Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 162, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ESTIMATES FOR DAMPED FRACTIONAL WAVE EQUATIONS AND APPLICATIONS

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Abstract. In our previous article [1] we estimated the L^p-norm ($p \geq 1$) of the solution to damped fractional wave equation. In this article, we prove other L^p estimates, with some emphasis on requiring less regularity of the initial data. We also study the Strichartz type estimate of this equation. Finally we present some application of these estimates, for proving existence of global solutions to semilinear damped fractional wave equations.

1. Introduction

We consider the Cauchy problem of linear damped fractional wave equation

$$\begin{gathered} \partial_{tt}u + 2u_t + (-\Delta)^\alpha u = 0, \quad \alpha > 0, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x) \end{gathered} \tag{1.1}$$

where $t > 0$, $x \in \mathbb{R}^n$ and $(-\Delta)^\alpha$ is defined as

$$(-\Delta)^\alpha f(x) = \mathcal{F}^{-1}\big(|\xi|^{2\alpha}\hat{f}(\xi)\big)(x). \tag{1.2}$$

Here and below, we denote $\hat{f}$ the Fourier transform of a distribution f and $\mathcal{F}^{-1}$ or $\check{f}$ the Fourier inverse transform of f. The solution to this Cauchy problem is formally given by

$$u(t,x) = \big\{e^{-t}\cosh(t\sqrt{\mathcal{L}})f + e^{-t}\frac{\sinh(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}(f+g)\big\}, \tag{1.3}$$

where $\mathcal{L}$ is the Fourier multiplier with symbol $1 - |\xi|^{2\alpha}$. When $\alpha = 1$, (1.1) becomes the damped wave equation, which is an important mathematical model in studying many physics problems. So it has attracted a lot of authors. One can easily find hundreds of papers addressing various research problems on this equation. For instance, the reader may refer to [6, 7, 11, 12, 13, 14, 19, 22] and the references therein to find results on the local and global well-posedness of the Cauchy problem, space-time estimates and asymptotic estimates etc. In a previous paper [1], we proved the below L^p-estimate for the solution (1.3).

2010 *Mathematics Subject Classification.* 35L05, 46E35, 42B37.
Key words and phrases. Damped fractional wave equation; L^p-estimate; Strichartz estimate.

Submitted November 30, 2014. Published June 16, 2015.

Theorem 1.1. *Let $\alpha > 0$, $1 \le r \le p \le \infty$ and $\beta > n\alpha|1/2 - 1/p|$ for $\alpha \neq 1$ or $\beta > (n-1)|1/2 - 1/p|$ for $\alpha = 1$. Then there exists some $\delta_p > 0$ such that*

$$\begin{aligned}\|u(t,x)\|_{L^p} &\preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{r}-\frac{1}{p})}\left(\|f\|_{L^r} + \|g\|_{L^r}\right)\\ &\quad + e^{-t}(1+t)^{\delta_p}\big(\|f\|_{L^p_\beta} + \|g\|_{L^p_{\beta-\alpha}}\big),\end{aligned}$$

where L^p_β and $L^p_{\beta-\alpha}$ are inhomogeneous Sobolev spaces.

Here and below, we use the notation $X \preceq Y$ to mean that there is some positive constant C, independent of all essential variables such that $X \le CY$. Before stating our new theorems, we first review some function spaces used in this paper. When $1 \le p \le \infty$ and $\alpha \in \mathbb{R}$, $\dot{L}^p_\alpha$ (or L^p_α) is defined to be all the tempered distributions such that the Fourier inverse of $|\xi|^\alpha \hat{f}(\xi)$ (or $(1+|\xi|)^\alpha \hat{f}(\xi)$) belongs to L^p. We also denote the norms

$$\|f\|_{\dot{L}^p_\alpha} = \|\mathcal{F}^{-1}\big(|\xi|^\alpha \hat{f}(\xi)\big)\|_{L^p}, \quad \|f\|_{L^p_\alpha} = \|\mathcal{F}^{-1}\Big((1+|\xi|)^\alpha \hat{f}(\xi)\Big)\|_{L^p}.$$

If α is some nonnegative integer, $\dot{L}^p_\alpha$ (or L^p_α) consists of all the tempered distributions such that $D^{\mathbf{k}} f \in L^p$ for all $|\mathbf{k}| = \alpha$ (or $|\mathbf{k}| \le \alpha$), where $\mathbf{k} = (k_1, k_2, \dots, k_n)$. It is not hard to see that $L^p = \dot{L}^p_0 = L^p_0$. When $p = 2$, we write $\dot{H}^\alpha = \dot{L}^2_\alpha$ and $H^\alpha = L^2_\alpha$, which are the Sobolev spaces we usually refer to them. Readers may consult [2, 3, 18] for more properties and applications on all above mentioned function spaces.

From [10] or [14], we already know that the solution to (1.1), when $\alpha = 1$, satisfies

$$\|u(t,x)\|_{L^p} \preceq (1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\left(\|f\|_{L^r} + \|g\|_{L^r} + \|f\|_{H^{[n/2]+1}} + \|g\|_{H^{[n/2]}}\right) \tag{1.4}$$

for $1 \le r \le 2 \le p \le \infty$. In Section 2, we will prove the below similar theorem for all $\alpha > 0$ and meantime, require less regularity on the initial data f and g.

Theorem 1.2. *Let $\alpha > 0$, $1 \le r \le p < \infty$ and $p > 2$. Then (1.3) satisfies*

$$\begin{aligned}\|u(t,x)\|_{L^p} &\preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{r}-\frac{1}{p})}\left(\|f\|_{L^r} + \|g\|_{L^r}\right)\\ &\quad + e^{-t}\left(\|f\|_{H^{n(1/2-1/p)}} + \|g\|_{H^{n(1/2-1/p)-\alpha}}\right).\end{aligned}$$

The estimate also holds if we substitute the inhomogeneous Sobolev spaces with the homogeneous ones.

Note that we require less regularity in Theorem 1.2 than in (1.4). As we did in [1], we may also estimate the norm $\|u(t,x)\|_{L^q(\mathbb{R}^+, L^p(\mathbb{R}^n))}$ by taking an integral of the above inequality. But in this way we have to assume the same regularity on the initial data f and g. In Section 3 we will study the Strichartz type estimate for (1.3) (Theorem 1.3 below), which shows that the $L^q(\mathbb{R}^+, L^p)$ estimate in fact requires less regularity on the initial data than the L^p estimate.

