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EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS FOR FRACTIONAL SEMILINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence and uniqueness of a local mild solution for a class of semilinear differential equations involving the Caputo fractional time derivative of order α ($0 < \alpha < 1$) and, in the linear part, a sectorial linear operator A. We put some conditions on a nonlinear term f and an initial data u_0 in terms of the fractional power of A. By applying Banach's Fixed Point Theorem, we obtain a unique local mild solution with smoothing effects, estimates, and a behavior at t close to 0. An example as an application of our results is also given.

1. Introduction

Some existing researches showed that, in diffusion process, there are particle's movements that can be no longer modelled by the (normal) diffusion equation. To see these phenomenons, one can refer to [1, 3, 8, 16] observing the dispersion in a heterogeneous aquifer, the transport of contaminants in geological formations, the dispersive transport of ions in column experiments, and the diffusion of water in sand, respectively. All of these processes follow the pattern

$$\langle x^2(t)\rangle \sim t^\alpha, \quad 0 < \alpha < 1, \tag{1.1}$$

where $\langle x^2(t)\rangle$ is the mean square displacement at time t. These processes are called subdiffusion and can be modelled by the equation

$$D_t^\alpha u(x,t) = D_\alpha \Delta u(x,t), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.2}$$

where $0 < \alpha < 1$, D_α is a subdiffusion coeficient, and D_t^α is the Caputo fractional derivative of order α. Reaction subdiffusion equation was also derived (see [2, 9, 10, 11, 17, 18, 22, 27, 28]). Subdifusion model can also be a formula for memory phenomenon (see [13, 21]). In [5], Du et al. also found that the order of fractional derivative is an index of memory. Thus a study to investigate a solution to this model is very useful. Recently, there are some researches studying a solution to fractional evolution equations, for instance, see [4, 6, 19, 20, 24, 26, 29, 31, 32, 33].

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In this article, we show the existence and uniqueness of a local mild solution to the fractional abstract Cauchy problem

$$\begin{gathered} D_t^\alpha u = Au + f(u),\ t > 0,\ 0 < \alpha < 1, \\ u(0) = u_0, \end{gathered} \tag{1.3}$$

where H is a Banach space, D_t^α is the Caputo fractional derivative of order α, $A : D(A) \to H$ is a sectorial linear operator, $u_0 \in H$, and $f : H \to H$. We use some conditions on f and u_0 in terms of the fractional power of A. The conditions are

(i) $f(0) = 0$,
(ii) there exist $C_0 > 0$, $\vartheta > 1$, and $0 < \beta < 1$ such that

$$\|f(u) - f(v)\| \le C_0(\|A^\beta u\| + \|A^\beta v\|)^{\vartheta-1}\|A^\beta u - A^\beta v\|$$

for all $u, v \in D(A^\beta)$,
(iii) $u_0 \in D(A^\nu)$ for some $0 < \nu < 1$.

These conditions are used to study the solvability and smoothing effect for some class of semilinear parabolic equations (see [14]). As in [14], we apply Banach's Fixed Point Theorem to construct a local mild solution to the problem (1.3) by employing the properties of solution operators generated by A and the fractional power of A. In this paper, we obtain the existence and uniqueness of the local mild solution with smoothing effects, estimates, and a behaviour at t close to 0 as the advantages of our results compared with the preceding related results.

This article is composed of four sections. In section 2, we introduce briefly the fractional integration and differentiation of Riemann-Liouville and Caputo operators. In this section, we also provides some properties of analytic solution operators for fractional evolution equations including some estimates involving the fractional power of sectorial operators. In next section, our main results are showed. Finally, in the last section, an application of our main results is given.

2. Preliminaries

2.1. **Fractional time derivative.** Let $0 < \alpha < 1$, $a \ge 0$ and $I = (a, T)$ for some $T > 0$. The *Riemann-Liouville fractional integral* of order α is defined by

$$J_{a,t}^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad f \in L^1(I),\ t > a. \tag{2.1}$$

We set $J_{a,t}^0 f(t) = f(t)$. The fractional integral operator (2.1) obeys the semigroup property

$$J_{a,t}^\alpha J_{a,t}^\beta = J_{a,t}^{\alpha+\beta}, \quad 0 \le \alpha,\ \beta < 1. \tag{2.2}$$

The *Riemann-Liouville fractional derivative* of order α is defined by

$$\mathcal{D}_{a,t}^\alpha f(t) = D_t \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)ds, \quad f \in L^1(I),\ t^{-\alpha} * f \in W^{1,1}(I),\ t > a, \tag{2.3}$$

where $*$ denotes the convolution of functions

$$(f * g)(t) = \int_a^t f(t-\tau)g(\tau)d\tau, \quad t > a,$$

and $W^{1,1}(I)$ is the set of all functions $u \in L^1(I)$ such that the distributional derivative of u exists and belongs to $L^1(I)$. The operator $\mathcal{D}^\alpha_{a,t}$ is a left inverse of $J^\alpha_{a,t}$; that is,

$$\mathcal{D}^\alpha_{a,t} J^\alpha_{a,t} f(t) = f(t), \quad t > a,$$

but it is not a right inverse, that is

$$J^\alpha_{a,t}\mathcal{D}^\alpha_{a,t} f(t) = f(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} J^{1-\alpha}_{a,t} f(a), \quad t > a.$$

The *Caputo fractional derivative* of order α is defined by

$$D^\alpha_{a,t} f(t) = D_t \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}(f(s) - f(0))ds, \ t > a, \tag{2.4}$$

if $f \in L^1(I),\ t^{-\alpha} * f \in W^{1,1}(I)$, or

$$D^\alpha_{a,t} f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} D_s f(s) ds, \ t > a, \tag{2.5}$$

if $f \in W^{1,1}(I)$. The operator $D^\alpha_{a,t}$ is also a left inverse of $J^\alpha_{a,t}$, that is

$$D^\alpha_{a,t} J^\alpha_{a,t} f(t) = f(t), \quad t > a, \tag{2.6}$$

but it is not also a right inverse, that is

$$J^\alpha_{a,t} D^\alpha_{a,t} f(t) = f(t) - f(a), \quad t > a. \tag{2.7}$$

The relation between the Riemann-Liouville and Caputo fractional derivative is

$$D^\alpha_{a,t} f(t) = \mathcal{D}^\alpha_{a,t} f(t) - \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a), \quad t > a. \tag{2.8}$$

For $a = 0$, we set $J^\alpha_{a,t} = J^\alpha_t$, $\mathcal{D}^\alpha_{a,t} = \mathcal{D}^\alpha_t$, and $D^\alpha_{a,t} = D^\alpha_t$. We refer to Kilbas et al [15] or Podlubny [25] for more details concerning the fractional integrals and derivatives.

