p'(z)^2 + \frac{c}{a} - \frac{1}{i}] = \frac{dp''(z)}{a} - \frac{d}{a}p'(z)^2 - \frac{c}{a} - \frac{1}{i}, \tag{3.9}$$

hence either $p(z)$ is a constant, then $f(z)$ is a constant or $T(r, e^{2p}) = S(r, e^{2p})$ which is impossible. Assume that $b \neq 0$, we can get $p'(z) = -\frac{c}{bi+d}$ and $a = 0$, by a simple computation, we have $f(z) = \frac{bi+d}{bc} sh(e^{\frac{c}{bi+d}z - A}) + B$, where $d + \frac{dc}{b^2 i + bd} = 0$ and A, B are constants.
Subcase 2.2. If $p(z)$ is a constant p, we have $e^{2p}(iba + bc) = iba - bc$ from (3.8), if $b = 0$, then f should be a constant. If $b \neq 0$, we have $e^{2p} = \frac{ia-c}{ia+c}$, then $f(z) = De^{-\frac{a}{b}z} + \frac{e^p + e^{-p}}{2a}$, then substitute $f(z)$ into (3.1) and (3.2), we have $a^2d + b^2c = 0$, which is a contradiction of the Case 2.

4. Proof of Theorem 1.13

We need the following result by Yang and Yi, [16, Theorem 1.56].

Lemma 4.1. *Let f_1, f_2, f_3 be meromorphic functions such that f_1 is not a constant. If $f_1 + f_2 + f_3 = 1$ and if*

$$\sum_{j=1}^{3} N(r, 1/f_j) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$ and $T(r) := \max_{1 \leq j \leq 3} T(r, f_j)$, then either $f_2 = 1$ or $f_3 = 1$.

Using the classical results of Fermat equations of in the beginning of the introductions, we get $Df(z) = \sin(h(z))$ and $Af(z+c) + Bf(z) = \cos(h(z))$. Thus, we have the equation

$$A\sin(h(z+c)) + B\sin(h(z)) = D\cos(h(z)).$$

So we have

$$A\sin(h(z+c)) = \sqrt{D^2 + B^2}\sin(h(z) + \varphi),$$

where $\tan\varphi = -\frac{D}{B}$. Thus, we have

$$\frac{e^{ih(z+c)} - e^{-ih(z+c)}}{2i} = \frac{\sqrt{D^2 + B^2}}{A} \frac{e^{ih(z)+i\varphi} - e^{-ih(z)-i\varphi}}{2i},$$

which implies that

$$e^{2ih(z+c)} - \frac{\sqrt{D^2 + B^2}}{A} e^{ih(z+c)+ih(z)+i\varphi} + \frac{\sqrt{D^2 + B^2}}{A} e^{ih(z+c)-ih(z)-i\varphi} = 1. \tag{4.1}$$

Using Lemma 4.1 and remark that $e^{2ih(z+c)}$ and $e^{ih(z+c)+ih(z)+i\varphi}$ are not constants, then we have

$$\frac{\sqrt{D^2+B^2}}{A}e^{ih(z+c)-ih(z)-i\varphi}=1,$$
$$e^{2ih(z+c)}-\frac{\sqrt{D^2+B^2}}{A}e^{ih(z+c)+ih(z)+i\varphi}=0.$$

For avoiding a contradiction from the above two equations, we should have $h(z)=az+b$, thus, we get $A^2=B^2+D^2$. Then $ac=\varphi+2k\pi$ or $ac=\varphi+\pi+2k\pi$. If $Df(z)=\cos(h(z))=\sin(h(z)+\frac{\pi}{2}+2k\pi)$, using the similar method as above.

5. Proof of Theorem 1.15

If one of a_1,a_2,a_3,a_4 is zero, then using Theorem 1.13, we get the conclusion. Next, we talk about all a_1,a_2,a_3,a_4 being non-zero constants. Using the factorization, we obtain

$$a_1f(z+c)+a_2f(z)=\frac{e^{p(z)}+e^{-p(z)}}{2}, \tag{5.1}$$
$$a_3f(z+c)+a_4f(z)=\frac{e^{p(z)}-e^{-p(z)}}{2i}. \tag{5.2}$$

Then we have the following two equations

$$\begin{gathered}(a_2a_3-a_1a_4)f(z)=a_3\frac{e^{p(z)}+e^{-p(z)}}{2}-a_1\frac{e^{p(z)}-e^{-p(z)}}{2i},\\ (a_2a_3-a_1a_4)f(z+c)=a_2\frac{e^{p(z)}-e^{-p(z)}}{2i}-a_4\frac{e^{p(z)}+e^{-p(z)}}{2}.\end{gathered} \tag{5.3}$$

If $a_2a_3-a_1a_4=0$, then we have $e^{2p(z)}=-1$, which implies that $a_1f(z+c)+a_2f(z)=0$ and $a_3f(z+c)+a_4f(z)=\pm1$, so $f(z)$ should be a constant.