A triplet (p, q, r) is called σ-admissible if

$$\frac{1}{q} \le \sigma(\frac{1}{r} - \frac{1}{p}), \tag{1.5}$$

where $0 < r \le p \le \infty$, $r \le q \le \infty$ and $\sigma > 0$. If the equality in (1.5) holds, then we call (p, q, r) sharp σ-admissible.

Theorem 1.3. *Let $\alpha > 0, 2 \le q \le \infty, 2 \le p < \infty$ and (p,q,r) be $\frac{n}{2\alpha}$-admissible. Then*
$$\Big(\int_0^\infty \|u(t,x)\|_{L^p}^q dt\Big)^{1/q} \preceq \|f\|_{L^r} + \|g\|_{L^r} + \|f\|_{H^\beta} + \|g\|_{H^{\beta-\alpha}}$$
holds in each of the following cases:

(i) *$(p,q,2)$ is $\frac{n}{2}$-admissible, $(p,q) \ne (\frac{2n}{n-2}, 2)$ and $\beta \ge n(\frac12 - \frac1p) - \frac{\alpha}{q}$;*
(ii) *$(p,q,2)$ is not $\frac{n}{2}$-admissible and $\beta \ge n(\frac12 - \frac1p)(1 - \frac{\alpha}{2})$.*

The estimate also holds if we substitute $H^\beta, H^{\beta-\alpha}$ by $\dot H^\beta, \dot H^{\beta-\alpha}$.

Note if $(p,q,2)$ is sharp $n/2$-admissible, then we have
$$n(\frac12 - \frac1p) - \frac{\alpha}{q} = n(\frac12 - \frac1p)(1 - \frac{\alpha}{2}),$$
which means the regularity requirement on initial data varies continuously over the sharp $\frac{n}{2}$-admissible line. Strichartz estimates for Schrödinger equation and wave equation have a long story ([4, 5, 8, 9, 15, 16, 20, 21]). They are closely related to some important problem in analysis. They can also be applied to study the well-posedness of some nonlinear equations.

In Section 4, we study the existence of small initial data time global solution to the semilinear equation
$$\begin{gathered}\partial_{tt}u + 2u_t + (-\Delta)^\alpha u = F(u), \quad \alpha > 0, \\ u(0,x) = f(x),\ u_t(0,x) = g(x)\end{gathered} \tag{1.6}$$
where $F(u) = \pm|u|^\sigma u$ or $\pm|u|^{\sigma+1}$. For $\alpha = 1$, the problem has been studied by many authors. Todorova-Yordanov[19] and Zhang[22] have shown that when $\sigma \le 2/n$, the solution blows up in finite time for any non-negative initial data f and g. Todorova and Yordanov also proved the global existence when
$$\frac{2}{n} < \sigma < \frac{2}{n-2},\ n \ge 3 \text{ or } \frac{2}{n} < \sigma,\ n = 1,2$$
for compactly supported initial data. If one removes the compactness restriction, the global existence when $n \le 5, \sigma > \frac{2}{n}$ has been proved by Ikehata-Miyaoka-Nakatake[7] ($n = 1,2$), Nishihara[14] ($n = 3$) and Narazaki[12] ($n = 4,5$). The theorem for general $n \ge 1$ has also been proved assuming some rapid decay on the initial data as $|x| \to \infty$, see [6]. In Section 4, we prove the following existence theorem for (1.6).

Theorem 1.4. *Let $\alpha > 0$, $n = 1,2$ and $\sigma > 2\alpha/n$. Take $p_0 > 1$ close to 1 such that $\frac{n}{2\alpha}(1/p_0 - 1/p_0')\sigma > 1$ where p_0' is the dual number of p_0. If $f \in \cap_{1<p<\infty} L^p_\alpha$, $g \in \cap_{1<p<\infty} L^p$ and*
$$\|f,g\|_0 = \sup_{1<p<p_0} \big(\|f\|_{L^p_\alpha} + \|g\|_{L^p}\big) + \sup_{p_0'<\tilde p<\infty} \Big(\|f\|_{L^{\tilde p}_\alpha} + \|g\|_{L^{\tilde p}}\Big)$$
is sufficiently small. Then there exists a unique solution $u(t,x)$ to (1.6) in the space $L^\infty(\mathbb{R}^+, \cap_{1<p<\infty} L^p)$ such that
$$\|u(t,x)\|_{L^p} \preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{p_0} - \frac{1}{p_0'})(1-\frac1p)} \|f,g\|_0, \quad 1 < p < \infty.$$

For the proof of this theorem, we get some ideas from [14, Theorem 1.2], but choose a different working space and use our own L^p estimates from Theorem 1.1 and Section 2. Note also that the theorem is stated for all $\alpha > 0$. When $\alpha = 1$, a minor modification of the proof of Theorem 1.4 leads to the global existence for $n \le 3$. The reason we can do this is we have slightly different estimate in Theorem 1.1 for $\alpha = 1$.

2. Proof of Theorem 1.2

To study the solution (1.3), we will focus on two fundamental operators

$$e^{-t}\cosh(t\sqrt{\mathcal{L}}), \quad e^{-t}\frac{\sinh(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}.$$

Denote their kernels by $K_\alpha(t)$ and $\Omega_\alpha(t)$. Then

$$K_\alpha(t)(x) = e^{-t}\int_{\mathbb{R}^n} \cosh(t\sqrt{1-|\xi|^{2\alpha}})e^{i\langle x,\xi\rangle}d\xi,$$
$$\Omega_\alpha(t)(x) = e^{-t}\int_{\mathbb{R}^n} \frac{\sinh(t\sqrt{1-|\xi|^{2\alpha}})}{\sqrt{1-|\xi|^{2\alpha}}}e^{i\langle x,\xi\rangle}d\xi.$$

