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LOCAL AND GLOBAL LOW-REGULARITY SOLUTIONS TO GENERALIZED LERAY-ALPHA EQUATIONS

NATHAN PENNINGTON

Abstract. It has recently become common to study approximating equations for the Navier-Stokes equation. One of these is the Leray-α equation, which regularizes the Navier-Stokes equation by replacing (in most locations) the solution u with $(1-\alpha^2\Delta)u$. Another is the generalized Navier-Stokes equation, which replaces the Laplacian with a Fourier multiplier with symbol of the form $-|\xi|^\gamma$ ($\gamma = 2$ is the standard Navier-Stokes equation), and recently in [16] Tao also considered multipliers of the form $-|\xi|^\gamma/g(|\xi|)$, where g is (essentially) a logarithm. The generalized Leray-α equation combines these two modifications by incorporating the regularizing term and replacing the Laplacians with more general Fourier multipliers, including allowing for g terms similar to those used in [16]. Our goal in this paper is to obtain existence and uniqueness results with low regularity and/or non-L^2 initial data. We will also use energy estimates to extend some of these local existence results to global existence results.

1. Introduction

The incompressible form of the Navier-Stokes equation is given by

$$\begin{gathered}\partial_t u + (u\cdot\nabla)u = \nu\Delta u - \nabla p,\\ u(0,x) = u_0(x), \quad \operatorname{div}(u) = 0\end{gathered} \tag{1.1}$$

where $u : I\times\mathbb{R}^n \to \mathbb{R}^n$ for some time strip $I = [0,T)$, $\nu > 0$ is a constant due to the viscosity of the fluid, $p : I\times\mathbb{R}^n \to \mathbb{R}^n$ denotes the fluid pressure, and $u_0 : \mathbb{R}^n \to \mathbb{R}^n$. The requisite differential operators are defined by $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial^2_{x_i}}$ and $\nabla = (\frac{\partial}{\partial_{x_i}}, \dots, \frac{\partial}{\partial_{x_n}})$.

In dimension $n = 2$, local and global existence of solutions to the Navier-Stokes equation are well known (see [11]; for a more modern reference, see [17, Chapter 17]). For dimension $n \geq 3$, the problem is significantly more complicated. There is a robust collection of local existence results, including [7], in which Kato proves the existence of local solutions to the Navier-Stokes equation with initial data in $L^n(\mathbb{R}^n)$; [9], where Kato and Ponce solve the equation with initial data in the Sobolev space $H^{n/p-1,p}(\mathbb{R}^n)$; and [10], where Koch and Tataru establish local existence with initial data in the space $BMO^{-1}(\mathbb{R}^n)$ (for a more complete accounting of local existence theory for the Navier-Stokes equation, see [12]). In all of these

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local results, if the initial datum is assumed to be sufficiently small, then the local solution can be extended to a global solution. However, the issue of global existence of solutions to the Navier-Stokes equation in dimension $n \geq 3$ for arbitrary initial data is one of the most challenging open problems remaining in analysis.

Because of the intractability of the Navier-Stokes equation, many approximating equations have been studied. One of these is the Leray-α model, which is

$$\begin{gathered}\partial_t(1-\alpha^2\Delta)u + \nabla_u(1-\alpha^2\Delta)u - \nu\Delta(1-\alpha^2\Delta)u = -\nabla p,\\ u(0,x)=u_0(x),\ \operatorname{div} u_0 = \operatorname{div} u = 0,\end{gathered}$$

where we recall that $\nabla_u v = (u\cdot\nabla)v$. Note that setting $\alpha = 0$ returns the standard Navier-Stokes equation. Like the Lagrangian Averaged Navier-Stokes (LANS) equation (which differs from the Leray-α in the presence of an additional nonlinear term), the system (1.2) compares favorably with numerical data; see [5], in which the authors compared the Reynolds numbers for the Leray-α equation and the LANS equation with the Navier-Stokes equation.

Another commonly studied equation is the generalized Navier-Stokes equation, given by

$$\begin{gathered}\partial_t u + (u\cdot\nabla)u = \nu\mathcal{L}u - \nabla p,\\ u(0,x) = u_0(x), \quad \operatorname{div}(u) = 0\end{gathered}$$

where $\mathcal{L}$ is a Fourier multiplier with symbol $m(\xi) = -|\xi|^\gamma$ for $\gamma > 0$. Choosing $\gamma = 2$ returns the standard Navier-Stokes equation. In [20], Wu proved (among other results) the existence of unique local solutions for this equation provided the data is in the Besov space $B^s_{p,q}(\mathbb{R}^n)$ with $s = 1 + n/p - \gamma$ and $1 < \gamma \leq 2$. If the norm of the initial data is sufficiently small, these local solutions can be extended to global solutions.

