Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 166, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF LOCAL AND GLOBAL SOLUTIONS FOR HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study a class of Hadamard fractional differential equations and give sufficient conditions on the existence of local and global of solutions.

1. Introduction

Let $0 \le \gamma < 1$, $1 < a < T$ and $\mathbb{G}$ be an open set in $\mathbb{R}$. Denote a Banach space by $C_{\gamma,\ln}[a,T] := \{\mu(x) : (\ln \frac{x}{a})^\gamma \mu(x) \in C[a,T]\}$ endowed with the norm $\|\mu\|_{C_{\gamma,\ln}} = \|(\ln \frac{x}{a})^\gamma \mu(x)\|_C$. In this article, we study the existence of local and global solutions to the Hadamard type fractional differential equation

$$\begin{gathered} {}_H D^\alpha_{a,x} y(x) = f(x, y(x)), \quad 0 < \alpha < 1,\ x \in \mathbb{J}, \\ {}_H D^{\alpha-1}_{a,x} y(a^+) = c, \quad c \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $\mathbb{J} = [a, a+h]$, $h > 0$ or $[a+\infty)$ and the symbol ${}_H D^\alpha_{a,x} y(x)$ is defined by

$${}_H D^\alpha_{a,x} y(x) = \frac{1}{\Gamma(1-\alpha)} \big(x \frac{d}{dx}\big) \int_a^x (\ln \frac{x}{\tau})^{-\alpha} y(\tau) \frac{d\tau}{\tau}.$$

We use the notation ${}_H D^{\alpha-1}_{a,x} y(a^+) = \lim_{x\to a^+} \mathcal{J}^{\alpha-1}_{a,x} y(x)$ and

$$\mathcal{J}^{\alpha-1}_{a,x} y(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\ln \frac{x}{t})^{\alpha-1} y(t)) \frac{dt}{t}\,.$$

Following [1, Theorem 3.28], the solution $y \in C_{1-\gamma,\ln}[a, a+h]$ of (1.1) satisfies

$$y(x) = y_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x (\ln \frac{x}{\tau})^{\alpha-1} f(\tau, y(\tau)) \frac{d\tau}{\tau}, \quad x \in (a, a+h] \tag{1.2}$$

where $y_0(x) = \frac{c}{\Gamma(\alpha)} (\ln \frac{x}{a})^{\alpha-1}$, if $f : (a, a+h] \times \mathbb{G} \to \mathbb{R}$ and $f(x,y) \in C_{\gamma,\ln}[a, a+h]$ for any $y \in \mathbb{G}$.

Inspired by the work in [1, 2, 3], we examine other explicit sufficient conditions on the nonlinear term f to guarantee the local existence of solutions in $C_{\gamma,\ln}[a, a+h]$ and global existence of solutions in $C_{\gamma,\ln}[a, +\infty)$.

2010 *Mathematics Subject Classification.* 26A33, 34A12.
Key words and phrases. Hadamard fractional differential equation; local solution; global solution.

Submitted February 2, 2015. Published June 17, 2015.

2. Main results

The following equality will be used in the sequel.

Lemma 2.1 ([4, p.296]). *Let $\alpha, \beta, \gamma, p > 0$, then*

$$\int_0^x (x^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{x^\theta}{\alpha} B\big[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\big], \quad x > 0,$$

where $\theta = p[\alpha(\beta-1)+\gamma-1]+1$ and $B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds$.

Let $\mathbb{B} = \{y \in \mathbb{R} : \|y - y_0(x)\|_{C_{\gamma,\ln}} \le b\}$ where b will be chosen latter. Define $\mathbb{D} = \{(x,y) \in R \times R : x \in \mathbb{J}, y \in \mathbb{B}\}$. We assume that $f : \mathbb{D} \to \mathbb{R}$ satisfies the following conditions:

- (H1) $f(x,y)$ is Lebesgue measurable with regard to x on $\mathbb{J}$ and $f(x,y)$ is continuous with respect to y on $\mathbb{B}$.
- (H2) there exists $m(\cdot) \in L^q(\mathbb{J})$, $q > 1$ such that $|f(x,y)| \le m(x)$, for arbitrary $x \in \mathbb{J}$, $y \in \mathbb{B}$.

