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POSITIVE SOLUTIONS FOR A NONLINEAR PARAMETER-DEPENDING ALGEBRAIC SYSTEM

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Abstract. Through variational methods, the existence of positive solutions for a nonlinear parameter-depending algebraic system is investigated. The main tools used are some very recent critical points theorems on finite dimensional Banach spaces and a new version of the weak and strong discrete maximum principle.

1. Introduction

In this article we study the following parameter-depending system of nonlinear algebraic equations

$$Bu = \lambda \underline{f}(u) \tag{1.1}$$

where $u = (u(1), \dots, u(N))^t$, $\underline{f}(u) := (f_1(u(1)), f_2(u(2)), \dots, f_N(u(N)))^t \in \mathbb{R}^N$ are two column vectors, $f_k : \mathbb{R} \to \mathbb{R}$ is a continuous function for every $k = 1, 2, \dots, N$, λ is a positive parameter and $B = [b_{ij}]_{N\times N}$ is a symmetric $Z-$matrix.

Our aim is to describe suitable intervals of parameters for which system (1.1) admits positive solutions. To this end, we use the critical point theorems established in [4] where, among the others, several existence results are obtained for a second-order nonlinear discrete Dirichlet boundary-value problem, namely

$$\begin{gathered} -\Delta^2 u(k-1) = \lambda f_k(u(k)), \quad k \in [1, N], \\ u(0) = u(N+1) = 0, \end{gathered} \tag{1.2}$$

where $[1, N]$ denotes the discrete interval $\{1, \dots, N\}$, and, for every $k \in [1, N]$, $\Delta u(k) := u(k+1) - u(k)$ is the forward difference operator, $\Delta^2 u(k-1) := u(k+1) - 2u(k) + u(k-1)$ is the second-order difference operator. An easy computation shows that problem (1.2) is a particular case of system (1.1) where the matrix B is given by

$$B := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{N\times N}.$$

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Arguing in a similar way, we can see that other difference boundary-value problems, as for instance, Neumann problem, three-point problem, can be considered as special cases of system (1.1).

We emphasize that the results obtained here are interesting also for many authors working in different research fields. Indeed, system (1.1) plays a crucial role to develop numerical schemes to find approximations of solutions of differential boundary-value problems, as the finite element method or the finite difference method, see for instance [8] and the listed references. Moreover, it appears in many mathematical models concerning the steady-states of reaction-diffusion processes, compartmental systems and complex dynamical networks. More details on these topics are contained in [1, 2, 11, 18].

It is worth remarking that in many frameworks mentioned before, for different reasons depending on the peculiarities of the problems investigated, it is important to know if a solution is positive. A list of problems, where a special meaning is ascribed to the positive solutions, is contained in [20].

Roughly speaking, here with respect to [4], we look for the existence of solutions for more general nonlinear discrete problems and, in addition, we give a more complete view on a new approach to get information on the sign of the solutions, see also [5, 6].

In this order of ideas, new formulations of the so called weak and strong discrete maximum principles are furnished (Theorems 2.1 and 2.3). Moreover, exploiting truncation techniques and using the variational formulation of the system (1.1), we can show how to get sign information on the solutions just looking for the sign of the nonlinearities f_k at zero (Lemmas 2.5 and 2.6). At the best of our knowledge, this is remarkable because in many papers, to obtain positive solutions, it requires that the nonlinearities f_k are positive [20], whereas here the nonlinearities can be sign-changing (Theorem 3.9) as well as the components of the right-hand side of the systems can have opposite sign (Remark 3.11). In order to deepen these arguments see also [8] and [10].

Usually the existence of a nontrivial solution for a nonlinear algebraic system is guaranteed combining two suitable growth conditions, one at zero and one at infinity, on the nonlinearities f_k and/or their primitives F_k. For instance, it is requested that they are superlinear both at zero and at infinity [15, 16, 18, 19, 20].

In this connection, we highlight that the results obtained here are mutually independent from those contained in the paper mentioned before because, for suitable intervals of parameters, we obtain a nontrivial solution for system (1.1) assuming just one sign condition at zero (Theorem 3.1), an appropriate behavior of the antiderivative F_k, either on a box of $\mathbb{R}^N$ (Theorem 3.2, Corollary 3.4) or along a direction of $\mathbb{R}^N$ (Corollaries 3.5 and 3.6). In particular, the existence of two positive solutions is ensured provided that the antiderivative F_k has superlinear growth at infinity (Theorem 3.7).

Finally, for completeness, we use a tridiagonal system to illustrate how we can apply successfully our main results (Theorem 3.9, Corollary 3.10 and Remark 3.11).