It is well known that if $\gamma \geq \frac{n+2}{2}$, then this equation has a unique global solution. In [16], Tao strengthened this result, proving global existence with the symbol $m(\xi) = -|\xi|^\gamma/g(|\xi|)$, with $\gamma \geq \frac{n+2}{2}$ and g a non-decreasing, positive function that satisfies

$$\int_1^\infty \frac{ds}{sg(s)^2} = +\infty.$$

Note that $g(|x|) = \log^{1/2}(2+|x|^2)$ satisfies the condition. Similar types of results involving g terms that are, essentially, logarithms have been proven for the nonlinear wave equation; see [16] for a more detailed description.

Here we consider a combination of these two models, called the generalized Leray-α equation, which is

$$\begin{gathered}\partial_t(1-\alpha^2\mathcal{L}_2)u + \nabla_u(1-\alpha^2\mathcal{L}_2)u - \nu\mathcal{L}_1(1-\alpha^2\mathcal{L}_2)u = -\nabla p,\\ u(0,x) = u_0(x), \quad \operatorname{div} u_0 = \operatorname{div} u = 0,\end{gathered} \tag{1.2}$$

with the operators $\mathcal{L}_i$ defined by

$$\mathcal{L}_i u(x) = \int -\frac{|\xi|^{\gamma_i}}{g_i(\xi)}\hat{u}(\xi)e^{ix\cdot\xi}d\xi,$$

where g_i are radially symmetric, nondecreasing, and bounded below by 1. Note that if $g_2 = 1$ and $\gamma_2 = 0$, then $\mathcal{L}_2 u(x) = -u(x)$, so choosing $g_1 = g_2 = 1$, $\gamma_1 = 2$, and $\gamma_2 = 0$ returns the Navier-Stokes equation (after absorbing $(1+\alpha^2)^{-1}$ into the pressure function p). Choosing $g_1 = g_2 = 1$ and $\gamma_1 = \gamma_2 = 2$ gives the Leray-α equation, and choosing $g_2 = \gamma_2 = 1$ returns the generalized Navier-Stokes equation.

In [1], the authors proved the existence of a smooth global solution to the generalized Leary-α equation with smooth initial data provided $\gamma_1 + \gamma_2 \geq n/2 + 1$, $g_2 = 1$, and g_1 is in a category similar to, though inclusive of, the type of g required in Tao's argument in [16].

In [22], Yamazaki obtains a unique global solution to equation (1.2) in dimension three provided $(1 - \alpha^2 \mathcal{L}_2)u_0$ is in the Sobolev space $H^{m,2}(\mathbb{R}^3)$, where u_0 is the initial data, $m > \max\{5/2, 1 + 2\gamma_1\}$, and provided γ_1 and γ_2 satisfy the inequality $2\gamma_1 + \gamma_2 \geq 5$ and that g_1 and g_2 satisfy

$$\int_1^\infty \frac{ds}{s g_1^2(s) g_2(s)} = \infty. \tag{1.3}$$

The goal of this article is to obtain a much wider array of existence results, specifically existence results for initial data with low regularity and for initial data outside the L^2 setting. We will also, where applicable, use the energy bound from [22] to extend these local solutions to global solutions. Our plan is to follow the general contraction-mapping based procedure outlined by Kato and Ponce in [9] for the Navier-Stokes equation, with two key modifications.

First, the approach used in [9] relies heavily on operator estimates for the heat kernel $e^{t\Delta}$. We will require similar estimates for our solution operator $e^{t\mathcal{L}_1}$ and some operator estimates for $(1-\alpha^2\mathcal{L}_2)$, and establishing these estimates is the purpose of Section 5 and Section 6. This will require some technical restrictions on the choices of g_1 and g_2 that will be more fully addressed below. We also note that these estimates should allow the application of this general technique to other similar equations, like the MHD equation in [23] and [22], the generalized MHD equation found in [21] (see [19] for a general study of the generalized MHD equation), the logarithmically super critical Boussinesq system in [6], and the Navier-Stokes like equation studied in [13].