Now we use Picard iterative approach to derive the existence of a local solutions to (1.1).

Theorem 2.2. *Assume that (H1)–(H2) hold for $\mathbb{J} = [a, a+h]$ and p, q, α satisfy $p(\alpha-1)+1 > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. Then (1.1) has a solution in $C_{\gamma,\ln}[a, a+h]$ for some $h > 0$.*

Proof. To achieve our aim, we divide our proof into three steps.

Step 1. Linking our assumptions and using Hölder inequality via $p(\alpha-1)+1 > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, one can obtain

$$\int_a^x |(\ln x - \ln \tau)^{\alpha-1} f(\tau, y(\tau))| \frac{dt}{\tau} \le \sqrt[1/p]{\frac{a^{1-p} h^{p(\alpha-1)+1}}{p(\alpha-1)+1}} \|m(\cdot)\|_{L^q[a,a+h]}, \tag{2.1}$$

where we use basic inequalities: $\ln u - \ln v \le u - v$ for $u \ge v > 1$ and [5, Lemma 2.2],

$$\int_a^x (\ln x - \ln \tau)^{p(\alpha-1)} \tau^{-p} d\tau \le \frac{a^{1-p}(\ln x - \ln a)^{p(\alpha-1)+1}}{p(\alpha-1)+1}. \tag{2.2}$$

This proves that $(\ln x - \ln \tau) f(\tau, y(\tau))$ is Lebesgue integrable with respect to $\tau \in [a, x]$ for arbitrary x on $\mathbb{J}$, provided that $y(\tau)$ is Lebesgue measurable on the interval $[a, a+h]$.

Step 2. For a given $M > 0$, there exists a $h' > 0$ satisfying

$$\int_a^{a+h'} m^q(\tau) d\tau \le M^q, \tag{2.3}$$

whenever $h = \min\{h', T, [\frac{b\Gamma(\alpha)(p(\alpha-1)+1)}{M a^{p-1}(\ln T - \ln a)^\gamma}]^{\frac{1}{p(\alpha-1)+1}}\}$. For δ to be chosen latter, define

$$y_n(x) = \begin{cases} 0, \quad \text{if } a \le x < a+\delta, \ 0 < \delta < \frac{h}{n}, \\ \frac{c}{\Gamma(\alpha)}(\ln \frac{x}{a})^{\alpha-1}, \quad \text{if } a+\delta \le x < a+\frac{h}{n}, \\ \frac{c}{\Gamma(\alpha)}(\ln \frac{x}{a})^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^{x-\frac{h}{n}} (\ln \frac{x}{\tau})^{\alpha-1} f(\tau, y_n(\tau)) \frac{d\tau}{\tau}, \\ \quad \text{if } a + \frac{h}{n} \le x \le a+h. \end{cases}$$

We show that $y_n(x)$ is continuous on $[a, a+\frac{2h}{n}]$ for all n.

Case 1. For $a < a+\delta \le x_1 < a+\frac{h}{n} < x_2 \le a+h$,

$$\begin{aligned}|y_n(x_2)-y_n(x_1)| &\le \frac{|c|}{\Gamma(\alpha)}\big|(\ln\frac{x_2}{a})^{\alpha-1}-(\ln\frac{x_1}{a})^{\alpha-1}\big| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^{x_2-\frac{h}{n}}(\ln\frac{x_2}{\tau})^{\alpha-1}|f(\tau,y_n(\tau))|\frac{d\tau}{\tau} \\ &:= I_1+I_2.\end{aligned}$$

For each $\varepsilon>0$, there exists $0<\delta_1<[\frac{a\varepsilon\Gamma(\alpha)\delta^{2-\alpha}}{2|c|(1-\alpha)}]$ such that for all $x_2-x_1\le\delta_1$ and for all n, we derive that

$$\begin{aligned}I_1 &= \frac{|c|}{\Gamma(\alpha)}|\chi(\frac{x_2}{a})-\chi(\frac{x_1}{a})| \le \frac{|c|}{\Gamma(\alpha)}|\chi'(\xi)||\frac{x_2}{a}-\frac{x_1}{a}|,\ \xi\in(\frac{x_1}{a},\frac{x_2}{a}) \\ &\le \frac{|c|(1-\alpha)(x_2-x_1)}{\Gamma(\alpha)(a+\delta)(\ln(a+\delta)-\ln a)^{2-\alpha}} \\ &< \frac{|c|(1-\alpha)(x_2-x_1)}{\Gamma(\alpha)a(\ln(a+\delta)-\ln a)^{2-\alpha}} < \varepsilon/2,\end{aligned}$$

where $\chi(x)=(\ln x-\ln a)^{\alpha-1}$ and $\chi'(x)=(\alpha-1)(\ln x-\ln a)^{\alpha-2}\frac{a}{x}$.