Now the solution (1.3) is written as

$$u(t,x) = K_\alpha(t) * f(x) + \Omega_\alpha(t) * (f+g)(x). \tag{2.1}$$

Let $H(\xi)$ be a real C^∞ radial function supported in $\{\xi : |\xi| > 100\}$ with $H(\xi) \equiv 1$ for $|\xi| > 150$. Define also $L(\xi) = 1 - H(\xi)$. We then decompose $K_\alpha(t)$ and $\Omega_\alpha(t)$ as

$$K_\alpha(t) = L(D)K_\alpha(t) + H(D)K_\alpha(t),$$
$$S_\alpha(t) = L(D)\Omega_\alpha(t) + H(D)\Omega_\alpha(t),$$

where $L(D)K_\alpha(t)$ is the low frequency part of the kernel $K_\alpha(t)$ defined as

$$L(D)K_\alpha(t)(x) = e^{-t}\int_{\mathbb{R}^n} L(\xi)\cosh(t\sqrt{1-|\xi|^{2\alpha}})e^{i\langle x,\xi\rangle}d\xi.$$

The other three terms are defined similarly. For the two low frequency parts, we have already proved their L^p estimates below (see [1, Propositions 8 and 10]).

Proposition 2.1. *Let $\alpha > 0$ and $1 \le r \le p \le \infty$. Then for any $t > 0$, we have*

$$\|L(D)K_\alpha(t) * h\|_{L^p} \preceq (1+t)^{-\frac{n}{2\alpha}(1/r-1/p)}\|h\|_{L^r},$$
$$\|L(D)\Omega_\alpha(t) * h\|_{L^p} \preceq (1+t)^{-\frac{n}{2\alpha}(1/r-1/p)}\|h\|_{L^r}.$$

For the high frequency parts, we have the following proposition.

Proposition 2.2. *Let $\alpha > 0$ and $1 \le p \le \infty$. Then there exists some $\delta_p > 0$ such that*

$$\|H(D)K_\alpha(t) * h\|_{L^p} \preceq e^{-t}(1+t)^{\delta_p}\|h\|_{L^p_\beta},$$
$$\|H(D)\Omega_\alpha(t) * h\|_{L^p} \preceq e^{-t}(1+t)^{\delta_p}\|h\|_{L^p_{\beta-\alpha}}$$

whenever $\beta > n\alpha|1/2-1/p|$ or $\beta > (n-1)|1/2-1/p|$ for $\alpha = 1$.

This proposition is a minor modification of [1, Propositions 11 and 12], where the estimates were stated with homogeneous Sobolev spaces $\dot{L}^p_\beta$ and $\dot{L}^p_{\beta-\alpha}$. The proof of Proposition 2.2 here is almost the same. One only has to notice the definitions of Sobolev spaces and the fact that when dealing with the high frequency parts of the two operators, we actually have $|\xi| \simeq 1+|\xi|$ (see also the proof of Proposition 2.3). It is easy to see that Theorem 1.1 is the combination of the above two propositions.

In a same manner, Theorem 1.2 will be proved if we substitute Proposition 2.2 with the following proposition.

Proposition 2.3. *Let $\alpha > 0$ and $2 \le p < \infty$. Then we have*

$$\|H(D)K_\alpha(t) * h\|_{L^p} \preceq e^{-t}\|h\|_{H^{n(1/2-1/p)}},$$
$$\|H(D)\Omega_\alpha(t) * h\|_{L^p} \preceq e^{-t}\|h\|_{H^{n(1/2-1/p)-\alpha}}.$$

Proof. Let us first prove the estimate for $K_\alpha(t)$. Since

$$H(D)K_\alpha(t) * h = e^{-t}\mathcal{F}^{-1}\Big(H(\xi)\cosh\Big(it\sqrt{|\xi|^{2\alpha}-1}\Big)\hat{h}(\xi)\Big),$$

by Plancherel's Theorem, we easily have

$$\begin{aligned}\|H(D)K_\alpha(t) * h\|_{L^2} &= e^{-t}\|\mathcal{F}^{-1}\Big(H(\xi)\cosh\big(it\sqrt{|\xi|^{2\alpha}-1}\big)\hat{h}(\xi)\Big)\|_{L^2}\\ &= e^{-t}\|\Big(H(\xi)\cosh\big(it\sqrt{|\xi|^{2\alpha}-1}\big)\hat{h}(\xi)\Big)\|_{L^2}\\ &\preceq e^{-t}\|\hat{h}(\xi)\|_{L^2} = e^{-t}\|h\|_{L^2}.\end{aligned}$$

On the other hand, by the Sobolev imbedding $\mathrm{BMO}(\mathbb{R}^n) \hookrightarrow H^{n/2}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|H(D)K_\alpha(t) * h\|_{BMO} &\preceq \|H(D)K_\alpha(t) * h\|_{H^{n/2}}\\ &= e^{-t}\|(1+|\xi|)^{n/2}H(\xi)\cosh\Big(it\sqrt{|\xi|^{2\alpha}-1}\Big)\,\hat{h}(\xi)\|_{L^2}\\ &\preceq e^{-t}\|(1+|\xi|)^{n/2}\hat{h}(\xi)\|_{L^2} = e^{-t}\|h\|_{H^{n/2}}.\end{aligned}$$

Now interpolating between the two estimates

$$\|H(D)K_\alpha(t) * h\|_{L^2} \preceq e^{-t}\|h\|_{L^2},$$
$$\|H(D)K_\alpha(t) * h\|_{BMO} \preceq e^{-t}\|h\|_{H^{n/2}}$$

yields that, for $2 \le p < \infty$,

$$\|H(D)K_\alpha(t) * h\|_{L^p} \preceq e^{-t}\|h\|_{H^{n(1/2-1/p)}}. \tag{2.2}$$

To prove the second estimate of the proposition, we note that

$$\begin{aligned}&H(D)\Omega_\alpha(t) * h\\ &= e^{-t}\mathcal{F}^{-1}\Big(H(\xi)\frac{\sinh(it\sqrt{|\xi|^{2\alpha}-1})}{i\sqrt{|\xi|^{2\alpha}-1}}\hat{h}(\xi)\Big)\\ &\simeq e^{-t}\mathcal{F}^{-1}\Big(H(\xi)\sinh(it\sqrt{|\xi|^{2\alpha}-1})\frac{(1+|\xi|)^\alpha}{i\sqrt{|\xi|^{2\alpha}-1}}\mathcal{F}((1+|D|)^{-\alpha}h)\Big).\end{aligned}$$

Since the support of $H(\xi)$ lies in $\{\xi:\ |\xi| > 100\}$, we know the term

$$H(\xi)\sinh(it\sqrt{|\xi|^{2\alpha}-1})\frac{(1+|\xi|)^\alpha}{i\sqrt{|\xi|^{2\alpha}-1}}$$

is still bounded. invoking the steps we prove (2.2), one obtains

$$\|H(D)\Omega_\alpha(t) * h\|_{L^p} \preceq e^{-t}\|\mathcal{F}((1+|D|)^{-\alpha}h)\|_{H^{n(1/2-1/p)}} = e^{-t}\|h\|_{H^{n(1/2-1/p)-\alpha}}.$$

$\square$

3. Strichartz estimate

We first prove the following high frequency part estimate.