2. Discrete maximum principle and variational framework

In the N-dimensional Banach space $\mathbb{R}^N$, we consider the two equivalent norms

$$\|u\|_2 := \Big(\sum_{k=1}^{N} u(k)^2\Big)^{1/2} \quad \text{and} \quad \|u\|_\infty := \max_{k\in[1,N]} |u(k)|,$$

for which we have

$$\|u\|_\infty \le \|u\|_2 \le \sqrt{N}\|u\|_\infty. \tag{2.1}$$

Let be $u \in \mathbb{R}^N$, we said that u is nonnegative ($u \ge 0$), if $u(k) \ge 0$ for every $k \in [1, N]$, while we said that u is positive ($u > 0$), if $u(k) > 0$ for every $k \in [1, N]$. Analogous meaning have the symbols $u \le 0$, $u < 0$ and $u \ne 0$. We recall that a matrix $A = [a_{ij}]_{N\times N}$ is said: *monotone*, if $Au \ge 0$ implies $u \ge 0$; *a $Z-$matrix*, if $a_{ij} \le 0$ for every $i \ne j$; *a $M-$matrix*, if A is a monotone $Z-$matrix; *positive definite*, if $u^tAu > 0$ for all $u \ne 0$; *positive semidefinite*, if $u^tAu \ge 0$ for all $u \in \mathbb{R}^N$. It is easy to show that the diagonal entries of any positive semidefinite matrix are nonnegative. Moreover, if $B = [b_{ij}]_{N\times N}$ denotes a positive semidefinite matrix with eigenvalues $\lambda_1, \dots, \lambda_N$ ordered as $\lambda_1 < \dots < \lambda_N$, we known that

$$\lambda_1\|u\|_2^2 \le u^tBu \le \lambda_N\|u\|_2^2, \quad \forall u \in \mathbb{R}^N, \tag{2.2}$$

from which we have that a real matrix B is positive semidefinite if and only if its eigenvalues are nonnegative and it is positive definite whenever its eigenvalues are all positive.

Let S be a proper subset of $[1, N]$, we denote with $B(S)$ the matrix resulting from deleting the rows and the columns complementary to those indicate by S from the matrix B, that is $B(S)$ is a principal submatrix of B. In this framework, we have the following weak maximum principle for problem (1.1).

Theorem 2.1. *Let $B = [b_{ij}]_{N\times N}$ be a positive definite real $Z-$matrix. If $u \in \mathbb{R}^N$ satisfies the condition*

(i) *either $u(k) > 0$ or $(Bu)(k) \ge 0$, for each $k \in [1, N]$.*

Then $u \ge 0$.

Proof. We argue by contradiction. So, assume that the set

$$I^- := \{k \in [1, N] : u(k) < 0\},$$

is non empty and, owing to (i), for every $k \in I^-$ one has $\sum_{j=1}^N b_{kj}u(j) \ge 0$. From this, denoted with $I^+ := \{k \in [1, N] : u(k) > 0\}$, being B a $Z-$matrix, we obtain

$$\sum_{j\in I^-} b_{kj}u(j) \ge -\sum_{j\in I^+} b_{kj}u(j) \ge 0, \quad \forall k \in I^-. \tag{2.3}$$

On the other hand, if we consider the principal submatrix $B(I^-)$ of the matrix B, taking into account that a principal submatrix of a definite positive matrix is positive definite, see [9, p. 397], as well as that a positive definite $Z-$matrix is monotone, see [13, Theorem 6.12 and Corollary 6.14], by (2.3), we obtain that $u(k) \ge 0$ for every $k \in I^-$. But, this is a contradiction. So, $u \ge 0$ and this completes the proof. □

Remark 2.2. We explicitly observe that whenever $Bu(k) \ge 0$, for each $k \in [1, N]$, Theorem 2.1 gives back the classical result that a positive definite $Z-$matrix is a $M-$matrix, see [13]. Consequently, Theorem 2.1 in an unpublished version of the weak maximum principle for system (1.1). In particular, the conclusion is obtained with a different proof with respect to [7]. To have an insight how these topics are involved to solve, by finite element and finite difference methods, an elliptic continuous Dirichlet boundary-value problem, see also [8] and [10].

A more precise conclusion is the following strong discrete maximum principle, which pays more attention to the entries of the matrix B.

Theorem 2.3. *Let $B = [b_{ij}]_{N\times N}$ be a positive definite real $Z-$matrix and let $u \in \mathbb{R}^N$ be a vector satisfying the condition (i) of Theorem 2.1. In addition, for each $k \in [2, N]$, assume that*

(ii) *there exists $j_k < k$ such that $b_{kj_k} < 0$;*
(iii) *there exists $i_k < k$ such that $b_{i_k k} < 0$.*

Then, either $u \equiv 0$ or $u > 0$.

Proof. Clearly, Theorem 2.1 ensures that $u \geq 0$. Moreover, it is easy to see that condition (i) produces

$$u(i) = 0 \quad \Rightarrow \quad \sum_{j=1}^{N} b_{ij}u(j) = 0. \tag{2.4}$$

Now, we distinguish two cases:

(a) *Claim: if $u(1) > 0$, then $u > 0$.* First we show that $u(2) > 0$. By contradiction assume that $u(2) = 0$. By (2.4) with $i = 2$, owing to (ii), we obtain

$$0 = \sum_{j=1}^{N} b_{2j}u(j) < 0.$$

So, we obtain a contradiction. Hence, $u(2) > 0$.