For any $\varepsilon>0$, there exists $0<\delta_2<[\frac{\varepsilon^p\Gamma^p(\alpha)(p(\alpha-1)+1)}{2^pa^{1-p}M^p}]^{\frac{1}{p(\alpha-1)+1}}$ such that for all $x_2-\frac{h}{n}-a\le x_2-x_1\le\delta_2$ and for all n, we use (2.2) and (2.3) to obtain

$$\begin{aligned}I_2 &= \frac{1}{\Gamma(\alpha)}\int_a^{x_2-\frac{h}{n}}(\ln\frac{x_2}{\tau})^{\alpha-1}|f(\tau,y_n(\tau))|\frac{d\tau}{\tau} \\ &\le \frac{1}{\Gamma(\alpha)}\int_a^{x_2-\frac{h}{n}}(\ln\frac{x_2}{\tau})^{\alpha-1}m(\tau)\frac{d\tau}{\tau} \\ &\le \frac{M}{\Gamma(\alpha)}\Big[\frac{a^{1-p}(\ln(x_2-\frac{h}{n})-\ln a)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\Big]^{1/p} \\ &\le \frac{M}{\Gamma(\alpha)}\Big[\frac{a^{1-p}(x_2-\frac{h}{n}-a)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\Big]^{1/p} < \varepsilon/2,\end{aligned}$$

where we use that $\ln u-\ln v\le u-v$, $u>v>1$ again in the last inequality.

From above, we can choose $\bar\delta=\min\{\delta_1,\delta_2,h/n\}$ such that for all $x_2-x_1\le\bar\delta$ and for all n, such that $|y_n(x_2)-y_n(x_1)|\le I_1+I_2<\varepsilon$.

Case 2. For $a+\frac{h}{n}\le x_1<x_2\le a+\frac{2h}{n}$. One has

$$\begin{aligned}&|y_n(x_2)-y_n(x_1)| \\ &\le S_1+\frac{1}{\Gamma(\alpha)}\int_a^{x_1-\frac{h}{n}}\Big((\ln\frac{x_2}{\tau})^{\alpha-1}-(\ln\frac{x_1}{\tau})^{\alpha-1}\Big)|f(\tau,y_n(\tau))|\frac{d\tau}{\tau} \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{x_1-\frac{h}{n}}^{x_2-\frac{h}{n}}(\ln\frac{x_2}{\tau})^{\alpha-1}|f(\tau,y_n(\tau))|\frac{d\tau}{\tau} \\ &:= S_1+S_2+S_3.\end{aligned}$$

For each $\varepsilon > 0$, there exists $0 < \bar{\delta_1} < \Big[\frac{a\varepsilon\Gamma(\alpha)(\frac{h}{n})^{2-\alpha}}{3|c|(1-\alpha)}\Big]$ such that for all $x_2 - x_1 \le \bar{\delta_1}$, we have

$$S_1 \le \frac{|c|(1-\alpha)(x_2 - x_1)}{\Gamma(\alpha)a(\ln(a+\frac{h}{n}) - \ln a)^{2-\alpha}} < \varepsilon/3.$$