Proposition 3.1. *Let $n \geq 2, \alpha > 0, 2 \leq q \leq \infty$ and $2 \leq p < \infty$. Then*

$$\|H(D)(e^{-t}e^{it\sqrt{-L}}f)\|_{L_t^q L_x^p} \preceq \|f\|_{H^\beta}$$

holds in eaach of the following cases,

(i) *$(p, q, 2)$ is $\frac{n}{2}$-admissible, $(p, q) \neq (\frac{2n}{n-2}, 2)$ and $\beta \geq n(\frac{1}{2} - \frac{1}{p}) - \frac{\alpha}{q}$;*
(ii) *$(p, q, 2)$ is not $\frac{n}{2}$-admissible and $\beta \geq n(\frac{1}{2} - \frac{1}{p})(1 - \frac{\alpha}{2})$.*

Keel-Tao[9] proved an abstract Strichartz estimate which applies to the case $e^{it(-\Delta)^{\alpha/2}}$. They actually proved that

$$\|e^{it(-\Delta)^{\alpha/2}}f\|_{L_t^q L_x^p} \preceq \|f\|_{\dot{L}_\beta^2} \quad \text{for } \sigma\text{-amdissible } (p, q, 2),$$

where $\beta = n(\frac{1}{2} - \frac{1}{p}) - \frac{\alpha}{q}$ and $\sigma = \frac{n}{2}$ $(\alpha \neq 1)$ or $\sigma = \frac{n-1}{2}$ $(\alpha = 1)$. Let us compare Proposition 3.1 to this result. First, β is taken to be larger than $n(\frac{1}{2} - \frac{1}{p}) - \frac{\alpha}{q}$ in our proposition. This is because we only estimate the high frequency part of the operator. Secondly, we always assume $(p, q, 2)$ to be $\frac{n}{2}$-admissible (ever when $\alpha = 1$), which is caused by the difference between $\sqrt{-L}$ and $(-\Delta)^{\alpha/2}$. Finally, we even have the space-time estimate for non-admissible index. This in fact is the contribution of the extra term e^{-t}. But this term also hinders us from applying Keel and Tao's theorem directly. So in the proof of Proposition 3.1, we will modify some of their argument to treat this extra term.

Lemma 3.2. *Let $\alpha > 0$ and Φ be be some C^∞ function supported in $\{\xi : 2 < |\xi| < 8\}$ with $\Phi(\xi) \geq c > 0$ for $3 \leq |\xi| \leq 5$. Then we have*

$$\Big|\int_{\mathbb{R}^n} e^{it\sqrt{|\xi|^{2\alpha}-1}}\Phi(\xi)e^{i\langle x,\xi\rangle}d\xi\Big| \preceq (1+t)^{-n/2}.$$

Proof. The lemma follows from standard stationary phase argument. But we still present the proof here for clarity. Let us denote

$$I(t, x) = \int_{\mathbb{R}^n} e^{it\sqrt{|\xi|^{2\alpha}-1}}\Phi(\xi)e^{i\langle x,\xi\rangle}d\xi.$$

If $t \leq 1$, then since $|I(t, x)| \preceq C$, the lemma follows. So we assume $t > 1$. By polar decomposition,

$$I(t, x) = \int_0^\infty e^{it\sqrt{r^{2\alpha}-1}}\Phi(r)r^{n-1}\int_{S^{n-1}} e^{i\langle rx,\xi'\rangle}d\sigma(\xi')dr. \tag{3.1}$$

Denote $g(r) = \sqrt{r^{2\alpha} - 1}$. Then it is easy to check that, in the support of $\Phi(r)$,

$$g'(r) = \frac{\alpha r^{2\alpha-1}}{\sqrt{r^{2\alpha} - 1}} \simeq 1.$$

By some further but elementary calculation, one also finds that $|g''(r)| \geq c > 0$.

When $|x| > 1/4$, using the asymptotic of Fourier transform on S^{n-1} (see [17] page 347-348), we have

$$I(t, x) \simeq |x|^{\frac{1-n}{2}} \int_0^\infty e^{i(tg(r)\pm|x|r)}\Phi(r)r^{\frac{n-1}{2}}dr\,.$$

If $|x| > 2t$, then integrating by parts, we have

$$\begin{aligned} I(t,x) &\simeq |x|^{\frac{1-n}{2}} \int_0^\infty e^{i(tg(r)\pm|x|r)} d\Big(\frac{\Phi(r)r^{\frac{n-1}{2}}}{tg'(r)\pm|x|}\Big) \\ &= |x|^{\frac{1-n}{2}} \int_0^\infty e^{i(tg(r)\pm|x|r)} \Big(\frac{(\Phi(r)r^{\frac{n-1}{2}})'}{tg'(r)\pm|x|} + \frac{-tg''(r)\Phi(r)r^{\frac{n-1}{2}}}{(tg'(r)\pm|x|)^2}\Big) dr\,. \end{aligned}$$