Now, arguing again by contradiction, we prove that $u(3) > 0$. Indeed, if we have $u(3) = 0$, taking into account that (ii) implies that $a_{31}u(1) + a_{32}u(2) < 0$, by (2.4) with $i = 3$, we arrive to the contradiction

$$0 = \sum_{j=1}^{N} b_{3j}u(j) < 0.$$

By repeating these reasoning for $k = 4, \dots, N$, claim (a) is proved.

(b)]*Claim: if $u(1) = 0$, then $u \equiv 0$.* We have that $u(2) = 0$. On the contrary, by using (2.4) with $i = 1$, owing to (iii), we obtain

$$0 = \sum_{j=2}^{N} b_{1j}u(j) < 0.$$

Hence, one has $u(2) = 0$. Now, we prove that also $u(3) = 0$. Otherwise, since condition (iii) holds, either with $i_3 = 1$ or $i_3 = 2$, that is either one has $b_{13} < 0$ or $b_{23} < 0$, by (2.4), we obtain

$$0 = \sum_{j=1}^{N} b_{i_3 j}u(j) < 0.$$

By iterating this arguments, by now it is clear how to complete the proof. □

Remark 2.4. It is interesting to point out that if we know a priori that $u \geq 0$ the conclusion of Theorem 2.3 continues to hold under the following hypothesis on the matrix B,

(iv) B is just a $Z-$matrix satisfying conditions (ii) and (iii) of Theorem 2.3 provided that u verifies (2.4).

It is easy to show that if (ii) and (iii) are not satisfied the conclusion of Theorem 2.3 does not hold, it is enough to take a diagonal matrix B positive definite.

To focalize the variational framework for the system (1.1) the first step is to introduce the energy function $I_\lambda : \mathbb{R}^N \to \mathbb{R}$ defined by putting

$$I_\lambda(u) := \frac{1}{2} u^t B u - \lambda \sum_{k=1}^{N} \int_0^{u(k)} f_k(t) dt, \quad \forall u \in \mathbb{R}^N.$$

A direct computation shows that

$$\nabla I_\lambda u = Bu - \lambda \underline{f}(u), \quad \forall u \ \mathbb{R}^N.$$

Hence, the directional derivative of I_λ at the point u in the direction v is given by

$$\frac{\partial I_\lambda(u)}{\partial v} = (\nabla I_\lambda u, v) = v^t B u - \lambda \sum_{k=1}^{N} f_k(u(k)) v(k), \quad \forall u,\ v \in \mathbb{R}^N. \tag{2.5}$$

Therefore, we have that $\nabla I_\lambda u \equiv 0$ if and only if

$$v^t B u - \lambda \sum_{k=1}^{N} f_k(u(k)) v(k) = 0, \quad \forall v \in \mathbb{R}^N. \tag{2.6}$$

In few words, (2.6) is the weak formulation of problem (1.1) which is the key to study the nonlinear system (1.1) via variational methods because we have that *the critical points of I_λ are exactly the solutions of problem* (1.1). Moreover, we point out that (2.6) is also very useful to study the sign of a solution of problem (1.1). To this end, here and in the sequel, we assume that

$$f_k(0) \geq 0, \quad \forall k \in [1, N], \tag{2.7}$$

Since, we are interested to obtain positive solutions for problem (1.1), without loss of generality, as the following results show, on the energy functional I_λ we can replace, for all $k \in [1, N]$, the nonlinearities f_k with

$$f_k^+(x) = \begin{cases} f_k(t), & \text{if } t \geq 0; \\ f_k(0), & \text{if } t < 0. \end{cases} \tag{2.8}$$

Let $r \in \mathbb{R}$ be, we put $r^\pm = \max\{\pm r, 0\}$.

Lemma 2.5. *Let $B = [b_{ij}]_{N\times N}$ be a real $Z-$matrix. Assume that* (2.7) *holds and that one of the following conditions are fulfilled:*

(I) *the matrix B is positive definite;*

(II) *the matrix B is semipositive definite and for every $u \in \mathbb{R}^N \setminus \{0\}$ such that $(u^-)^t B u^- = 0$ with $u^- \neq 0$, one has $\sum_{k=1}^N f_k(0) u^-(k) > 0$.*

Then, every critical point u of the functional I_λ turns out to be a nonnegative solution of problem (1.1). *In particular, if $f_k(0) > 0$ one has $u(k) > 0$.*

Proof. Clearly, we can suppose that $u \in \mathbb{R}^N$ is a nontrivial critical point of I_λ. Since (2.7) holds, we observe that condition (i) of Theorem 2.1 is verified. Indeed, if we have $u(k) \leq 0$ for some $k \in [1, N]$, by (2.7) one has $(Bu)(k) = f_k(0) \geq 0$. So, if (I) is satisfied, Theorem 2.1 guarantees that u is nonnegative. Therefore, u turns out to be a nontrivial solution of system (1.1).