2.2. **Analytic solution operators.** In this section, we provide briefly some results concerning solution operators for the fractional Cauchy problem

$$\begin{gathered} D^\alpha_t u(t) = Au(t) + f(t), \ t > 0, \\ u(0) = u_0. \end{gathered} \tag{2.9}$$

For more details, we refer to Guswanto [7].

Henceforth, we assume that the linear operator $A : D(A) \subset H \to H$ satisfies the properties that there is a constant $\theta \in (\pi/2, \pi)$ such that

$$\rho(A) \supset S_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}, \tag{2.10}$$

$$\|R(\lambda; A)\| \leq \frac{M}{|\lambda|}, \quad \lambda \in S_\theta, \tag{2.11}$$

where $R(\lambda; A) = (\lambda - A)^{-1}$ and $\rho(A)$ are the resolvent operator and resolvent set of A, respectively. We call A as a sectorial operator. Every operator satisfying this property is closed since its resolvent set is not empty. The linear operator A generates solution operators for the problem (2.9), those are

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1} R(\lambda^\alpha; A) d\lambda, \quad t > 0, \tag{2.12}$$

$$P_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda^\alpha; A) d\lambda, \quad t > 0, \tag{2.13}$$

where $r > 0$, $\pi/2 < \omega < \theta$, and

$$\Gamma_{r,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \omega, |\lambda| = r\}$$

is oriented counterclockwise. By the Cauchy's theorem, the integral form (2.12) and (2.13) are independent of $r > 0$ and $\omega \in (\pi/2, \theta)$.

Let $B(H)$ be the set of all bounded linear operators on H. The properties of the families $\{S_\alpha(t)\}_{t>0}$ and $\{P_\alpha(t)\}_{t>0}$ are given in the following theorems.

Theorem 2.1. *Let A be a sectorial operator and $S_\alpha(t)$ be an operator defined by* (2.12)*. Then the following statements hold.*

(i) *$S_\alpha(t) \in B(H)$ and there exists a constant $C_1 = C_1(\alpha) > 0$ such that*

$$\|S_\alpha(t)\| \leq C_1, \quad t > 0,$$

(ii) *$S_\alpha(t) \in B(H; D(A))$ for $t > 0$, and if $x \in D(A)$ then $AS_\alpha(t)x = S_\alpha(t)Ax$. Moreover, there exists a constant $C_2 = C_2(\alpha) > 0$ such that*

$$\|AS_\alpha(t)\| \leq C_2 t^{-\alpha}, \quad t > 0,$$

(iii) *The function $t \mapsto S_\alpha(t)$ belongs to $C^\infty((0,\infty); B(H))$ and it holds that*

$$S_\alpha^{(n)}(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{t\lambda} \lambda^{\alpha+n-1} R(\lambda^\alpha; A) d\lambda, \ n = 1, 2, \dots$$

and there exist constants $M_n = M_n(\alpha) > 0, n = 1, 2, \dots$ such that

$$\|S_\alpha^{(n)}(t)\| \leq M_n t^{-n}, \quad t > 0,$$

Moreover, it has an analytic continuation $S_\alpha(z)$ to the sector $S_{\theta-\pi/2}$ and, for $z \in S_{\theta-\pi/2}$, $\eta \in (\pi/2, \theta)$, it holds that

$$S_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{\lambda z} \lambda^{\alpha-1} R(\lambda^\alpha; A) d\lambda.$$

Theorem 2.2. *Let A be a sectorial operator and $P_\alpha(t)$ be an operator defined by* (2.13)*. Then the following statements hold.*

(i) *$P_\alpha(t) \in B(H)$ and there exists a constant $L_1 = L_1(\alpha) > 0$ such that*

$$\|P_\alpha(t)\| \leq L_1 t^{\alpha-1}, \quad t > 0,$$

(ii) *$P_\alpha(t) \in B(H; D(A))$ for all $t > 0$, and if $x \in D(A)$ then $AP_\alpha(t)x = P_\alpha(t)Ax$. Moreover, there exists a constant $L_2 = L_2(\alpha) > 0$ such that*

$$\|AP_\alpha(t)\| \leq L_2 t^{-1}, \quad t > 0,$$

(iii) *The function $t \mapsto P_\alpha(t)$ belongs to $C^\infty((0,\infty); B(H))$ and it holds that*

$$P_\alpha^{(n)}(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{t\lambda} \lambda^n R(\lambda^\alpha; A) d\lambda, \ n = 1, 2, \dots$$

and there exist constants $K_n = K_n(\alpha) > 0, n = 1, 2, \dots$ such that

$$\|P_\alpha^{(n)}(t)\| \leq K_n t^{\alpha-n-1}, \quad t > 0,$$

Moreover, it has an analytic continuation $P_\alpha(z)$ to the sector $S_{\theta-\pi/2}$ and, for $z \in S_{\theta-\pi/2}$, $\eta \in (\pi/2, \theta)$, it holds that

$$P_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda^\alpha; A) d\lambda.$$

The following theorem states some identities concerning the operators $S_\alpha(t)$ and $P_\alpha(t)$ including the semigroup-like property.

Theorem 2.3. *Let A be a sectorial operator, $S_\alpha(t)$ and $P_\alpha(t)$ be operators defined by (2.12) and (2.13), respectively. Then the following statements hold.*

(i) *For $x \in H$ and $t > 0$,*
$$S_\alpha(t)x = J_t^{1-\alpha} P_\alpha(t)x, \quad D_t S_\alpha(t)x = AP_\alpha(t)x,$$
(ii) *For $x \in D(A)$ and $s, t > 0$,*
$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x,$$
$$S_\alpha(t+s)x = S_\alpha(t)S_\alpha(s)x - A\int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau)P_\alpha(r)x\,dr\,d\tau.$$

The next theorem shows us the behavior of the operator $S_\alpha(t)$ at t close to 0^+.

Theorem 2.4. *Let A be a sectorial operator and $S_\alpha(t)$ be an operator defined by (2.12). Then the following statements hold.*

(i) *If $x \in \overline{D(A)}$ then $\lim_{t\to 0^+} S_\alpha(t)x = x$.*
(ii) *For every $x \in D(A)$ and $t > 0$,*
$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau \in D(A),$$
$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(\tau)x d\tau = S_\alpha(t)x - x,$$
(iii) *If $x \in D(A)$ and $Ax \in \overline{D(A)}$ then*
$$\lim_{t\mapsto 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} = \frac{1}{\Gamma(\alpha+1)} Ax.$$

The representation of the solution to (2.9) in term of $S_\alpha(t)$ and $P_\alpha(t)$ is given in the following theorem.