Using integration by parts N times and by induction,

$$I(t,x) \simeq |x|^{\frac{1-n}{2}} \int_0^\infty \sum_{j=N}^{2N} \frac{t^{j-N}\phi_j^N(r)}{(tg'(r)\pm|x|)^j} e^{i(tg(r)\pm|x|r)} dr, \tag{3.2}$$

where $\phi_j^N(r)$ are C_0^∞ functions. Since

$$|tg'(r)\pm|x|| \ge 1/2|x|,$$

taking N large enough, we have

$$|I(t,x)| \preceq |x|^{\frac{1-n}{2}} \sum_{j=N}^{2N} \int_0^\infty \frac{|x|^{j-N}|\phi_j^N(r)|}{|x|^j} dr \le |x|^{-N+\frac{1-n}{2}} \preceq t^{-n/2}.$$

If $1/4 < |x| < t/2$, then

$$|tg'(r)\pm|x|| \ge t/2,$$

and by (3.2), we also have

$$|I(t,x)| \le |x|^{\frac{1-n}{2}} \sum_{j=N}^{2N} t^{-N} \int_0^\infty |\phi_j^N(r)| dr \preceq |t|^{-N} \preceq t^{-n/2}.$$

If $t/2 \le |x| \le 2t$, since

$$|(tg(r)\pm|x|r)'' = |tg''(r)| \ge t/100,$$

by the Van de Coupt lemma [17, page 334], we have

$$|I(t,x)| \preceq |x|^{\frac{1-n}{2}} t^{-\frac{1}{2}} \preceq t^{-n/2}.$$

When $|x| \le 1/4$, we have $r|x| \le 1$ so

$$\int_{S^{n-1}} e^{i\langle rx,\xi'\rangle} d\sigma(\xi') = O(1)$$

and consequently

$$I(t,x) \simeq \int_0^\infty e^{itg(r)}\Phi(r)r^{n-1} dr$$

by (3.1). Integrating by parts as above, we complete the proof. □

Remark 3.3. By checking the above proof carefully, we find that the lemma holds uniformly for all

$$I(t,x) = e^{-t} \int_{\mathbb{R}^n} e^{it\sqrt{|\xi|^{2\alpha}-h}} \Phi(\xi) e^{t\langle x,\xi\rangle} d\xi$$

with $0 < h < 1$. This enables us to estimate

$$I_j(t,x) = \int_{\mathbb{R}^n} e^{it\sqrt{|\xi|^{2\alpha}-1}} \Phi(2^{-j}\xi) e^{i\langle x,\xi\rangle} d\xi, \quad j \ge 1.$$

In fact by Lemma 3.2 and a change of variable,

$$|I_j(t,x)| = \big|2^{jn} \int_{\mathbb{R}^n} e^{i2^{j\alpha}t\sqrt{|\xi|^{2\alpha}-2^{-2j\alpha}}} \Phi(\xi) e^{i\langle 2^j x,\xi\rangle} d\xi\big|$$

$$= 2^{jn}|I(2^{j\alpha}t, 2^j x)| \preceq 2^{jn}(1 + 2^{j\alpha}t)^{-n/2}.$$

Proof of Proposition 3.1. Let Φ be as in Lemma 3.2 and require that $\sum_{j\in\mathbb{Z}} \Phi(2^j\xi) = 1$ for all $\xi \neq 0$. Noting the support of $H(\xi)$, we have the decomposition,

$$\begin{aligned} H(D)(e^{-t}e^{it\sqrt{-L}}f)(x) &= e^{-t}\sum_{j=6}^{+\infty}\int_{\mathbb{R}^n} e^{it\sqrt{|\xi|^{2\alpha}-1}}\Phi(2^{-j}\xi)H(\xi)\hat{f}(\xi)e^{i\langle x,\xi\rangle}d\xi \\ &:= \sum_{j=6}^{\infty} U_j(t)f(x). \end{aligned}$$

Since $H(\xi) \equiv 1$ for $2 < 2^{-j}\xi < 8$ after some $j_0 > 0$, we may omit this term in the above expression for simplicity. If we obtain

$$\|U_j(t)f(x)\|_{L_x^p L_t^q} \preceq 2^{j\beta}\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.3}$$

then Proposition 3.1 follows from some standard arguments involving Littlewood-Paley theory, see also [9].

Now we prove (3.3) using the bilinear method of [9]. First we notice that the dual of (3.3) is

$$\|\int_0^\infty U_j^*(s)F(s,\cdot)(x)ds\|_{L^2(\mathbb{R}^n)} \preceq 2^{j\beta}\|F(t,x)\|_{L_t^{q'}L_x^{p'}},$$

which by the TT^* method, is further equivalent to the bilinear form

$$\begin{aligned} &\Big|\int_0^\infty\int_0^\infty \langle U_j^*(s)F(s), U_j^*(t)G(t)\rangle\, ds\, dt\Big| \\ &\preceq 2^{2j\beta}\|F(t,x)\|_{L_t^{q'}L_x^{p'}}\|G(t,x)\|_{L_t^{q'}L_x^{p'}}. \end{aligned} \tag{3.4}$$

Here $U_j^*(t)$ denotes the adjoint operator of $U_j(t)$. By checking the definition, it is easy to see that

$$U_j^*(t)f(x) = e^{-t}\int_{\mathbb{R}^n} e^{-it\sqrt{|\xi|^{2\alpha}-1}}\Phi(2^{-j}\xi)\hat{f}(\xi)e^{i\langle x,\xi\rangle}d\xi.$$

Therefore,

$$U_j(s)U_j^*(t)g = e^{-t}e^{-s}\int_{\mathbb{R}^n} e^{i(s-t)\sqrt{|\xi|^{2\alpha}-1}}\Phi(2^{-j}\xi)\hat{g}(\xi)e^{i\langle x,\xi\rangle}d\xi.$$

By Young's inequality and Remark 3.3, we obtain

$$\|U_j(s)U_j^*(t)g\|_{L^\infty} \preceq e^{-t}e^{-s}2^{jn}\left(1 + 2^{j\alpha}|t-s|\right)^{-n/2}\|g\|_{L^1}.$$

We rewrite it in the bilinear form as

$$\begin{aligned} &|\langle U_j^*(s)F(s), U_j^*(t)G(t)\rangle| \\ &\preceq e^{-t}e^{-s}2^{jn}\left(1 + 2^{j\alpha}|t-s|\right)^{-n/2}\|F(s)\|_{L_x^1}\|G(t)\|_{L_x^1}. \end{aligned} \tag{3.5}$$

On the other hand, by Prancherel's Theorem, we have

$$\|U_j(t)f\|_{L^2} \preceq e^{-t}\|f\|_{L^2}.$$

Again we take its bilinear form

$$|\langle U_j^*(s)F(s), U_j^*(t)G(t)\rangle| \preceq e^{-t}e^{-s}\|F(s)\|_{L_x^2}\|G(t)\|_{L_x^2}. \tag{3.6}$$

Interpolating between (3.5) and (3.6) yields

$$|\langle U_j^*(s)F(s), U_j^*(t)G(t)\rangle| \preceq \frac{e^{-t}e^{-s}2^{j\theta n}}{(1+2^{j\alpha}|t-s|)^{n\theta/2}}\|F(s)\|_{L_x^{p'}}\|G(t)\|_{L_x^{p'}}, \tag{3.7}$$

where $\theta = 1-(2/p)$.