In the case in which (II) holds, arguing by contradiction suppose that there exists some $k \in [1, N]$ such that $u^-(k) > 0$. Taking $-u^-$ as test function in (2.6), we have that

$$
\begin{aligned}
0 = I'_\lambda(u)(-u^-) &= (-u^-)^t Bu + \lambda \sum_{k=1}^{N} f_k(0) u^-(k), \\
&= (-u^-)^t Bu^+ + (u^-)^t Bu^- + \lambda \sum_{k=1}^{N} f_k(0) u^-(k), \\
&\geq - \sum_{i,j=1, i\neq j}^{N} b_{ij} u^-(i) u^+(j) + (u^-)^t Bu^- + \lambda \sum_{k=1}^{N} f_k(0) u^-(k), \\
&\geq (u^-)^t Bu^- + \lambda \sum_{k=1}^{N} f_k(0) u^-(k) > 0,
\end{aligned}
$$

which clearly is a contradiction. So, u is nonnegative.

Moreover, if for some $k \in [1, N]$ one has that $f_k(0) > 0$, we can not have $u(k) = 0$, as the following inequality shows

$$
0 \geq \sum_{j=1, j\neq k}^{N} b_{kj} u(j) = f_k(0) > 0.
$$

□

Lemma 2.6. *Let $B = [b_{ij}]_{N\times N}$ be a real $Z-$matrix. Assume that one of the following conditions are satisfied:*

(III) *The matrix B satisfies the conditions (ii) and (iii) of Theorem 2.3.*

(IV) *The matrix B is semipositive definite and $f_k(0) > 0$ for every $k \in [1, N]$.*

Then, every nontrivial critical point of the functional I_λ turns out to be a positive solution of problem (1.1).

Proof. Let $u \in \mathbb{R}^N$ be a nontrivial critical point of I_λ. Bearing in mind (2.7), we observe that condition (i) of Theorem 2.1 is verified. So, by (III), Theorem 2.3 guarantees that u is positive and consequently u is a positive solution of system (1.1). If (IV) holds, our conclusion follows at once by Lemma 2.5. □

Remark 2.7. It is worth noticing that Lemma 2.6 highlights the novelty introduced by Theorems 2.1 and 2.3 in studying positive solutions for nonlinear discrete problems. Indeed, it is well known that in order to apply the classical version of the discrete maximum principle, usually, it is assumed that $f_k(s) \geq 0$ for every $s \geq 0$ and for every $k \in [1, N]$, see [12, 20]. Instead, here, roughly speaking, it is enough to look at the sign of $f_k(0)$.

Clearly, Theorems 2.1 and 2.3 can be used to study positive solution for a nonlinear algebraic system for which variational methods can not applied.

Now, we introduce our main tools used to investigate problem (1.1) which are based on variational methods on finite dimensional Banach spaces. For their proofs and for a full treatment of these arguments, we refer the interested reader to [4], see also [3].

Let $(X, \|\cdot\|)$ be a finite dimensional Banach space and let $I_\lambda : X \to \mathbb{R}$ be a function satisfying the structure hypothesis

(H1) $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$, and λ is a real positive parameter.

The first one describes a suitable interval of parameters for which a functional I_λ, satisfying condition $(H1)$, admits a local minimum.

Theorem 2.8. *Assume that* (H1) *holds and let* $r > 0$. *Then, for each* $\lambda \in \Lambda := \left]0, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right[$, *the function* $I_\lambda = \Phi - \lambda\Psi$ *admits at least a local minimum* $\overline{u} \in X$ *such that* $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}([0, r])$ *and* $I'_\lambda(\overline{u}) = 0$.

The second one shows that an appropriate restriction from below of the interval Λ given by Theorem 2.8 guarantees a priori a nonzero critical point.

Theorem 2.9. *Assume that* (H1) *holds. In addition, suppose that there exist* $r \in \mathbb{R}$ *and* $w \in X$, *with* $0 < \Phi(w) < r$, *such that*

$$\frac{\sup_{\Phi^{-1}([0,r])} \Psi}{r} < \frac{\Psi(w)}{\Phi(w)}. \tag{2.9}$$

Then, for each

$$\lambda \in \Lambda_w := \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right[,$$

the function $I_\lambda = \Phi - \lambda\Psi$ *admits at least one local minimum* $\overline{u} \in X$ *such that* $\overline{u} \neq 0$, $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}([0, r])$ *and* $I'_\lambda(\overline{u}) = 0$.

To introduce the third result, we need to recall the well known Palais-Smale condition, in brief (PS). We say that I_λ satisfies the (PS)-condition whenever one has that any sequence $\{u_n\}$ such that

(1) $\{I_\lambda(u_n)\}$ is bounded;
(2) $\{I'_\lambda(u_n)\}$ is convergent at 0 in X^*

admits a subsequence which is converging in X. Obviously, being X a finite dimensional Banach space, it is enough that any sequence $\{u_n\}$ satisfying (1) and (2) admits a bounded subsequence.

Theorem 2.10. *Assume that* (H1) *holds and fix* $r > 0$. *Assume that for each*

$$\lambda \in \Lambda := \left]0, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right[,$$

the function $I_\lambda = \Phi - \lambda\Psi$ *satisfies the (PS)-condition and is unbounded from below. Then, for each* $\lambda \in \Lambda$, *the function* I_λ *admits at least two distinct critical points.*

3. Main results

In this section, we present our main results, where we focalize our attention to obtain the existence of positive solutions for problem (1.1) whenever B is a symmetric real $Z-$matrix fulfilling the conditions (ii) and (iii) of Theorem 2.3 and the functions f_k satisfy condition (2.7). The first result gives the existence of one positive solution requiring just one sign condition on the functions f_k at 0.