Theorem 2.5. *Let $u \in C^1((0,\infty); H) \cap L^1((0,\infty); H)$, $u(t) \in D(A)$ for $t \in [0,\infty)$, $Au \in L^1((0,\infty); H)$, $f \in L^1((0,\infty); D(A))$, and $Af \in L^1((0,\infty); H)$. If u is a solution to the problem (2.9) then*
$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(s)ds, \quad t > 0. \tag{2.14}$$

Now, we consider the fractional power of operator A
$$A^{-\beta}x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{-\beta} R(\lambda; A)x d\lambda, \quad x \in H,\ \beta > 0,$$
and
$$A^\beta x = A(A^{\beta-1}x) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda; A)Ax d\lambda, \quad x \in D(A),\ 0 < \beta < 1.$$
Some estimates involving A^β and the operators families $\{S_\alpha(t)\}_{t>0}$, $\{P_\alpha(t)\}_{t>0}$ generated by the sectorial operator A are provided by the following theorem. These estimates are analogous to those as stated in [23, Theorem 6.13] for analytic semigroups.

Theorem 2.6. *For each $0 < \beta < 1$, there exist positive constants $C_1' = C_1'(\alpha,\beta)$, $C_2' = C_2'(\alpha,\beta)$, and $C_3' = C_3'(\alpha,\beta)$ such that for all $x \in H$,*

$$\|A^\beta S_\alpha(t)x\| \le C_1' t^{-\alpha}(t^{-\alpha(\beta-1)}+1)\|x\|, \quad t>0, \tag{2.15}$$

$$\|A^\beta P_\alpha(t)x\| \le C_2' t^{-\alpha(\beta-1)-1}\|x\|, \quad t>0. \tag{2.16}$$

Moreover, for all $x \in D(A^\beta)$,

$$\|S_\alpha(t)x - x\| \le C_3' t^{\alpha\beta}\|A^\beta x\|, \quad t>0. \tag{2.17}$$

Now, let $\xi_\zeta = \alpha(\zeta-1)+1$, for $0<\zeta<1$, and $x^+ = \max\{0,x\}$, for $x \in \mathbb{R}$. Thus we have the following result.

Corollary 2.7. *For each $\beta > (2-1/\alpha)^+$ or $\beta = 2-1/\alpha > 0$ and $x \in H$,*

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \le 2C_1'\|x\|, \quad 0<t\le 1, \tag{2.18}$$

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \le 2C_1' t^{1-\alpha}\|x\|, \quad t>1, \tag{2.19}$$

$$t^{\xi_\beta}\|A^\beta P_\alpha(t)x\| \le C_2'\|x\|, \quad t>0, \tag{2.20}$$

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \to 0, \quad \text{as } t \to 0^+. \tag{2.21}$$

Furthermore, we have the same result as Theorem 2.3 (ii) with weaker condition.

Theorem 2.8. *Let $0<\beta<1$. Then, for $x \in D(A^\beta)$ and $s,t>0$,*

$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x, \tag{2.22}$$

$$S_\alpha(t+s)x = S_\alpha(t)S_\alpha(s)x - A\int_0^t\int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau)P_\alpha(r)x\,dr\,d\tau. \tag{2.23}$$

3. Main results

In this section, we show the existence and uniqueness of a mild solution for the problem (1.3) under certain conditions by applying Banach's Fixed Point Theorem. Based on Theorem 2.5, we define a mild solution to the problem (1.3) as follows.

Definition 3.1. A continuous function $u : (0,T] \to H$ is a mild solution to the problem (1.3) if it satisfies

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(u(s))ds, \quad 0<t\le T.$$

The conditions on f are:

(i) $f(0)=0$,
(ii) there exist $C_0>0$, $\vartheta>1$, and $0<\beta<1$ such that
$$\|f(u)-f(v)\| \le C_0(\|A^\beta u\| + \|A^\beta v\|)^{\vartheta-1}\|A^\beta u - A^\beta v\|,$$
for all $u,v \in D(A^\beta)$.

Let $BC((0,T];D(A^\beta))$ be the set of all bounded and continuous functions $w : (0,T] \to D(A^\beta)$. Under the conditions on f above, we obtain the following results.

Theorem 3.2. *Let $u_0 \in D(A^\nu)$ with*

$$\beta-\nu > (2-1/\alpha)^+, \quad 1-\alpha\nu-\vartheta\xi_{\beta-\nu} \ge 0, \quad 0<\vartheta\xi_{\beta-\nu}<1, \tag{3.1}$$

where

$$\xi_\zeta = \alpha(\zeta-1)+1, \quad 0<\zeta<1; \quad x^+ = \max\{0,x\}, \quad x \in \mathbb{R}.$$

Then there exits $T > 0$ sufficiently small such that the problem (1.3) *has a unique mild solution u satisfying*

$$t^{\xi_{\eta-\nu}} u \in BC((0,T]; D(A^\eta)), \quad \lim_{t\to 0^+} t^{\xi_{\eta-\nu}} A^\eta u(t) = 0,$$
$$\|A^\eta u(t)\| \le C t^{-\xi_{\eta-\nu}} \|A^\nu u_0\|, \quad t \in (0,T],$$

for every $\eta \in (\nu + (2 - 1/\alpha)^+, \beta]$.

Theorem 3.3. *Let u be the mild solution to the problem* (1.3) *in Theorem 3.2. If $f(u(t)) \in D(A)$, for $t \in (0,\infty)$, then*

$$t^{\xi_{1-\nu}} u \in BC((0,T]; D(A))$$

with

$$\|Au(t)\| \le C t^{-\xi_{1-\nu}} \|A^\nu u_0\|, \quad t \in (0,T].$$

3.1. **Proof of Theorem 3.2.** We define first the Banach space

$$E_{\beta,T} = \{u : [0,T] \to H : t^{\xi_{\beta-\nu}} u \in BC((0,T]; D(A^\beta))\}$$

equipped with the norm

$$|||u|||_{\beta,T} = \sup_{0<t\le T} t^{\xi_{\beta-\nu}} \|A^\beta u(t)\|, \tag{3.2}$$

and define a closed ball $B_{\beta,T}$ in $E_{\beta,T}$ by

$$B_{\beta,T} = \{u \in E_{\beta,T} : |||u|||_{\beta,T} \le K\},$$

where T and K are some constants which will be specified later.