For each $\varepsilon > 0$, there exists $0 < \bar{\delta_2} < \Big[\frac{\varepsilon^p\Gamma(\alpha)^p(p(\alpha-1)+1)}{3^pM^pa^{1-p}}\Big]^{\frac{1}{p(\alpha-1)+1}}$ such that for all $x_2 - x_1 \le \bar{\delta_2}$, by using the similar estimation methods of I_2 we have

$$\begin{aligned}
S_2 &= \frac{1}{\Gamma(\alpha)}\int_a^{x_1-\frac{h}{n}} \Big((\ln\frac{x_2}{\tau})^{\alpha-1} - (\ln\frac{x_1}{\tau})^{\alpha-1}\Big)|f(\tau, y_n(\tau))|\frac{d\tau}{\tau} \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^{x_1-\frac{h}{n}} \Big((\ln\frac{x_2}{\tau})^{\alpha-1} - (\ln\frac{x_1}{\tau})^{\alpha-1}\Big)m(\tau)\frac{d\tau}{\tau} \\
&\le \frac{M}{\Gamma(\alpha)}\Big[\Big(\int_a^{x_1-\frac{h}{n}}(\ln\frac{x_2}{\tau})^{p(\alpha-1)}\tau^{-p}d\tau\Big)^{1/p} - \Big(\int_a^{x_1-\frac{h}{n}}(\ln\frac{x_1}{\tau})^{p(\alpha-1)}\tau^{-p}d\tau\Big)^{1/p}\Big] \\
&\le \frac{M}{\Gamma(\alpha)}\Big[\frac{(\ln x_2 - \ln x_1)^{p(\alpha-1)+1}}{a^{p-1}(p(\alpha-1)+1)}\Big]^{1/p} \\
&\le \frac{M}{\Gamma(\alpha)}\Big[\frac{(x_2 - x_1)^{p(\alpha-1)+1}}{a^{p-1}(p(\alpha-1)+1)}\Big]^{1/p} < \varepsilon/3.
\end{aligned}$$

For each $\varepsilon > 0$, there exists $0 < \bar{\delta_3} < [\frac{\varepsilon^p\Gamma(\alpha)^p(p(\alpha-1)+1)}{3^pM^pa^{1-p}}]^{\frac{1}{p(\alpha-1)+1}}$, such that for all $x_2 - x_1 \le \bar{\delta_3}$, by using the similar estimation methods of I_2 we have

$$\begin{aligned}
S_3 &= \frac{1}{\Gamma(\alpha)}\int_{x_1-\frac{h}{n}}^{x_2-\frac{h}{n}} (\ln\frac{x_2}{\tau})^{\alpha-1}|f(\tau, y_n(\tau))|\frac{d\tau}{\tau} \\
&\le \frac{1}{\Gamma(\alpha)}\int_{x_1-\frac{h}{n}}^{x_2-\frac{h}{n}} (\ln\frac{x_2}{\tau})^{\alpha-1}m(\tau)\frac{d\tau}{\tau} \\
&\le \frac{M}{\Gamma(\alpha)}\big[\frac{a^{1-p}(x_2 - x_1)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\big]^{1/p} < \varepsilon/3.
\end{aligned}$$

From the above, we choose $\bar{\bar{\delta}} = \min\{\bar{\delta_1}, \bar{\delta_2}, \bar{\delta_3}, h/n\}$ such that $x_2 - x_1 \le \bar{\bar{\delta}}$, then $|y_n(x_2) - y_n(x_1)| \le S_1 + S_2 + S_3 < \varepsilon$. Therefore, we choose $\delta = \min\{\bar{\delta}, \bar{\bar{\delta}}\}$ will lead to $y_n(x)$ is continuous with regard to x on $[a, a+\frac{2h}{n}]$ for all positive integers n. Note that $(\ln x - \ln a)^\gamma$ is continuous function, so $y_n(x)(\ln x - \ln a)^\gamma$ is also continuous.

Nevertheless, for all $x \in [a+\delta, a+\frac{h}{n}]$, one has $|y_n(x) - \frac{c}{\Gamma(\alpha)}(\ln\frac{x}{a})^{\alpha-1}| = 0$, and for all $x \in [a+\frac{h}{n}, a+h]$, using Hölder inequality again,

$$\begin{aligned}
&(\ln\frac{x}{a})^\gamma|y_n(x) - \frac{c}{\Gamma(\alpha)}(\ln\frac{x}{a})^{\alpha-1}| \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^{x-\frac{h}{n}}(\ln\frac{x}{a})^\gamma(\ln\frac{x}{\tau})^{\alpha-1}|f(\tau, y_n(\tau))|\frac{d\tau}{\tau} \qquad (2.4)\\
&\le \frac{(\ln(a+h) - \ln a)^\gamma a^{1-p}Mh^{p(\alpha-1)+1}}{\Gamma(\alpha)(p(\alpha-1)+1)} \le b,
\end{aligned}$$

which implies that $(x, y_n(x)) \in \mathbb{D}$ for all n. Therefore, $\{y_n(x)\}_{n=1}^\infty$ defined on $[a, a+h]$ is equicontinuous and uniformly bounded.