Theorem 3.1. *Let c be a positive constant. Assume that*

$$f_k(0) \neq 0 \quad \text{for some } k \in [1, N]. \tag{3.1}$$

Then, for each

$$\lambda \in \Lambda_1 := \Big]0, \frac{\lambda_1}{2} \frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}\Big[,$$

problem (1.1) *admits at least one positive solution $\bar{u} \in \mathbb{R}^N$ such that $\|\bar{u}\|_\infty < c$.*

Proof. First, we observe that the interval Λ_1 is well-posed by (3.1) and (2.7). Next, we apply Theorem 2.8 by putting

$$\Phi(u) := \frac{1}{2} u^t B u, \quad \Psi(u) := \sum_{k=1}^N \int_0^{u(k)} f_k(t)dt, \quad \forall u \in \mathbb{R}^N, \tag{3.2}$$

and $I_\lambda := \Phi - \lambda\Psi$. Bearing in mind (2.2), it is clear that Φ is coercive and by calculations we can verify that I_λ fulfils condition $(H1)$. Moreover, taking $r = \frac{\lambda_1}{2} c^2$, by (2.1) and (2.2), we observe that

$$\Phi(u) \leq r \Longrightarrow \|u\|_\infty \leq c. \tag{3.3}$$

Therefore,

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \leq \frac{2}{\lambda_1} \frac{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}{c^2}. \tag{3.4}$$

Hence, for every $\lambda \in \Lambda_1$, we obtain

$$\lambda < \frac{\lambda_1}{2} \frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)} \leq \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)},$$

that is $\Lambda_1 \subset \Lambda$. Consequently, Theorem 2.8 ensures that the functional I_λ admits at least one critical point in $\mathbb{R}^N$, namely $\bar{u}$, such that $\Phi(\bar{u}) < r$. Putting together (3.1) and (3.3), we have that $0 < \|\bar{u}\|_\infty < c$. By Lemma 2.6, our conclusion follows. □

The second result ensures the existence of at least one positive solution without requiring asymptotic conditions neither at zero nor at infinity.

Theorem 3.2. *Let $w \in \mathbb{R}^N$ be a vector with $0 < w^t B w < \lambda_1 c^2$ and let c be a positive constant such that*

$$\frac{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}{c^2} < \lambda_1 \frac{\sum_{k=1}^N F_k(w(k))}{\sum_{i,j=1}^N b_{ij} w(i) w(j)}. \tag{3.5}$$

Then, for each

$$\lambda \in \Lambda_2 := \Big]\frac{1}{2} \frac{\sum_{i,j=1}^N b_{ij} w(i) w(j)}{\sum_{k=1}^N F_k(w(k))}, \frac{\lambda_1}{2} \frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}\Big[,$$

problem (1.1) *admits at least one positive solution $\bar{u}$ such that $\|\bar{u}\|_\infty < c$.*

Proof. In the same variational setting of the proof of Theorem 3.1, our aim is to use Theorem 2.9 to show that I_λ admits a nontrivial critical point. Putting again $r = \frac{\lambda_1}{2}c^2$, it is clearly that (3.4) holds. On the other hand, we observe that

$$\frac{\Psi(w)}{\Phi(w)} = 2\frac{\sum_{k=1}^N F_k(w(k))}{\sum_{i,j=1}^N b_{ij}w(i)w(j)}. \tag{3.6}$$

Hence, owing to (3.5), combining (3.4) and (3.6), we obtain

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)}.$$

Clearly, one has $0 < \Phi(w) < r$. So, since one has that $\Lambda_2 \subseteq \Lambda_w$, our claim is true. Therefore, I_λ has a non trivial critical point u. Taking into account Lemma 2.6, we can complete the proof. □

Remark 3.3. We highlight that the conclusion of Theorem 3.2 holds also in the case in which condition (3.1) of Theorem 3.1 does not work, that is whenever one has that $f_k(0) = 0$ for every $k \in [1, N]$.

Meaningful consequences of Theorem 3.2 are the following results. Roughly speaking, Corollaries 3.4 establishes the existence of a positive solution when the energy's field $(\underline{F(u)}) = (F_k(u(k))_{k\in[1,2,\dots,N]})$ of the diagonal field $\underline{f}(u)$ has a suitable behavior on a box of $\mathbb{R}^N$. While, Corollary 3.5 gives the same conclusion when a component F_k satisfies an appropriate superquadratic growth condition at zero along a direction of $\mathbb{R}^N$.