Next, we define a mapping F on $B_{\beta,T}$ by

$$Fu(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(u(s)) ds.$$

First, we prove the continuity of $A^\beta Fu(t)$ with respect to t in $(0,T]$. Since A^β is a bounded operator on $D(A)$ and, for each $x \in H$, $S_\alpha(t)x$ is continuous with respect to t in $(0,\infty)$, then, for each $x \in H$, $A^\beta S_\alpha(t)x$ is continuous with respect to t in $(0,\infty)$. Thus it remains to show the continuity of

$$A^\beta \int_0^t P_\alpha(t-s) f(u(s)) ds, \quad 0 < t \le T.$$

Note that

$$\begin{aligned}
&A^\beta \int_0^{t+h} P_\alpha(t+h-s) f(u(s)) ds - A^\beta \int_0^t P_\alpha(t-s) f(u(s)) ds \\
&= A^\beta \int_{-h}^t P_\alpha(t-s) f(u(s+h)) ds - A^\beta \int_0^t P_\alpha(t-s) f(u(s)) ds \\
&= A^\beta \int_0^t P_\alpha(t-s)(f(u(s+h)) - f(u(s))) ds \\
&\quad + A^\beta \int_0^h P_\alpha(t+h-s) f(u(s)) ds.
\end{aligned}$$

Observe that, for $u \in E_{\beta,T}$,

$$\|f(u(t+h)) - f(u(t))\| \le C_0 2^{\vartheta-1} K^{\vartheta-1} t^{-(\vartheta-1)\xi_{\beta-\nu}} \|A^\beta u(t+h) - A^\beta u(t)\| \tag{3.3}$$

and

$$\|f(u(t))\| \le C_0\|A^\beta u(t)\|^\vartheta \le C_0 t^{-\vartheta\xi_{\beta-\nu}}|||u|||_{\beta,T}^\vartheta \le C_0 K^\vartheta t^{-\vartheta\xi_{\beta-\nu}}, \tag{3.4}$$

for $0 < t \le T$. Next, we have

$$\begin{aligned}
&\int_0^t \|A^\beta P_\alpha(t-s)(f(u(s+h)) - f(u(s)))\|ds \\
&\le 2^{\vartheta-1}C_0C_2K^{\vartheta-1}\int_0^t (t-s)^{-\xi_\beta}s^{-(\vartheta-1)\xi_{\beta-\nu}}\|A^\beta u(s+h) - A^\beta u(s)\|ds.
\end{aligned}$$

Now, consider that, for $0 < s < t \le T$,

$$\begin{gathered}
(t-s)^{-\xi_\beta}s^{-(\vartheta-1)\xi_{\beta-\nu}}\|A^\beta u(s+h) - A^\beta u(s)\| \le 2K(t-s)^{-\xi_\beta}s^{-\vartheta\xi_{\beta-\nu}}, \\
s \mapsto 2K(t-s)^{-\xi_\beta}s^{-\vartheta\xi_{\beta-\nu}} \in L^1((0,t);H), \quad 0 < t \le T, \\
\|A^\beta u(s+h) - A^\beta u(s)\| \to 0, \text{ as } h \to 0.
\end{gathered}$$

Hence, by the Dominated Convergence theorem,

$$\int_0^t (t-s)^{-\xi_\beta}s^{-(\vartheta-1)\xi_{\beta-\nu}}\|A^\beta u(s+h) - A^\beta u(s)\|ds \to 0, \quad \text{as } h \to 0.$$

This implies

$$\int_0^t \|A^\beta P_\alpha(t-s)(f(u(s+h)) - f(u(s)))\|ds \to 0, \quad \text{as } h \to 0.$$

Next, observe that

$$\begin{aligned}
&\int_0^h \|A^\beta P_\alpha(t+h-s)\|\|f(u(s)\|)ds \\
&\le C_0C_2'(\alpha,\beta)K^\vartheta\int_0^h (t+h-s)^{-\xi_\beta}s^{-\vartheta\xi_{\beta-\nu}}ds \\
&= C_0C_2'(\alpha,\beta)K^\vartheta(t+h)^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}}\int_0^{\frac{h}{t+h}}(1-r)^{-\xi_\beta}r^{-\vartheta\xi_{\beta-\nu}}dr \\
&= C_0C_2'(\alpha,\beta)K^\vartheta(t+h)^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}}\frac{1}{1-\vartheta\xi_{\beta-\nu}} \\
&\quad\times\Big(\frac{h}{t+h}\Big)^{1-\vartheta\xi_{\beta-\nu}}H\Big(1-\vartheta\xi_{\beta-\nu},\xi_\beta;2-\vartheta\xi_{\beta-\nu};\frac{h}{t+h}\Big) \\
&= \frac{C_0C_2'(\alpha,\beta)K^\vartheta}{1-\vartheta\xi_{\beta-\nu}}h^{1-\vartheta\xi_{\beta-\nu}}(t+h)^{-\xi_\beta}H\Big(1-\vartheta\xi_{\beta-\nu},\xi_\beta;2-\vartheta\xi_{\beta-\nu};\frac{h}{t+h}\Big),
\end{aligned}$$

where

$$H(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a}, \quad c-b-a > 0,\ |x| \le 1$$

is hypergeometric function (see [15]). Thus

$$\int_0^h \|A^\beta P_\alpha(t+h-s)\|\|f(u(s)\|ds \to 0, \quad \text{as } h \to 0.$$

Therefore the continuity of $A^\beta Fu(t)$ with respect to t in $(0,T]$ is obtained.

Next, we prove that the mapping F is well-defined and maps $B_{\beta,T}$ into itself. Consider

$$\begin{aligned}
&\int_0^t \|A^\beta P_\alpha(t-s)\|\|f(u(s))\|ds \\
&\le C_0 C_2'(\alpha,\beta) K^{\vartheta-1} |||u|||_{\beta,T} \int_0^t (t-s)^{-\xi_\beta} s^{-\vartheta\xi_{\beta-\nu}} ds \\
&\le C_0 C_2'(\alpha,\beta) K^{\vartheta-1} B(1-\vartheta\xi_{\beta-\nu}, 1-\xi_\beta) |||u|||_{\beta,T} t^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}},
\end{aligned}$$

where

$$B(a,b) = \int_0^1 r^{a-1}(1-r)^{b-1} dr, \quad a,b>0,$$

is Beta function. Therefore

$$t^{\xi_{\beta-\nu}} \|A^\beta F u(t)\| \le t^{\xi_{\beta-\nu}} \|A^\beta S_\alpha(t) u_0\| + C_4 K^{\vartheta-1} |||u|||_{\beta,T} t^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}+\xi_{\beta-\nu}}, \tag{3.5}$$

where $C_4 = C_0 C_2'(\alpha,\beta) B(1-\xi_\beta, 1-\vartheta\xi_{\beta-\nu})$, implying

$$|||Fu|||_{\beta,T} \le \sup_{0<t\le T} t^{\xi_{\beta-\nu}} \|A^\beta S_\alpha(t) u_0\| + C_4 K^{\vartheta-1} T^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}+\xi_{\beta-\nu}} |||u|||_{\beta,T}. \tag{3.6}$$