Step 3. By using Arzelò-Ascoli lemma and Step 2, there must exist $\{y_{n_k}(x)\}_{k=1}^{\infty} := \{y_k(x)\}_{k=1}^{\infty}$ contained in $\{y_n(x)\}_{n=1}^{\infty}$, such that $\{y_k(x)\}_{k=1}^{\infty}$ is uniformly convergent to $y(x)$ which is continuous with regard to x on $[a, a+h]$. Now we only need to prove that this limit function $y(x)$ is a solution of (1.2).

For each $\varepsilon > 0$, there exists $K_1 > 0$, such that for all $k > K_1$, and $x \in [a, a+h]$, we have

$$|f(x, y_k(x)) - f(x, y(x))| < \frac{\Gamma(\alpha+1)\varepsilon}{2(\ln(a+h) - \ln a)^{\gamma} h^{\alpha}}. \tag{2.5}$$

Note that

$$\begin{aligned}
&(\ln x - \ln a)^{\gamma}|y_k(x) - y(x)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_a^x (\ln\frac{x}{a})^{\gamma}(\ln\frac{x}{\tau})^{\alpha-1}|f(\tau, y_k(\tau)) - f(\tau, y(\tau))|\frac{d\tau}{\tau} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{x-\frac{h}{k}}^x (\ln\frac{x}{a})^{\gamma}(\ln\frac{x}{\tau})^{\alpha-1}|f(\tau, y_k(\tau))|\frac{d\tau}{\tau} \\
&:= S_4 + S_5.
\end{aligned}$$

Using (2.5) one obtains,

$$\begin{aligned}
S_4 &= \frac{1}{\Gamma(\alpha)} \int_a^x (\ln\frac{x}{a})^{\gamma}(\ln\frac{x}{\tau})^{\alpha-1}|f(\tau, y_k(\tau)) - f(\tau, y(\tau))|\frac{d\tau}{\tau} \\
&\le \frac{(\ln(a+h) - \ln a)^{\gamma}}{\Gamma(\alpha)} \int_x^a (\ln\frac{x}{\tau})^{\alpha-1}|f(\tau, y_k(\tau)) - f(\tau, y(\tau))|d(\ln x - \ln\tau) \\
&\le \frac{(\ln(a+h) - \ln a)^{\gamma} h^{\alpha}}{a\Gamma(\alpha)} \frac{\Gamma(\alpha+1)\varepsilon}{2(\ln(a+h) - \ln a)^{\gamma} h^{\alpha}} < \varepsilon/2.
\end{aligned}$$

Also there exists

$$0 < K_2 = h\Big[\frac{a^{p-1}\varepsilon^p \Gamma(\alpha)^p (p(\alpha-1)+1)}{2^p M^p (\ln(a+h) - \ln a)^{p\gamma}}\Big]^{\frac{-1}{p(\alpha-1)+1}},$$

such that for all $k > K_2$,

$$\begin{aligned}
S_5 &= \frac{1}{\Gamma(\alpha)} \int_{x-\frac{h}{k}}^x (\ln\frac{x}{a})^{\gamma}(\ln\frac{x}{\tau})^{\alpha-1}|f(\tau, y_k(\tau))|\frac{d\tau}{\tau} \\
&\le \frac{(\ln(a+h) - \ln a)^{\gamma} M}{\Gamma(\alpha)} \Big(\int_{x-\frac{h}{k}}^x (\ln\frac{x}{\tau})^{p(\alpha-1)}\tau^{-p} dt\Big)^{1/p} \\
&\le \frac{(\ln(a+h) - \ln a)^{\gamma} M}{\Gamma(\alpha)} \Big(\frac{(\frac{h}{k})^{p(\alpha-1)+1}}{a^{p-1}(p(\alpha-1)+1)}\Big)^{1/p} < \varepsilon/2.
\end{aligned}$$