Extending the definition of $F(s,x), G(t,x)$ such that $F(s,x) = G(t,x) \equiv 0$ whenever $s,t \leq 0$. Note also that

$$e^{-t}e^{-s}\chi_{\{s>0,t>0\}}(s,t) \leq e^{-|s-t|}.$$

Denoting the left side of (3.4) by J, then by (3.7) and Hölder's inequality, we have

$$\begin{aligned} J &\preceq 2^{j\theta n}\int_{\mathbb{R}}\|F(s)\|_{L_x^{p'}}\int_{\mathbb{R}}\frac{e^{-|s-t|}}{(1+2^{j\alpha}|t-s|)^{n\theta/2}}\|G(t)\|_{L_x^{p'}}dtds \\ &\preceq 2^{j\theta n}\|F(s)\|_{L_s^{q'}L_x^{p'}}\|\int_{\mathbb{R}}\frac{e^{-|s-t|}}{(1+2^{j\alpha}|t-s|)^{n\theta/2}}\|G(t)\|_{L_x^{p'}}dt\|_{L_s^q}. \end{aligned}$$

When $\frac{1}{q} = \frac{1}{q'} - (1-\frac{n\theta}{2})$ and $q' > q$, by the Hardy-Littlewood-Sobolev inequality [18, Section V.1.2], the second norm above is less than

$$2^{-j\frac{n\theta\alpha}{2}}\|\int_{\mathbb{R}}\frac{\|G(t)\|_{L_x^{p'}}}{|t-s|^{n\theta/2}}dt\|_{L_s^q} \preceq 2^{-j\frac{n\theta\alpha}{2}}\|G(t)\|_{L_t^{q'}L_x^{p'}}.$$

Due to the restriction that $q' > q$, we have to exclude the point $(p,q) = (\frac{2n}{(n-2)}, 2)$ in Proposition 3.1. Note also that $n\theta/2 = 2/q$ in this case. So plugging the above inequality into the estimate of J we easily reach

$$J \preceq 2^{2j\beta}\|F(s)\|_{L_s^{q'}L_x^{p'}}\cdot\|G(t)\|_{L_t^{q'}L_x^{p'}}.$$

When $\frac{1}{q} \neq \frac{1}{q'} - (1-\frac{n\theta}{2})$, we apply Young's inequality and obtain

$$J \preceq 2^{j\theta n}\|F(s)\|_{L_s^{q'}L_x^{p'}}\|G(t)\|_{L_t^{q'}L_x^{p'}}\|\frac{e^{-|t|}}{(1+2^{j\alpha}|t|)^{n\theta/2}}\|_{L_t^r},$$

where $r = q/2$. In order to prove (3.4), we need to show

$$M = 2^{j\theta n}\|\frac{e^{-|t|}}{(1+2^{j\alpha}|t|)^{n\theta/2}}\|_{L_t^r} \preceq 2^{2j\beta}.$$

Let us first compute

$$N = \int_{\mathbb{R}}\Big(\frac{e^{-|t|}}{(1+2^{j\alpha}|t|)^{n\theta/2}}\Big)^r dt = 2\int_0^\infty \frac{e^{-rt}}{(1+2^{j\alpha}t)^{\frac{nr\theta}{2}}}dt.$$

Since $\frac{1}{q} \neq \frac{1}{q'} - (1-\frac{n\theta}{2})$, we have $\frac{nr\theta}{2} \neq 1$. If $\frac{nr\theta}{2} > 1$, by change of variable, we have

$$N = 2\cdot 2^{-j\alpha}\int_0^\infty \frac{e^{-2^{-j\alpha}rt}}{(1+t)^{\frac{nr\theta}{2}}}dt \preceq 2^{-j\alpha}\int_0^\infty \frac{1}{(1+t)^{\frac{nr\theta}{2}}}dt \preceq 2^{-j\alpha}.$$

Thus

$$M \leq 2^{j\theta n}N^{1/r} \preceq 2^{2j(n(\frac{1}{2}-\frac{1}{p})-\frac{\alpha}{q})}.$$

If (p,q) is not $\frac{n}{2}$-admissible, i.e. $\frac{nr\theta}{2} < 1$, we note that

$$1+2^{j\alpha}t \simeq 1 \text{ if } 0 < t < 2^{-j\alpha}, \text{ and } 1+2^{j\alpha}t \simeq 2^{j\alpha}t \text{ if } t \geq 2^{-j\alpha}.$$

Therefore,

$$\begin{aligned} N &\preceq \int_0^{2^{-j\alpha}} e^{-rt}dt + 2^{-j\frac{nr\theta}{2}} \int_{2^{-j\alpha}}^{\infty} e^{-rt} t^{-\frac{nr\theta}{2}} dt \\ &\preceq 2^{-j\alpha} + 2^{-j\frac{nr\theta}{2}} \Big(\int_{2^{-j\alpha}}^{1} t^{-\frac{nr\theta}{2}} dt + \int_1^{\infty} e^{-rt} t^{-\frac{n\alpha\theta}{2}} dt \Big) \\ &\preceq 2^{-j\alpha} + 2^{-j\alpha\frac{n\theta r}{2}}. \end{aligned}$$