Note that $1-\xi_\beta-\vartheta\xi_{\beta-\nu}+\xi_{\beta-\nu} = 1-\alpha\nu-\vartheta\xi_{\beta-\nu} \ge 0$ by (3.1). By (2.18), we can find $0<T\le 1$ such that

$$t^{\xi_{\beta-\nu}} \|A^{\beta-\nu} S_\alpha(t) A^\nu u_0\| \le 2C_1'(\alpha,\beta-\nu)\|A^\nu u_0\|, \quad 0<t\le T.$$

Then, for $u \in B_{\beta,T}$, we have

$$\begin{aligned}
|||Fu|||_{\beta,T} &\le \sup_{0<t\le T} t^{\xi_{\beta-\nu}} \|A^{\beta-\nu} S_\alpha(t) A^\nu u_0\| + C_4 K^\vartheta T^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}} \\
&\le 2C_1'(\alpha,\beta-\nu)\|A^\nu u_0\| + C_4 K^\vartheta T^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}}.
\end{aligned} \tag{3.7}$$

Next, we choose $K>0$ such that

$$2C_1'(\alpha,\beta-\nu)\|A^\nu u_0\| + C_4 K^\vartheta T^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}} \le K. \tag{3.8}$$

For the case $1-\alpha\nu-\vartheta\xi_{\beta-\nu}>0$, we can get such a K by taking T sufficiently small. For the case $1-\alpha\nu-\vartheta\xi_{\beta-\nu}=0$, we choose $K>0$ sufficiently small such that

$$C_4 K^\vartheta < K,$$

and then take T such that

$$\sup_{0<t\le T} t^{\xi_{\beta-\nu}} \|A^{\beta-\nu} S_\alpha(t) A^\nu u_0\| \le K - C_4 K^\vartheta. \tag{3.9}$$

Note that in both cases, we can find $C = C(\alpha,\beta) > 0$ such that

$$K \le C\|A^\nu u_0\|. \tag{3.10}$$

Hence $|||Fu|||_{\beta,T} \le K$. Thus the mapping F is well-defined and maps $B_{\beta,T}$ into itself.

Next, we show that the mapping $F : B_{\beta,T} \to B_{\beta,T}$ is a strict contraction. Note that, if $u,v \in B_{\beta,T}$, we have

$$\begin{aligned}
&\|A^\beta F u(t) - A^\beta F v(t)\| \\
&\le \int_0^t \|A^\beta P_\alpha(t-s)\|\|f(u(s)) - f(v(s))\|ds
\end{aligned}$$

$$\begin{aligned}
&\le C_0 C_2'(\alpha,\beta)\int_0^t (t-s)^{-\xi_\beta}\big(|||u|||_{\beta,T}+|||v|||_{\beta,T}\big)^{\vartheta-1}\\
&\quad\times s^{-(\vartheta-1)\xi_{\beta-\nu}}|||u-v|||_{\beta,T}s^{-\xi_{\beta-\nu}}ds\\
&\le C_0 C_2'(\alpha,\beta)2^{\vartheta-1}K^{\vartheta-1}\int_0^t (t-s)^{-\xi_\beta}s^{-\vartheta\xi_{\beta-\nu}}ds|||u-v|||_{\beta,T}\\
&\le C_4 2^{\vartheta-1}K^{\vartheta-1}|||u-v|||_{\beta,T}t^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}}.
\end{aligned}$$

Then

$$\begin{aligned}
t^{\xi_{\beta-\nu}}\|A^\beta Fu(t)-A^\beta Fv(t)\| &\le C_4 2^{\vartheta-1}K^{\vartheta-1}|||u-v|||_{\beta,T}t^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}}\\
&\le C_4 2^{\vartheta-1}K^{\vartheta-1}|||u-v|||_{\beta,T}T^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}}.
\end{aligned}$$

Note that we can select $K>0$ and $T>0$ sufficiently small such that

$$C_5 = C_4 2^{\vartheta-1}K^{\vartheta-1}T^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}} < 1. \tag{3.11}$$

Consequently,

$$|||Fu-Fv||| \le C_5|||u-v|||_{\beta,T}.$$

It means the mapping $F: B_{\beta,T}\to B_{\beta,T}$ is a strict contraction. Thus, by Banach's Fixed Point Theorem, we can get a unique $u\in B_{\beta,T}$ which is a mild solution to the problem (1.3). Furthermore, by (3.7), (3.8), (3.9), and (3.10), for this u, we have

$$|||u|||_{\beta,T} \le \sup_{0<t\le T} t^{\xi_{\beta-\nu}}\|A^{\beta-\nu}S_\alpha(t)A^\nu u_0\| + C_4K^\vartheta T^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}+\xi_{\beta-\nu}} \le C\|A^\nu u_0\|.$$

Then, by (3.2),

$$\|A^\beta u(t)\| \le Ct^{-\xi_{\beta-\nu}}\|A^\nu u_0\|, \quad 0<t\le T.$$

Now, we check the continuity of u at $t=0$. Note that

$$\begin{aligned}
t^{\xi_{\beta-\nu}}\|A^\beta u(t)\| &\le t^{\xi_{\beta-\nu}}\|A^\beta S_\alpha(t)u_0\| + t^{\xi_{\beta-\nu}}\int_0^t \|A^\beta P_\alpha(t-s)\|\|f(u(s))\|\\
&\le t^{\xi_{\beta-\nu}}\|A^{\beta-\nu}S_\alpha(t)A^\nu u_0\| + C_4K^\vartheta t^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}}.
\end{aligned} \tag{3.12}$$

Thus, if $1-\alpha\nu-\vartheta\xi_{\beta-\nu}>0$, letting $t\to 0^+$ on both sides of (3.12), we obtain

$$\lim_{t\to 0^+} t^{\xi_{\beta-\nu}}A^\beta u(t) = 0.$$

For the case $1-\alpha\nu-\vartheta\xi_{\beta-\nu}=0$, consider first that, from (3.6), we have

$$|||u|||_{\beta,T'} \le \sup_{0<t\le T'} t^{\xi_{\beta-\nu}}\|A^{\beta-\nu}S_\alpha(t)A^\nu u_0\| + C_4K^{\vartheta-1}|||u|||_{\beta,T'},$$

for any $0<T'\le T$. Since $C_5<1$, then $C_4K^{\vartheta-1}<1$. Hence there exists $C_6>0$ such that

$$|||u|||_{\beta,T'} \le C_6\sup_{0<t\le T'} t^{\xi_{\beta-\nu}}\|A^{\beta-\nu}S_\alpha(t)A^\nu u_0\|.$$

By taking $T'\to 0$, thus we also have

$$\lim_{t\to 0^+} t^{\xi_{\beta-\nu}}A^\beta u(t) = 0,$$

for the case $1-\alpha\nu-\vartheta\xi_{\beta-\nu}=0$. We can also conclude that the results above also hold for every $\eta\in(\nu+(2-1/\alpha)^+,\beta)$ since such a η satisfies the condition (3.1).