Hence, taking $K = \max\{K_1, K_2\}$ and for all $k > K$, one arrives at $\|y_k(x) - y(x)\|_{C_{\gamma,\ln}} < \varepsilon$. Consequently, $y(x)$ satisfies (1.2) which means that there at least exists a solution of (1.1). □

Next, we give an existence and uniqueness theorem, using the assumption

- (H3) there exists a $\mu(\cdot) \in L^q(\mathbb{J})$, $\frac{1}{q} = 1 - \frac{1}{p}$, $p > 1$ such that $|f(x, y) - f(x, z)| \le \mu(x)|y - z|$ for $x \in \mathbb{J}$ and $y, z \in \mathbb{B}$,

Theorem 2.3. *Let $0 \le \gamma \le \min\{\alpha - 1 + \frac{1}{p}, \frac{1}{2p}\}$, $p > 1$. Assume that* (H1)–(H3) *are satisfied for $\mathbb{J} = [a, a+h]$. Then* (1.1) *has a unique solution in $C_{\gamma,\ln}[a, a+h]$ for some $h > 0$.*

Proof. There exits $h^*>0$, such that for all $x\in[a,a+h^*]$,
$$\int_a^x \mu^q(\tau)d\tau \le \int_a^{a+h^*} \mu^q(\tau)d\tau < g^q. \tag{2.6}$$

Let $\Psi_h=\{y\in C_{\gamma,\ln}[a,a+h]:\|y(x)-y_0(x)\|_{C_{\gamma,\ln}}\le b,\ x\in[a,a+h]\}$, where h is the smaller one between h^* and h obtained in Theorem 2.2. Note Ψ_h endowed with $\|\cdot\|_{C_{r\ln}}$ is a Banach space. Define
$$\psi(y)=y_0(x)+\frac{1}{\Gamma(\alpha)}\int_a^x(\ln\frac{x}{\tau})^{\alpha-1}f(\tau,y(\tau))\frac{d\tau}{\tau},\quad \forall x\in[a,a+h].$$

Firstly, assume that $a\le x_1<x_2\le a+h$, then according to the proof in Step 2 of Theorem 2.2, we obtain that $\psi(y)$ is continuous with regard to x on $[a,a+h]$. Note that $(\ln x-\ln a)^\gamma$ is continuous function, so $\psi(y)(\ln x-\ln a)^\gamma$ is also continuous. Secondly, it following (2.4) that $\psi(y)\in\Psi_h$ with $y\in\Psi_h$. Thirdly, the condition $0\le\gamma\le\min\{\alpha-1+\frac{1}{p},\frac{1}{2p}\}$, $p>1$ implies that $p(\alpha-1-\gamma)+1>0$, $0\le\gamma<\frac{1}{2p}$. For any $y_1,y_2\in\Psi_h$, then using Hölder inequality, (2.6) and Lemma 2.1, we have
$$\begin{aligned}
&(\ln x-\ln a)^\gamma|\psi(y_2)-\psi(y_1)|\\
&\le\frac{(\ln x-\ln a)^\gamma}{\Gamma(\alpha)}\int_a^x(\ln\frac{x}{\tau})^{\alpha-1}\mu(\tau)|y_1(\tau)-y_2(\tau)|\frac{d\tau}{\tau}\\
&\le\frac{(\ln(a+h)-\ln a)^\gamma}{\Gamma(\alpha)}\int_a^x(\ln\frac{x}{\tau})^{\alpha-1}(\ln\frac{\tau}{a})^{-\gamma}\mu(\tau)\frac{d\tau}{\tau}\|y_1-y_2\|_{C_{r\ln}}\\
&\le\frac{(\ln(a+h)-\ln a)^\gamma g a^{1-p}}{\Gamma(\alpha)}\Big(\int_0^{\ln x-\ln a}(\ln\frac{x}{a}-t)^{p(\alpha-1)}t^{-p\gamma}dt\Big)^{1/p}\|y_1-y_2\|_{C_{\gamma\ln}}\\
&\le\frac{(\ln(a+h)-\ln a)^\gamma g a^{1-p}}{\Gamma(\alpha)}\Big((\ln\frac{x}{a})^{p(\alpha-1-\gamma)+1}B[1-p\gamma,p(\alpha-1)+1]\Big)^{1/p}\\
&\quad\times\|y_1-y_2\|_{C_{\gamma\ln}}\\
&\le a^{1-p}\frac{h^\gamma g}{\Gamma(\alpha)}(h^{p(\alpha-1-\gamma)+1}B[\frac{1}{2},p(\alpha-1)+1])^{1/p}\|y_1-y_2\|_{C_{\gamma\ln}}\\
&=a^{1-p}\Big[\frac{h^{\alpha-1+\frac{1}{p}}g}{\Gamma(\alpha)}\Big(B[\frac{1}{2},p(\alpha-1)+1]\Big)^{1/p}\Big]\|y_1-y_2\|_{C_{\gamma\ln}},
\end{aligned}$$
where we use that
$$\gamma<\frac{1}{2p}\Rightarrow 1-p\gamma>\frac{1}{2}\Rightarrow t^{1-pr}\le t^{\frac{1}{2}}\ (0\le t\le1),$$
$$\begin{aligned}
B[1-p\gamma,p(\alpha-1)+1]&=\int_0^1 t^{-p\gamma}(1-t)^{p(\alpha-1)}dt\\
&\le\int_0^1 t^{-\frac{1}{2}}(1-t)^{p(\alpha-1)}dt=B[\frac{1}{2},p(\alpha-1)+1].
\end{aligned}$$
Obviously, one can choose
$$h\le\Big(\frac{\Gamma(\alpha)}{g(B[\frac{1}{2},p(\alpha-1)+1])^{1/p}}\Big)^{\frac{p}{p(\alpha-1)+1}},$$
then
$$\frac{h^{\alpha-1+\frac{1}{p}}g}{\Gamma(\alpha)}\Big(B[\frac{1}{2},p(\alpha-1)+1]\Big)^{1/p}\le1.$$