If $a_2a_3-a_1a_4\neq0$, we have

$$\begin{aligned}&a_3\frac{e^{p(z+c)}+e^{-p(z+c)}}{2}-a_1\frac{e^{p(z+c)}-e^{-p(z+c)}}{2i}\\ &=a_2\frac{e^{p(z)}-e^{-p(z)}}{2i}-a_4\frac{e^{p(z)}+e^{-p(z)}}{2}.\end{aligned} \tag{5.4}$$

Furthermore, we get

$$[a_3+a_1i]e^{p(z+c)+p(z)}+[a_3-a_1i]e^{p(z)-p(z+c)}+[a_4+a_2i]e^{2p(z)}=a_2i-a_4. \tag{5.5}$$

From Lemma 4.1, we have $p(z)-p(z+c)$ should be a constant, it implies that $p(z)=az+b$, where a is a non-zero constant and we have

$$[a_3+a_1i]e^{ac}+[a_4+a_2i]=0, \tag{5.6}$$
$$[a_3-a_1i]e^{-ac}=a_2i-a_4, \tag{5.7}$$

which implies that $a_1^2+a_3^2=a_2^2+a_4^2$. Then $f(z)=\frac{a_3\cos(aiz+bi)+a_1\sin(aiz+bi)}{a_2a_3-a_1a_4}$ follows. Thus, we have the proof of Theorem 1.15.

6. Further discussions

We remark that the finite order transcendental entire solutions should be of order one in Fermat differential or difference equations that we have considered. Of course, we always using the theory of factorization to get the expressions. We raise the following conjecture for the further studying on Fermat type differential or difference equations.

Conjecture 6.1. If there exist transcendental entire solutions $f(z)$ of Fermat differential or difference equations $P(f)^2+Q(f)^2=1$, then the order $\sigma(f)=1$, where $P(f), Q(f)$ are two linear differential or difference polynomials with constant coefficients.

Wiman-Valiron theory [14] and the difference Wiman-Valiron theory [1] can be used to get $\sigma(f)\geq 1$ in some cases. However, we have no ideas to remove the case $\sigma(f)>1$.

Finally, let us remind some results on the equation

$$f^n+g^m+h^p=1\,. \tag{6.1}$$

Gundersen [3] summed up some results, and recent results can be found in [18], [17]. Considering Fermat type difference equation with three terms, such as

$$f(z)^n+f(z+c_1)^m+f(z+c_2)^p=1. \tag{6.2}$$

It is easy to see that the necessary conditions of existence of transcendental entire solutions of finite order is $m=n=p$. So the classical result in Fermat functional equations show that there is no any entire solutions when $m=n=p\geq 7$. Here, we consider when (6.2) can admit entire solutions. It is easy to see that if $m=n=p=1$, then $f(z)=e^z+\frac{1}{3}$ and $e^{c_1}=\frac{1}{2}$ and $e^{c_1}=-\frac{3}{2}$ is a solution of $f(z)+f(z+c_1)+f(z+c_2)=1$. We also can find an entire solution of $f(z)^2+f(z+c_1)^2+f(z+c_2)^2=1$. For example, $f(z)=e^z+\frac{\sqrt{3}}{3}$ and $e^{c_1}=\frac{-1+\sqrt{3}i}{2}$ and $e^{c_2}=\frac{-1-\sqrt{3}i}{2}$.

For our further studying, we raise the following questions about the equation

$$f(z)^n+f(z+c_1)^n+f(z+c_2)^n=1, \tag{6.3}$$

where c_1 and c_2 are non-zero constants.

Question 6.2. Does there exist a transcendental entire solution with finite order of (6.3), provided that $3\leq n\leq 6$.

Question 6.3. Does there exist a transcendental meromorphic solution with finite order of (6.3), provided that $3\leq n\leq 8$.

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