Remark 3.4. From (2.19), for $T > 1$, we have

$$t^{\xi_{\beta-\nu}}\|A^\beta S_\alpha(t)u_0\| \le 2C_1'(\alpha,\beta-\nu)t^{1-\alpha}\|A^\nu u_0\|, \quad t\in(0,T].$$

Then, it follows that (3.8) becomes

$$2C_1'(\alpha,\beta-\nu)\|A^\nu u_0\|T^{1-\alpha} + C_4K^\vartheta T^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}} \le K. \tag{3.13}$$

Observe that we can not get $K > 0$ satisfying (3.11) and (3.13) for T sufficiently large although K is taken to be sufficiently small. Thus the problem (1.3) has no a global mild solution u on $(0,\infty)$.

Remark 3.5. If we assume that f is a nonlinear operator in H satisfying

(i) $f(0) = 0$,

(ii) there exist $C_0 > 0$, $\vartheta > 1$, and $0 < \beta < 1$ such that
$$\|f(u)-f(v)\| \le C_0(1+(\|A^\beta u\|+\|A^\beta v\|)^{\vartheta-1})\|A^\beta u - A^\beta v\|,$$
for all $u,v\in D(A^\beta)$,

then Theorem 3.2 remains valid.

3.2. **Proof of Theorem 3.3.** We verify first the following lemma.

Lemma 3.6. *Let $u\in B_{\beta,T}$ be a mild solution to (1.3). Then, by the condition (3.1), $A^\beta u(t)$ is Hölder continuous in $[\varepsilon,T]$ for each $\varepsilon>0$.*

Proof. First, consider that, by (2.23),

$$\begin{aligned}
&A^\beta S_\alpha(t+h)u_0 - A^\beta S_\alpha(t)u_0\\
&= A^\beta(S_\alpha(h)-I)S_\alpha(t)u_0 - A^\beta\int_0^t\int_0^h \frac{(t+h-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}AP_\alpha(\tau)P_\alpha(r)u_0\,dr\,d\tau
\end{aligned}$$

and

$$\begin{aligned}
&A^\beta\int_0^{t+h}P_\alpha(t+h-s)f(u(s))ds - A^\beta\int_0^t P_\alpha(t-s)f(u(s))ds\\
&= A^\beta\int_{-h}^t P_\alpha(t-s)f(u(s+h))ds - A^\beta\int_0^t P_\alpha(t-s)f(u(s))ds\\
&= A^\beta\int_0^t P_\alpha(t-s)(f(u(s+h))-f(u(s)))ds\\
&\quad + A^\beta\int_0^h P_\alpha(t+h-s)f(u(s))ds.
\end{aligned}$$

Now, let $\varepsilon\le t<t+h\le T$ with $\varepsilon>0$. Observe that

$$\begin{aligned}
&\int_0^h \frac{(t+h-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\tau^{-\xi_{1-\delta}}d\tau\\
&= \frac{h^{1-\xi_{1-\delta}}(t+h-r)^{-\alpha}}{\Gamma(1-\alpha)(1-\xi_{1-\delta})}H\Big(1-\delta,\alpha;2-\delta;\frac{h}{t+h-r}\Big)\\
&\le \frac{\Gamma(2-\xi_{1-\delta})B(\alpha,1-\alpha)}{(1-\xi_{1-\delta})\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(2-\xi_{1-\delta}-\alpha)}h^{1-\xi_{1-\delta}}(t+h-r)^{-\alpha}
\end{aligned}$$

implying

$$\int_0^t\int_0^h \frac{(t+h-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\tau^{-\xi_{1-\delta}}r^{-\xi_{\beta+\delta-\nu}}\,dr\,d\tau$$

$$
\begin{aligned}
&\le \frac{\Gamma(2-\xi_{1-\delta})B(\alpha,1-\alpha)h^{1-\xi_{1-\delta}}}{(1-\xi_{1-\delta})\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(2-\xi_{1-\delta}-\alpha)}\int_0^t (t+h-r)^{-\alpha} r^{-\xi_{\beta+\delta-\nu}}dr\\
&= \frac{\Gamma(2-\xi_{1-\delta})B(\alpha,1-\alpha)h^{1-\xi_{1-\delta}}(t+h)^{1-\alpha-\xi_{\beta+\delta-\nu}}}{(1-\xi_{1-\delta})\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(2-\xi_{1-\delta}-\alpha)}\int_0^{\frac{t}{t+h}} (1-s)^{-\alpha} s^{-\xi_{\beta+\delta-\nu}}ds\\
&\le C_7 h^{1-\xi_{1-\delta}}(t+h)^{1-\alpha-\xi_{\beta+\delta-\nu}}
\end{aligned}
$$

where

$$
C_7 = \frac{\Gamma(2-\xi_{1-\delta})B(\alpha,1-\alpha)B(1-\xi_{\beta+\delta-\nu},1-\alpha)}{(1-\xi_{1-\delta})\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(2-\xi_{1-\delta}-\alpha)}.
$$

Then, for every $0<\delta<1-\beta$,

$$
\begin{aligned}
&\|A^\beta S_\alpha(t+h)u_0 - A^\beta S_\alpha(t)u_0\|\\
&\le \|(S_\alpha(h)-I)A^\beta S_\alpha(t)u_0\|\\
&\quad + \int_0^t\int_0^h \frac{(t+h-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\|A^{1-\delta}P_\alpha(\tau)A^{\beta+\delta-\nu}P_\alpha(r)A^\nu u_0\|\,dr\,d\tau\\
&\le C_3'(\alpha,\delta)h^{\alpha\delta}\|A^{\beta+\delta-\nu}S_\alpha(t)A^\nu u_0\| + C_2'(\alpha,1-\delta)C_2'(\alpha,\beta+\delta-\nu)\\
&\quad \times \int_0^t\int_0^h \frac{(t+h-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\tau^{-\xi_{1-\delta}}r^{-\xi_{\beta+\delta-\nu}}\,dr\,d\tau\|A^\nu u_0\|\\
&\le C_1'(\alpha,\beta+\delta-\nu)C_3'(\alpha,\delta)h^{1-\xi_{1-\delta}}t^{-\alpha}(t^{1-\xi_{\beta+\delta-\nu}}+1)\|A^\nu u_0\|\\
&\quad + C_2'(\alpha,1-\delta)C_2'(\alpha,\beta+\delta-\nu)C_7 h^{1-\xi_{1-\delta}}t^{1-\alpha-\xi_{\beta+\delta-\nu}}\|A^\nu u_0\|\\
&\le C_1'(\alpha,\beta+\delta-\nu)C_3'(\alpha,\delta)h^{1-\xi_{1-\delta}}t^{-\alpha}(t^{1-\xi_{\beta+\delta-\nu}}+1)\|A^\nu u_0\|\\
&\quad + C_8 h^{1-\xi_{1-\delta}}t^{-\alpha}(t^{1-\xi_{\beta+\delta-\nu}}+1)\|A^\nu u_0\|\\
&\le C_9 h^{1-\xi_{1-\delta}}t^{-\alpha}(t^{1-\xi_{\beta+\delta-\nu}}+1)\|A^\nu u_0\|,
\end{aligned}
$$