Therefore,
$$\|\psi(y_2)-\psi(y_1)\|_{C_{\gamma\,\ln}} \le a^{1-p}\|y_1-y_2\|_{C_{\gamma\,\ln}}.$$
Obviously, $a^{1-p}<1$ due to $a,p>1$, applying the Banach Contractive Mapping Principle, one concludes that there exists a unique $y^*(x)\in\Psi_h$, such that (1.2). The proof is compete. □

Next, we give the existence of a global solution, using the assumption

(H2') there exist $\omega,\nu>0$ such that $|f(x,y)|\le\omega+\nu|y|$ for $x\in(a,+\infty)$ and $y\in\mathbb{R}$.

Theorem 2.4. *Assume that* (H1), (H2'), (H3) *hold for* $\mathbb{J}=(a,+\infty)$. *Further, choose* $\gamma=1-\alpha\le\min\{\alpha-1+\frac{1}{p},\frac{1}{2p}\}$, $p>1$. *Then* (1.1) *has a unique solution in* $C_{\gamma,\ln}[a,+\infty)$.

Proof. It follows (H2') that f is locally bounded in the domain $\mathbb{D}$. By Theorem 2.3, (1.1) has a unique solution in $C_{\gamma,\ln}[a,a+h]$. Next, we present proof by contradiction. Assume that the solution $y(x)$ admits a maximal existence interval, denoted by $(a,T)\subset(a,+\infty)$. To achieve our aim, it is sufficient to verify that $\|y\|_{C_{\gamma,\ln}}$ is bounded. In fact,
$$\begin{aligned}
&(\ln x-\ln a)^{\gamma}|y(x)|\\
&\le \frac{|c|}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_a^x(\ln\frac{x}{a})^{\gamma}(\ln\frac{x}{\tau})^{\alpha-1}|f(\tau,y(\tau))|\frac{d\tau}{\tau}\\
&\le \frac{|c|}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_a^x(\ln\frac{x}{a})^{\gamma}(\ln\frac{x}{\tau})^{\alpha-1}(\omega+\nu|y(\tau)|)\frac{d\tau}{\tau}\\
&\le \frac{|c|}{\Gamma(\alpha)}+\frac{\omega(\ln T-\ln a)^{1-\alpha}(T-a)^{\alpha}}{\Gamma(a+1)}\\
&\quad+\frac{\nu}{\Gamma(\alpha)}\int_a^x(\ln\frac{x}{\tau})^{\alpha-1}(\ln x-\ln a)^{\gamma}|y(\tau)|\frac{d\tau}{\tau}\\
&\le \frac{|c|}{\Gamma(\alpha)}+\frac{\omega(T-a)}{\Gamma(\alpha+1)}+\frac{\nu}{\Gamma(\alpha)}\int_a^x(\ln\frac{x}{\tau})^{\alpha-1}(\ln x-\ln a)^{\gamma}|y(\tau)|\frac{d\tau}{\tau}.
\end{aligned}$$