So $N \preceq 2^{-j\alpha n\theta r/2}$ and

$$M \le 2^{j\theta n} N^{1/r} \preceq 2^{2jn(1-\alpha/2)(1/2-1/p)}. \tag{3.8}$$

□

From Proposition 3.1, one easily has

$$\begin{gathered} \|H(D)K_\alpha(t) * f\|_{L_t^q L_x^p} \preceq \|f\|_{H^\beta}, \\ \|H(D)\Omega_\alpha(t) * f\|_{L_t^q L_x^p} \preceq \|f\|_{H^{\beta-\alpha}} \end{gathered}$$

whenever p, q, β satisfies the conditions of Proposition 3.1. On the hand, taking integral (with variable t) on both sides of the two estimates in Proposition 2.1, we easily have (see also [1, Eq. 115])

$$\begin{gathered} \|L(D)K_\alpha(t) * h\|_{L_t^q L_x^p} \preceq \|h\|_{L^r}, \\ \|L(D)\Omega_\alpha(t) * h\|_{L_t^q L_x^p} \preceq \|h\|_{L^r} \end{gathered}$$

for all $\frac{n}{2\alpha}$-admissible triplet (p, q, r). Theorem 1.3 is then an easy combination of the above four estimates.

4. Some global existence theorems

Let us first prove Theorem 1.4. Set

$$\|u(t,x)\|_X = \sup_t \Big(\sup_{1<p<p_0} \|u(t,x)\|_{L^p} + (1+t)^{\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})} \sup_{p_0'<\tilde p<\infty} \|u(t,x)\|_{L^{\tilde p}} \Big)$$

and define a map

$$\mathcal{D}u(t,x) = u_l(t,x) + \int_0^t \Omega_\alpha(t-\tau) * F(u(\tau,\cdot))(x)d\tau,$$

where $u_l(t,x)$ is the solution to linear equation (1.1). Now we estimate the term $\|\mathcal{D}u\|_X$.

When $1 < p < p_0$, we have $n\alpha|1/p - 1/2| < \alpha$ by the assumptions of Theorem 1.4. So taking $\beta = \alpha$ and $r = p$ in Theorem 1.1, we obtain

$$\|u_l(t,x)\|_{L^p} \preceq (\|f\|_{L^p} + \|g\|_{L^p}) + e^{-t}(1+t)^{\delta_p}(\|f\|_{L^p_\alpha} + \|g\|_{L^p}) \preceq \|f\|_{L^p_\alpha} + \|g\|_{L^p}\,.$$

For $p_0' < \tilde p < \infty$, we still have $n\alpha|1/\tilde p - 1/2| < \alpha$ thus

$$\begin{aligned} \|u_l(t,x)\|_{L^{\tilde p}} &\preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{\tilde p})} (\|f\|_{L^p} + \|g\|_{L^p}) \\ &\quad + e^{-t}(1+t)^{\delta_p}(\|f\|_{L^{\tilde p}_\alpha} + \|g\|_{L^{\tilde p}}) \\ &\preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})}(\|f\|_{L^p} + \|g\|_{L^p} + \|f\|_{L^{\tilde p}_\alpha} + \|g\|_{L^{\tilde p}}). \end{aligned}$$

Combining the above two estimates, we reach

$$\|u_l(t,x)\|_X \preceq \sup_{1<p<p_0} \big(\|f\|_{L^p_\alpha} + \|g\|_{L^p}\big) + \sup_{p_0'<\tilde p<\infty} \Big(\|f\|_{L^{\tilde p}_\alpha} + \|g\|_{L^{\tilde p}}\Big) = \|f, g\|_0.$$

Next we bound the term

$$\| \int_0^t \Omega_\alpha(t-\tau) * F(u(\tau,\cdot))(x) d\tau \|_X.$$

When $1 < p < p_0$, by applying Proposition 2.1 and Proposition 2.2 with $\beta = \alpha$, we have

$$\begin{aligned}
&\| \int_0^t \Omega_\alpha(t-\tau) * F(u) d\tau \|_{L^p} \\
&\le \int_0^t \|\Omega_\alpha(t-\tau) * F(u)\|_{L^p} d\tau \\
&\le \int_0^t \|L(D)\Omega_\alpha(t-\tau) * F(u)\|_{L^p} + \|H(D)\Omega_\alpha(t-\tau) * F(u)\|_{L^p} d\tau \\
&\preceq \int_0^t \Big(1 + e^{-(t-\tau)}(1+(t-\tau))^{\delta_p}\Big) \|F(u)\|_{L^p} d\tau \\
&\preceq \int_0^t \|F(u(\tau,x))\|_{L^p} d\tau.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
\|F(u)\|_{L^p} &= \Big(\int_{\mathbb{R}^n} |u|^{\sigma p} |u|^p dx \Big)^{1/p} \\
&\le \Big(\int_{\mathbb{R}^n} |u|^{p(1+\epsilon)} dx \Big)^{\frac{1}{p(1+\epsilon)}} \Big(\int_{\mathbb{R}^n} |u|^{\sigma p(1+\epsilon)'} dx \Big)^{\frac{\sigma}{\sigma p(1+\epsilon)'}}.
\end{aligned}$$

By the fact that

$$\sigma p(1+\epsilon)' = \sigma p \frac{1+\epsilon}{\epsilon} \ge \frac{\sigma}{\epsilon} > \frac{2\alpha}{n\epsilon},$$

we choose sufficiently small $\epsilon > 0$ such that both $p(1+\epsilon) < p_0$ and $\sigma p(1+\epsilon)' > p_0'$ hold. Thus

$$\begin{aligned}
\|F(u(\tau,x))\|_{L^p} &\le \sup_{1<p<p_0} \|u(\tau,x)\|_{L^p} \sup_{p_0' < \tilde{p} < \infty} \|u(\tau,x)\|_{L^{\tilde{p}}}^\sigma \\
&\preceq (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0} - \frac{1}{p_0'})\sigma} \|u(\tau,x)\|_X^{\sigma+1},
\end{aligned}$$

and consequently (noting that $\frac{n}{2\alpha}(\frac{1}{p_0} - \frac{1}{p_0'})\sigma > 1$) we have

$$\| \int_0^t \Omega_\alpha(t-\tau) * F(u) d\tau \|_{L^p} \preceq \|u\|_X^{\sigma+1} \int_0^t (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0} - \frac{1}{p_0'})\sigma} d\tau \preceq \|u\|_X^{\sigma+1}.$$