for some constants $C_8, C_9>0$. Next, note that

$$
\begin{aligned}
&\int_0^t \|A^\beta P_\alpha(t-s)(f(u(s+h))-f(u(s)))\|ds\\
&\le 2^{\vartheta-1}C_0C_2'(\alpha,\beta)K^{\vartheta-1}\int_0^t (t-s)^{-\xi_\beta}s^{-(\vartheta-1)\xi_{\beta-\nu}}\|A^\beta u(s+h)-A^\beta u(s)\|ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^h \|A^\beta P_\alpha(t+h-s)f(u(s))\|ds\\
&\le C_0C_2'(\alpha,\beta)K^\vartheta\int_0^h (t+h-s)^{-\xi_\beta}s^{-\vartheta\xi_{\beta-\nu}}ds\\
&\le C_{10}(t+h)^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}}\int_0^{\frac{h}{t+h}}(1-r)^{-\xi_\beta}r^{-\vartheta\xi_{\beta-\nu}}dr\\
&\le \frac{C_{10}}{1-\vartheta\xi_{\beta-\nu}}(t+h)^{1-\xi_\beta-\vartheta\xi_{\beta-\nu}}\Big(\frac{h}{t+h}\Big)^{1-\vartheta\xi_{\beta-\nu}}H\Big(1-\vartheta\xi_{\beta-\nu},\xi_\beta;2-\vartheta\xi_{\beta-\nu};\frac{h}{t+h}\Big)\\
&\le C_{11}h^{1-\vartheta\xi_{\beta-\nu}}t^{-\xi_\beta},
\end{aligned}
$$

for some constants C_{10}, $C_{11}>0$. Thus we obtain

$$
\begin{aligned}
&\|A^\beta u(t+h)-A^\beta u(t)\|\\
&\le C_9 h^{1-\xi_{1-\delta}}t^{-\alpha}(t^{1-\xi_{\beta+\delta-\nu}}+1)\|A^\nu u_0\| + C_{11}h^{1-\vartheta\xi_{\beta-\nu}}t^{-\xi_\beta}
\end{aligned}
$$

$$+ 2C_0C_2K^{\vartheta-1}\int_0^t (t-s)^{-\xi_\beta} s^{-(\vartheta-1)\xi_{\beta-\nu}} \|A^\beta u(s+h) - A^\beta u(s)\| ds$$

By the Gronwall's inequality, it implies that $A^\beta u(t)$ is Hölder continuous on $[\varepsilon, T]$ for any $\varepsilon > 0$. $\square$

Next, by the Lemma 3.6, $f(u(t))$ is also Hölder continuous on $[\varepsilon, T]$ for any $\varepsilon > 0$; that is,

$$\begin{aligned}\|f(u(t+h)) - f(u(t))\| \le C_{12}\big\{&h^{1-\xi_{1-\delta}} t^{-\alpha-(\vartheta-1)\xi_{\beta-\nu}} (t^{1-\xi_{\beta+\delta-\nu}} + 1)\|A^\nu u_0\| \\ &+ h^{1-\vartheta\xi_{\beta-\nu}} t^{-\xi_\beta-(\vartheta-1)\xi_{\beta-\nu}}\big\},\end{aligned}$$

for some constant $C_{12} > 0$. Note that the assumption (3.1) assures that $0 < 1 - \vartheta\xi_{\beta-\nu}$ and, for each $0 < \delta < 1-\beta$, it holds that $0 < 1 - \xi_{1-\delta}$. Furthermore, consider that, for $t \in (0, T]$, we have

$$t^{\xi_{1-\nu}} \|AS_\alpha(t)u_0\| \le C_1'(\alpha, 1-\nu) t^{\xi_{1-\nu}-\alpha} (t^{1-\xi_{1-\nu}} + 1)\|A^\nu u_0\|$$

with

$$\xi_{1-\nu} - \alpha > 0, \quad \xi_{1-\nu} - \alpha + 1 - \xi_{1-\nu} = 1 - \alpha > 0.$$

It follows, for T sufficiently small, that

$$t^{\xi_{1-\nu}} \|AS_\alpha(t)u_0\| \le 2C_1'(\alpha, 1-\nu)\|A^\nu u_0\|.$$

Now, observe that

$$\begin{aligned}&A\int_0^t P_\alpha(t-s) f(u(s)) ds \\ &= \int_0^{t/2} AP_\alpha(t-s) f(u(s)) ds \\ &\quad + \int_{t/2}^t AP_\alpha(t-s)(f(u(s)) - f(u(t))) ds + (S_\alpha(t/2) - I) f(u(t)) \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Next, we note that

$$t^{\xi_{1-\nu}} \|(S_\alpha(t/2) - I) f(u(t))\| \le C_{13} t^{\xi_{1-\nu}} \|f(u(t))\| \le C_0 C_{13} K^\vartheta t^{\xi_{1-\nu}-\vartheta\xi_{\beta-\nu}},$$

for some constant $C_{13} > 0$, and

$$\xi_{1-\nu} - \vartheta\xi_{\beta-\nu} = 1 - \alpha\nu - \vartheta\xi_{\beta-\nu}.$$