Corollary 3.4. *Let c, d be two positive constant with $d < c$ such that*

$$\frac{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}{c^2} < \frac{\lambda_1}{\sum_{i,j=1}^N b_{ij}}\frac{\sum_{k=1}^N F_k(d)}{d^2}. \tag{3.7}$$

Then, for each

$$\lambda \in \Lambda_3 := \Big]\frac{\sum_{i,j=1}^N b_{ij}}{2}\frac{d^2}{\sum_{k=1}^N F_k(d)}, \frac{\lambda_1}{2}\frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}\Big[,$$

problem (1.1) *admits at least one positive solution $\bar u$ such that $\|\bar u\|_\infty < c$.*

Proof. We apply Theorem (3.2) by choosing $w(k) = d$ for every $k \in [1, N]$. Clearly, to get our conclusion it is enough to verify that $w^tBw < \lambda_1 c^2$, that is $d < \sqrt{\frac{\lambda_1}{\sum_{i,j=1}^N b_{ij}}}c$. Arguing, by contradiction, we have that $c > d \ge \sqrt{\frac{\lambda_1}{\sum_{i,j=1}^N b_{ij}}}c$, from which it follows that

$$\frac{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}{c^2} \ge \frac{\sum_{k=1}^N F_k(d)}{c^2} \ge \frac{\lambda_1}{d^2}\frac{\sum_{k=1}^N F_k(d)}{\sum_{i,j=1}^N b_{ij}},$$

which contradicts our assumption (3.7). □

Corollary 3.5. *Let c, d be two positive constant with $d < c$. Assume that there exists $\overline{k} \in [1, N]$ such that*

$$\frac{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}{c^2} < \frac{\lambda_1}{b_{\overline{k}\overline{k}}}\frac{F_{\overline{k}}(d)}{d^2}. \tag{3.8}$$

Then, for each

$$\lambda \in \Lambda_4 := \Big]\frac{b_{\overline{k}\overline{k}}}{2}\frac{d^2}{F_{\overline{k}}(d)}, \frac{\lambda_1}{2}\frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}\Big[,$$

problem (1.1) *admits at least one positive solution* $\bar{u}$ *such that* $\|\bar{u}\|_\infty < c$.

Proof. We apply Theorem (3.2) arguing as in the proof of Corollary 3.4, by choosing $w(\overline{k}) = d$ and $w(k) = 0$ for every $k \in [1, N]$ with $k \neq \overline{k}$. □

In particular it is meaningful to point out the following consequence of Theorem (3.2) which improves [4, Corollary 6.1].

Corollary 3.6. *Assume that there exists* $\overline{k} \in [1, N]$ *such that*

$$\limsup_{s\to 0^+} \frac{F_{\overline{k}}(s)}{s^2} = +\infty. \tag{3.9}$$

Then, for each

$$\lambda \in \Big]0, \frac{\lambda_1}{2}\sup_{c>0}\frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}\Big[,$$

problem (1.1) *admits at least one positive solution.*

Proof. Fix λ as in the conclusion and $\bar{c} > 0$ such that $\lambda < \frac{\lambda_1}{2}\frac{\bar{c}^2}{\sum_{k=1}^N \max_{s\in[0,\bar{c}]} F_k(s)}$. By (3.9), there exists $d < \bar{c}$ such that,

$$\frac{F_{\overline{k}}(d)}{d^2} > \frac{b_{\overline{k}\overline{k}}}{2\lambda} > \frac{2}{\lambda_1}\frac{b_{\overline{k}\overline{k}}}{2}\frac{\sum_{k=1}^N \max_{s\in[0,\bar{c}]} F_k(s)}{\bar{c}^2},$$

for which we have

$$\frac{\sum_{k=1}^N \max_{s\in[0,\bar{c}]} F_k(s)}{\bar{c}^2} < \frac{\lambda_1}{b_{\overline{k}\overline{k}}}\frac{F_{\overline{k}}(d)}{d^2},$$

that is, condition (3.8) of Corollary 3.5 is verified and in addition we have that $\lambda \in \Lambda_4$. So, the proof is complete. □

Now, put

$$L_\infty(k) := \liminf_{s\to+\infty}\frac{F_k(s)}{s^2} \quad L_\infty := \min_{1\le k\le N} L_\infty(k).$$

Theorem 3.7. *Assume that* (3.1) *holds and*

$$\inf_{c>0}\frac{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}{c^2} < \frac{\lambda_1}{\lambda_N} L_\infty. \tag{3.10}$$

Then, for each

$$\lambda \in \widetilde{\Lambda} := \Big]\frac{\lambda_N}{2L_\infty}, \frac{\lambda_1}{2}\sup_{c>0}\frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}\Big[,$$

problem (1.1) *admits at least two positive solutions.*

Proof. Our aim is to apply Theorem 2.10 to show that I_λ admits at least two nontrivial critical points for every $\lambda \in \widetilde{\Lambda}$. To this end, fix $\lambda \in \widetilde{\Lambda}$ there exists $c = c(\lambda) > 0$ such that

$$\lambda < \frac{\lambda_1}{2}\frac{c^2}{\sum_{k=1}^N \max_{s\in[0,c]} F_k(s)}.$$

Of course, arguing as in the proof of Theorem 3.1, see (3.4), putting $r = \frac{\lambda_1}{2}c_2$, one has $\widetilde{\Lambda} \subseteq \Lambda$.