By applying the generalized Gronwall inequality from [3, Corollary 3.4], one can conclude that there exists $l:=\mathbb{E}_\alpha(\nu(\ln T)^{\alpha})>0$ ($\mathbb{E}_\alpha$ denotes Mittag-Leffler function) such that
$$(\ln x-\ln a)^{\gamma}|y(x)|\le l(\frac{|c|}{\Gamma(\alpha)}+\frac{\omega(T-a)}{\Gamma(\alpha+1)}):=\rho<+\infty.$$
This implies that $\|y\|_{C_{\gamma,\ln}}<b$ on $[a,T)$ when b is chosen as a larger number than
$$b=\rho+\frac{|c|}{\Gamma(\alpha)}. \tag{2.7}$$
This contradicts the assumption that (a,T) is the maximal existence interval. The proof is complete. □

To finish this article, we give an example that illustrates our theoretical results. Consider
$$\begin{gathered}
{}_HD^{3/4}_{e,x}y(x)=x^2+\frac{4|y|}{1+|y|}\sin x,\quad x\in\mathbb{J}=(e,e^2]\text{ or }(e,+\infty),\\
{}_HD^{-1/4}_{e,x}y(e^+)=1,
\end{gathered}\tag{2.8}$$

where $\alpha = 3/4$, $T = e^2$, $\gamma = 1/4$, $a = e$, $c = 1$, and $p = q = 2$.

Define $f(x, y) = x^2 + \frac{4|y|}{1+|y|} \sin x$, $\mu(x) = 4$ and $\omega = e^2 + 4$ and $\nu = 0$. Thus $|f(x, y) - f(x, z)| \leq \mu(x)|y - z|$ and $|f(x, y)| \leq \omega$. Then $l := \mathbb{E}_\alpha(0) = 1$ (see [6, Lemma 2]) and $b = \frac{2}{\Gamma(3/4)} + \frac{(e^2+4)(e^2-e)}{\Gamma(7/4)}$ (see (2.7)).

Let $h' = h^* = e$. Set $M^2 = \int_e^{2e} (e^4 + 4)^2 dx = e(e^4 + 4)^2$ (see (2.3)) and $g^2 = \int_e^{2e} 16dx = 16e$ (see (2.6)). Moreover, one can find $\gamma = 1 - \alpha = \min\{\alpha - 1 + \frac{1}{p}, \frac{1}{2p}\}$.

- According to Theorem 2.3, (2.8) admits a unique solution $y \in C_{\frac{1}{4},\ln}(e, e + h]$ where

$$
\begin{aligned}
h = \min\Big\{ & h', T, \Big[\frac{b\Gamma(\alpha)(p(\alpha-1)+1)}{Ma^{p-1}}\Big]^{\frac{1}{p(\alpha-1)+1}}, \\
& \Big(\frac{\Gamma(\alpha)}{g(B[\frac{1}{2}, p(\alpha-1)+1])^{1/p}}\Big)^{\frac{p}{p(\alpha-1)+1}}\Big\}
\end{aligned}
$$

$$
= \Big\{e, e^2, \Big[\frac{(\frac{2}{\Gamma(3/4)} + \frac{(e^2+4)(e^2-e)}{\Gamma(7/4)})\Gamma(3/4)}{2\sqrt{e}(e^4+e)e}\Big]^2, \Big[\frac{\Gamma(3/4)}{4\sqrt{e}\sqrt{B[\frac{1}{2}, \frac{1}{2}]}}\Big]^4\Big\}.
$$

- According to Theorem 2.4, (2.8) has a unique solution $y \in C_{\frac{1}{4},\ln}(e, +\infty)$.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (11201091) and by the Outstanding Scientific and Technological Innovation Talent Award of Education Department of Guizhou Province ([2014]240).

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