When $p_0' < \tilde{p} < \infty$, we have

$$\begin{aligned}
&\| \int_0^t \Omega_\alpha(t-\tau) * F(u) d\tau \|_{L^{\tilde{p}}} \\
&\le \int_0^t \|L(D)\Omega_\alpha(t-\tau) * F(u)\|_{L^{\tilde{p}}} d\tau + \int_0^t \|H(D)\Omega_\alpha(t-\tau) * F(u)\|_{L^{\tilde{p}}} d\tau \\
&:= I_1 + I_2.
\end{aligned}$$

Applying Proposition 2.2 with $\beta = \alpha$ we have

$$I_2 \preceq \int_0^t e^{-(t-\tau)} (1+(t-\tau))^{\delta_p} \|F(u)\|_{L^{\tilde{p}}} d\tau$$

$$
\begin{aligned}
&\preceq \int_0^t (1+(t-\tau))^{-N} \|u\|_{L^{\tilde{p}(\sigma+1)}}^{\sigma+1} d\tau \\
&\preceq \|u\|_X^{\sigma+1} \int_0^t (1+(t-\tau))^{-N} (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})(\sigma+1)} d\tau.
\end{aligned}
$$

Splitting the integral, we have

$$
\begin{aligned}
&\int_0^{t/2} (1+(t-\tau))^{-N} (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})(\sigma+1)} d\tau \\
&\preceq (1+t/2)^{-N} \int_0^{t/2} (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})(\sigma+1)} d\tau \\
&\preceq (1+t)^{-N}
\end{aligned}
$$

for any positive N, and

$$
\begin{aligned}
&\int_{t/2}^t (1+(t-\tau))^{-N} (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})(\sigma+1)} d\tau \\
&\preceq (1+t/2)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})(\sigma+1)} \int_{t/2}^t (1+(t-\tau))^{-N} d\tau \\
&\preceq (1+t/2)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})} \int_0^{t/2} (1+\tau)^{-N} d\tau \\
&\preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})}.
\end{aligned}
$$

Let us turn to I_1. Note we always assume $1<p<p_0$ and $p_0'<\tilde{p}<\infty$. So splitting the integral and applying Proposition 2.2 we have

$$
I_1 \preceq \int_0^{t/2} (1+(t-\tau))^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{\tilde{p}})} \|F(u)\|_{L^p} d\tau + \int_{t/2}^t \|F(u)\|_{L^{\tilde{p}}} d\tau := J_1 + J_2.
$$

Plugging the estimate for $\|F(u)\|_{L^p}$ above implies

$$
\begin{aligned}
J_1 &\preceq \int_0^{t/2} (1+(t-\tau))^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{\tilde{p}})} (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})\sigma} \|u\|_X^{\sigma+1} d\tau \\
&\preceq (1+t/2)^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{\tilde{p}})} \int_0^{t/2} (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})\sigma} d\tau \|u\|_X^{\sigma+1} \\
&\preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})} \|u\|_X^{\sigma+1}.
\end{aligned}
$$

For J_2, we have

$$
\begin{aligned}
J_2 &= \int_{t/2}^t \|u\|_{L^{\tilde{p}(\sigma+1)}}^{\sigma+1} d\tau \\
&\preceq \int_{t/2}^t (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})(\sigma+1)} \|u\|_X^{\sigma+1} d\tau \\
&\preceq (1+t/2)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})} \int_{t/2}^t (1+\tau)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})\sigma} d\tau \cdot \|u\|_X^{\sigma+1} \\
&\preceq (1+t)^{-\frac{n}{2\alpha}(\frac{1}{p_0}-\frac{1}{p_0'})} \|u\|_X^{\sigma+1}.
\end{aligned}
$$

Combining all, we have proved that

$$
\|\mathcal{D}u\|_X \preceq \|f,g\|_0 + \|u\|_X^{\sigma+1}.
$$

We can similarly show that

$$\|\mathcal{D}u - \mathcal{D}v\|_X \preceq \frac{1}{2}\|u - v\|_X$$

whenever $\|u\|_X$ and $\|v\|_X$ are small. Theorem 1.4 then follows by a standard contraction map argument.

Remark 4.1. Throughout the above proof, we applied many times the estimates

$$\|H(D)\Omega_\alpha(t) * h\|_{L^p} \preceq e^{-t}(1+t)^{\delta_p}\|h\|_{L^p}, \quad 1 < p < \infty \tag{4.1}$$

which is derived from Proposition 2.2 by taking

$$\alpha = \beta > n\alpha|1/2 - 1/p| \text{ or } 1 = \beta > (n-1)|1/2 - 1/p|. \tag{4.2}$$

So we have to assume $n \le 2$ (or $n \le 3$ when $\alpha = 1$). But if we let $n = 1$ (or $n \le 2$ when $\alpha = 1$), then (4.2), thus (4.1) holds for $p = 1$ and $p = \infty$.

By a similar proof as above, we obtain the following Theorem.

Theorem 4.2. *Let $\alpha > 0$, $n = 1$ (or $n = 1, 2$ if $\alpha = 1$) and $\sigma > 2\alpha/n$. If $f \in L^1_\alpha \cap L^\infty_\alpha$, $g \in L^1 \cap L^\infty$ and*

$$\|f, g\|_0 = \|f\|_{L^1_\alpha} + \|f\|_{L^\infty_\alpha} + \|g\|_{L^1} + \|g\|_{L^\infty}$$

is sufficiently small. Then there exists a unique solution $u(t, x)$ to (1.6) in the space $L^\infty(\mathbb{R}^+, L^1 \cap L^\infty)$ such that

$$\|u(t, x)\|_{L^p} \preceq (1+t)^{(1-\frac{1}{p})}\|f, g\|_0, \quad 1 \le p \le \infty.$$

The Strichartz estimate in Section 3 could be applied to solve certain semilinear equations in the space $L^q(\mathbb{R}^+, L^p(\mathbb{R}^n))$ for some admissible pairs (p, q). But to do this, we need further Strichartz type estimate on semilinear equations. We will explore this issue in our upcoming paper.

Acknowledgments. This research is supported by the NSF of China (11271330, 11201103, and 11471288).

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