Now, we show that any (PS) sequence of I_λ is bounded in $\mathbb{R}^N$. Arguing by contradiction, suppose that there exists a sequence $\{u_n\}$ such that

$$\lim_{n\to+\infty} I_\lambda(u_n) = c,\ c \in \mathbb{R}, \quad \lim_{n\to+\infty} \sup_{\|v\|_2\le 1} I'_\lambda(u_n)(v) = 0, \quad \lim_{n\to+\infty} \|u_n\|_2 = +\infty. \tag{3.11}$$

Denoted by $u_n^{\pm} := \max\{\pm u, 0\}$, for every $n \in \mathbb{N}$, an easy computation produces

$$-I'_\lambda(u_n)(u_n^-) = (-u_n^-)^t B u_n + \lambda \sum_{k=1}^N f_k(0) u_n^-(k) \ge \lambda_1 \|u_n^-\|_2^2;$$

that is,

$$\lambda_1 \|u_n^-\|_2 \le -I'_\lambda(u_n)\big(\frac{u_n^-}{\|u_n^-\|_2}\big), \quad \forall n \in \mathbb{N}. \tag{3.12}$$

Thus, by (3.11), we obtain $\lim_{n\to+\infty} \|u_n^-\|_2 = 0$, which implies that $\{u_n^-\}$ is bounded in $\mathbb{R}^n$. In addition, by (2.1), there exists $M > 0$ such that $0 \le u_n^-(k) \le M$ for all $k \in [1, N]$ and $n \in \mathbb{N}$.

Now, we prove that $\{u_n^+\}$ is bounded. To this end, fixed $\rho = \rho(\lambda) > 0$ such that

$$\frac{\lambda_N}{2\lambda} < \rho < L_\infty, \tag{3.13}$$

for every $k \in [1, N]$, there is $\delta_k > 0$ such that

$$F_k(s) > \rho s^2, \quad \forall s > \delta_k.$$

More precisely, an easy computation shows that for every $k \in [1, N]$ there exists $\eta_k > 0$ such that

$$F_k(s) > \rho s^2 - \eta_k, \quad \forall\ s > -M. \tag{3.14}$$

By using the previous inequality, for every $n \in \mathbb{N}$, we obtain

$$\Psi(u_n) = \sum_{k=1}^N F_k(u_n(k)) \ge \rho \sum_{k=1}^N |u_n(k)|^2 - \sum_{k=1}^N \eta_k = \rho \|u_n\|_2^2 - \eta.$$

On the other hand, for every $n \in \mathbb{N}$, one has

$$\begin{aligned} I_\lambda(u_n) &= \Phi(u_n) - \lambda\Psi(u_n) \\ &\le \frac{\lambda_N}{2}\|u_n\|_2^2 - \lambda\rho\|u_n\|_2^2 + \lambda\eta \\ &= \big(\frac{\lambda_N}{2} - \lambda\rho\big)\|u_n\|_2^2 + \lambda\eta. \end{aligned}$$

Since $\|u_n\|_2 \to +\infty$, by (3.13), it follows that $\lim_{n\to+\infty} I_\lambda(u_n) = -\infty$, against (3.11). Hence, I_λ satisfies the (PS)-condition.

Moreover, by (3.13) and (3.14), working with an unbounded sequence $\{u_n\}$ of positive vector, i.e. $u_n = u_n^+$ for every $n \in \mathbb{N}$, with $\|u_n^+\| \to +\infty$, similar arguments to those developed to verify (PS) prove also that I_λ is unbounded from below.

Since all the assumptions of Theorem 2.10 are satisfied, we obtain two critical points for I_λ, which by (j), and Lemma 2.6, are two positive solutions for (1.1). □

Remark 3.8. If we assume that $f_k(t) \geq 0$ for all $t \geq 0$ and for all $k \in [1, N]$, with $f_k(0) > 0$ for some $k \in [1, N]$, the conclusion of Theorem 3.7 is again true provided that

$$\lim_{s \to +\infty} \frac{f_k(s)}{s} = +\infty \quad \text{for every } k \in [1, N]. \tag{3.15}$$

Of course, in a such case the interval of parameters is $\widetilde{\Lambda} := \big]0, \frac{\lambda_1}{2} \sup_{c>0} \frac{c^2}{\sum_{k=1}^N F_k(c)}\big[$.

Now we point some special cases of our main results concerning tridiagonal nonlinear symmetric systems which could be of some interest for people working in the applications of mathematics to the real world and to numerical methods for solving differential equations. More precisely, we deal with systems

$$T_N(a, b, b) = \lambda \underline{f}(u), \tag{3.16}$$

where $T_N(a, b, b)$, is the following matrix,

$$T_N(a, b, b) := \begin{pmatrix} a & b & 0 & \dots & 0 \\ b & a & b & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & \dots & b & a & b \\ 0 & \dots & 0 & b & a \end{pmatrix}_{N \times N}$$

with $a, b \in \mathbb{R}$. To guarantee that $T_N(a, b, b)$ turns out to be a positive definite matrix, bearing in mind that the eigenvalues are

$$\lambda_k = a + 2b \cos\Big(\frac{k\pi}{N+1}\Big), \quad k = 1, 2, \dots, N,$$

whenever $b < 0$, as you can see, for instance in [12, Theorem 2.2], we suppose that

$$a > 2|b| \cos\big(\frac{\pi}{N+1}\big). \tag{3.17}$$

Theorem 3.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let a, b and c be three constants with $a > 0$, $b < 0$ and $c > 0$. Assume that (3.17) holds. We have:*