Therefore, for $t \in (0, T]$ with $T > 0$ sufficiently small, we have

$$t^{\xi_{1-\nu}} \|(S_\alpha(t/2) - I) f(u(t))\| \le 2C_0 C_{14} K^\vartheta t^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}},$$

for some constant $C_{14} > 0$. Hence, by (3.10), we obtain

$$t^{\xi_{1-\nu}} \|I_3\| \le C_{15}\|A^\nu u_0\|,$$

for some constant $C_{15} > 0$. Furthermore,

$$\begin{aligned}t^{\xi_{1-\nu}} \|I_1\| &\le L_2(\alpha) C_0 K^\vartheta t^{\xi_{1-\nu}} \int_0^{t/2} (t-s)^{-1} s^{-\vartheta\xi_{\beta-\nu}} ds \\ &\le C_{16} t^{\xi_{1-\nu}-\vartheta\xi_{\beta-\nu}} \le C_{16} t^{1-\alpha\nu-\vartheta\xi_{\beta-\nu}},\end{aligned}$$

for some constant $C_{16} > 0$. Thus, for T sufficiently small, we find that

$$t^{\xi_{1-\nu}} \|I_1\| \le C_{17}\|A^\nu u_0\|,$$

for some constant $C_{17} > 0$. Now, consider

$$\begin{aligned}
&\|AP_\alpha(t-s)(f(u(s)) - f(u(t)))\| \\
&\le L_2(\alpha)C_{12}\Big\{(t-s)^{-\xi_{1-\delta}} s^{-\alpha-(\vartheta-1)\xi_{\beta-\nu}}(s^{1-\xi_{\beta+\delta-\nu}} + 1)\|A^\nu u_0\| \\
&\quad + (t-s)^{-\vartheta\xi_{\beta-\nu}} s^{-\xi_\beta-(\vartheta-1)\xi_{\beta-\nu}}\Big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{t/2}^t \|AP_\alpha(t-s)(f(u(s)) - f(u(t)))\|ds \\
&\le C_{18}\big\{t^{1-\xi_{1-\delta}-\alpha-(\vartheta-1)\xi_{\beta-\nu}}(t^{1-\xi_{\beta+\delta-\nu}} + 1)\|A^\nu u_0\| + t^{1-\vartheta\xi_{\beta-\nu}-\xi_\beta-(\vartheta-1)\xi_{\beta-\nu}}\big\},
\end{aligned}$$

for some constant $C_{18} > 0$. Furthermore, by using the assumption (3.1),

$$\begin{gathered}
\xi_{1-\nu} + 1 - \xi_{1-\delta} - \alpha - (\vartheta-1)\xi_{\beta-\nu} > 1 - \alpha\nu - \vartheta\xi_{\beta-\nu} \ge 0, \\
1 - \vartheta\xi_{\beta-\nu} - \xi_\beta - (\vartheta-1)\xi_{\beta-\nu} = 2(1 - \alpha\nu - \vartheta\xi_{\beta-\nu}) \ge 0.
\end{gathered}$$

Note also that $1 - \xi_{\beta+\delta-\nu} > 0$. Then, for T sufficiently small,

$$t^{\xi_{1-\nu}}\|I_2\| \le C_{19}\|A^\nu u_0\|,$$

for some constant $C_{19} > 0$. Thus we conclude that

$$\|Au(t)\| \le C_{20}t^{-\xi_{1-\nu}}\|A^\nu u_0\|, \quad t \in (0, T],$$

for some constant $C_{20} > 0$.

4. Applications

We consider the parabolic initial-value problem

$$\begin{gathered}
D_t^\alpha u = \Delta u + |u|^{p-1}u, \quad \text{in } \Omega \times (0, T) \\
u|_{\partial\Omega} = 0, \\
u(0) = u_0, \quad \text{in } \Omega
\end{gathered} \tag{4.1}$$

where $\Omega \in \mathbb{R}^N$ with C^2 boundary and $p > 1$. The abstract formulation of the problem (4.1) is

$$\begin{gathered}
D_t^\alpha u = Au + f(u), \quad \text{in } \Omega \times (0, T) \\
u(0) = u_0, \quad \text{in } \Omega,
\end{gathered} \tag{4.2}$$

where

$$A = \Delta, \quad f(u) = |u|^{p-1}u.$$

Here, we set $H = L^2(\Omega)$ and $D(A) = H_D^2 = \{u \in H^2(\Omega) : u = 0 \text{ on } \partial\Omega\}$. Note that A is sectorial in H.

Next, for $\beta \ge N(1 - 1/p)/4$ and $p > 1$, we have

$$\|u\|_{2p} \le C\|A^\beta u\|_2, \quad u \in D(A^\beta)$$

(see [12] for more details). By the mean value theorem and the Hölder inequality, for $u, v \in D(A^\beta)$, one can obtain that

$$\|f(u) - f(v)\|_2^2 \le p^2(\|u\|_{(p-1)q} + \|v\|_{(p-1)q})^{2(p-1)}\|u - v\|_r^2$$

where $2/p + 2/r = 1$. It implies

$$\|f(u) - f(v)\|_2 \le p(\|u\|_{2p} + \|v\|_{2p})^{p-1}\|u - v\|_{2p}$$

by taking $r = 2p$ such that $(p-1)q = 2p$. Thus we get

$$\|f(u) - f(v)\|_2 \le p(\|A^\beta u\|_2 + \|A^\beta v\|_2)^{p-1}\|A^\beta u - A^\beta v\|_2$$

for $u, v \in D(A^\beta)$. We find that

$$D(A^\beta) = H_D^{2\beta}, \quad H_D^{2\beta} = \{u \in H^{2\beta}(\Omega) : u|_{\partial\Omega} = 0\}, \quad 1/4 < \beta < 1$$

(see [30] for more details). Thus, for

$$\begin{gathered}
\frac{1}{4} < \beta < 1, \quad \text{if } N\big(1 - \frac{1}{p}\big) \le 1, \\
\frac{N}{4}\big(1 - \frac{1}{p}\big) \le \beta < 1, \quad \text{if } 1 < N\big(1 - \frac{1}{p}\big) < 4,
\end{gathered}$$

and $u_0 \in D(A^\nu)$ with

$$\begin{gathered}
\frac{p\xi_\beta - 1}{\alpha(p-1)} \le \nu < \beta - (2 - \frac{1}{\alpha})^+, \quad \text{if } p\xi_\beta > 1, \\
0 < \nu < \beta - (2 - \frac{1}{\alpha})^+, \quad \text{if } p\xi_\beta \le 1,
\end{gathered}$$

by Theorem 3.2, problem (4.1) has a unique mild solution u satisfying

$$\begin{gathered}
t^{\xi_{\eta-\nu}} u \in BC((0,T]; D(A^\eta)), \quad \lim_{t\to 0^+} t^{\xi_{\eta-\nu}} A^\eta u(t) = 0, \\
\|A^\eta u(t)\|_H \le C t^{-\xi_{\eta-\nu}} \|A^\nu u_0\|_H, \quad t \in (0,T]
\end{gathered}$$

for every $\eta \in (\nu + (2 - 1/\alpha)^+, \beta]$ with T sufficiently small.

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