(1) If $f(0) > 0$, then, for every

$$\lambda \in \Lambda_{abc} := \big]0, \big(\frac{a}{2} + b \cos\frac{\pi}{N+1}\big) \frac{c^2}{N \max_{s \in [0,c]} F(c)}\big[,$$

problem (3.16) admits at least one positive solution u_1, with $\|u\|_1 < c$. If, in addition, we have

$$\lim_{s \to +\infty} \frac{f(s)}{s} = +\infty, \tag{3.18}$$

then there exists a second positive solution.

(2) If $f(0) = 0$, $f(s) \leq 0$ for every $0 \leq s \leq c$ and (3.18) holds, then problem (3.16) admits at least one positive solution for every $\lambda > 0$.

Proof. Since (3.17) holds and $f(0) > 0$, put $f_k = f$ for every $k \in [1, N]$, it is clear that conclusion (α_1) is a direct consequence of Theorems 3.1 and 3.7, see also Remark 3.8. While, conclusion (α_2) follows from Theorem 3.7, taking into account that $\max_{s \in [0,c]} F(c) = 0$. □

To show that conditions (3.7) and (3.8) are mutually independent, first we write a version of Corollaries (3.4) and (3.5) for system (3.16).

Corollary 3.10. *Let a, b, c and d be four constants with $a > 0$, $b < 0$ and $0 < d < c$ such that* (3.17) *holds. Assume that one of the following two conditions is satisfied:*

$$\frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} < \frac{a + 2b\cos(\frac{\pi}{N+1})}{aN + 2(N-1)b} \frac{\sum_{k=1}^{N} F_k(d)}{d^2}, \tag{3.19}$$

or there exists $\overline{k} \in [1, N]$ such that

$$\frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F(s)}{c^2} < \frac{a + 2b\cos\left(\frac{\pi}{N+1}\right)}{a} \frac{F_{\overline{k}}(d)}{d^2}. \tag{3.20}$$

Then, for every

$$\lambda \in \Lambda_5 := \big]\Theta, \big(\frac{a}{2} + b\cos(\frac{\pi}{N+1})\big) \frac{c^2}{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}\big[,$$

problem (3.16) *admits a positive solution u such that $0 < \|u\|_\infty < c$, being*

$$\Theta = \frac{aN + 2(N-1)b}{2} \frac{d^2}{\sum_{k=1}^{N} F_k(d)} \quad \text{or} \quad \Theta = \frac{a}{2} \frac{d^2}{F_{\overline{k}}(d)},$$

according to either (3.19) *or* (3.20) *holds.*

Remark 3.11. It is worth noticing that condition (3.19) or (3.20) are mutually independent. Indeed, if we consider the special case where $f_k = f$ for every $k \in [1, N]$ being $f : \mathbb{R} \to \mathbb{R}$ a continuous function with $f(0) \geq 0$, it is easy to show that (3.19) is more general of (3.20), because for system (3.16) one has $\sum_{i,j=1}^{N} b_{ij} < N \min_{k\in[1,N]} b_{kk}$. On the other hand, it is a simple matter to prove that in the following framework (3.20) is verified, whereas (3.19) does not work. There exist two positive constant c and d with $0 < d < c$ such that:

(l) there exists $\overline{k} \in [1, N]$ such that

$$\frac{\max_{s\in[0,c]} F_{\overline{k}}(s)}{c^2} < \frac{a + 2b\cos(\frac{\pi}{N+1})}{a} \frac{F_{\overline{k}}(d)}{d^2};$$

(ll) $F_k(s) \leq 0$ for every $k \in [1, N]$ with $k \neq \overline{k}$ and $s \geq 0$;

(lll) $\sum_{k=1}^{N} F_k(d) \leq 0$.

Taking into account the above conditions, it is easy to prove that the system

$$2x - \frac{1}{10} y = \lambda e^x,$$
$$-\frac{1}{10} x + 2y = -2\lambda y^2 e^y,$$

admits a positive solution $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\max\{x, y\} \leq 1$ for every $\lambda \in \big]\frac{1}{100(e^{1/10}-1)}, \frac{19}{20}\frac{1}{e-1}\big[$. To see this, it is enough to choice $a = 2$, $b = -1/10$, $c = 1$, $d = 1/10$ and

$$B = \begin{pmatrix} 2 & -1/10 \\ -1/10 & 2 \end{pmatrix},$$

and to apply Corollary 3.10 with $F_1(s) = e^s - 1$ and $F_2(s) = -e^s(s^2 - 2s + 2) + 2$, for every $s \geq 0$. To be precise, bearing in mind Corollary 3.6, the above system as at least one positive solution for every $\lambda \in \big]0, \frac{19}{20}\frac{1}{e-1}\big[$.

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