The second modification is in how we will deal with the nonlinear term. For the first set of results, we will use the standard Leibnitz-rule estimate to handle the nonlinear terms. Our second set of results rely on a product estimate (due to Chemin in [3]) which will allow us to obtain lower regularity existence but will (among other costs) require us to work in Besov spaces. The advantages and disadvantages of each approach will be detailed later in this introduction. The product estimates themselves are stated as Proposition 2.2 and Proposition 2.3 in Section 2. We will also need bounds on the terms $(1 - \mathcal{L}_2)$ and $(1 - \mathcal{L}_2)^{-1}$, and establishing these bounds is the subject of Section 5.

The rest of this paper is organized as follows. The remainder of this introduction is devoted to stating and contextualizing the main results of the paper. Section 2 reviews the basic construction of Besov spaces and states some foundational results, including our two product estimates. In Section 3 we carry out the existence argument using the standard product estimate, and in Section 4 we obtain existence results using the other product estimate. As stated above, Sections 5 and 6 contain the proofs of the operator estimates that are central to the arguments used in Sections 3 and 4.

Our last task before stating the main results is to establish some notation. First, we denote Besov spaces by $B^s_{p,q}(\mathbb{R}^n)$, with norm denoted by $\|\cdot\|_{B^s_{p,q}} = \|\cdot\|_{s,p,q}$ (a complete definition of these spaces can be found in Section 2). We define the space

$$C^T_{a;s,p,q} = \{f \in C((0,T) : B^s_{p,q}(\mathbb{R}^n)) : \|f\|_{a;s,p,q} < \infty\},$$

where

$$\|f\|_{a;s,p,q} = \sup\{t^a \|f(t)\|_{s,p,q} : t \in (0,T)\},$$

$T > 0$, $a \geq 0$, and $C(A : B)$ is the space of continuous functions from A to B. We let $\dot{C}^T_{a;s,p,q}$ denote the subspace of $C^T_{a;s,p,q}$ consisting of f such that

$$\lim_{t\to 0^+} t^a f(t) = 0 \quad (\text{in } B^s_{p,q}(\mathbb{R}^n)).$$

Note that while the norm $\|\cdot\|_{a;s,p,q}$ lacks an explicit reference to T, there is an implicit T dependence. We also say $u \in BC(A : B)$ if $u \in C(A : B)$ and $\sup_{a\in A} \|u(a)\|_B < \infty$.

Now we are ready to state the existence results. As expected in these types of arguments, the full result gives unique local solutions provided the parameters satisfy a large collection of inequalities. Here we state special cases of the full results. Our first Theorem uses the standard product estimate (Proposition 2.2 in Section 2).

Theorem 1.1. *Let $\gamma_1 > 1$, $\gamma_2 > 0$, $q \geq 1$, $p \geq 2$, and let s_1, s_2 be real numbers such that $s_2 > \gamma_2$, $0 < s_2 - s_1 < \min\{\gamma_1/2, 1\}$ and $\gamma_1 \geq s_2 - s_1 + 1 + n/p$. We also assume that g_1 and g_2 are Mikhlin multipliers (see inequality* (5.1)*). Then for any divergence free $u_0 \in B^{s_1}_{p,q}(\mathbb{R}^n)$, there exists a unique local solution u to the generalized Leray-alpha equation* (1.2)*, with*

$$u \in BC([0,T) : B^{s_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s_2,p,q},$$

where $a = (s_2-s_1)/\gamma_1$. T can be chosen to be a non-increasing function of $\|u_0\|_{B^{s_1}_{p,q}}$ with $T = \infty$ if $\|u_0\|_{B^{s_1}_{p,q}}$ is sufficiently small.

Before stating our second theorem, we remark that this result also holds if the Besov spaces are replaced by Sobolev spaces. This is not true of the next theorem, which is a special case of the more general Theorem 4.1, and relies on our second product estimate (Proposition 2.3 in Section 2).

Theorem 1.2. *Let $\gamma_1 > 1$, $\gamma_2 > 0$, $q \geq 1$, $p \geq 2$, s_1, and s_2 satisfy*

$$\begin{gathered} 0 < s_2 - s_1 < \gamma_1/2, \\ s_1 > \gamma_2 - n/p - 1, \\ \gamma_1 \geq 2s_2 - s_1 - \gamma_2 + n/p + 1, \\ n/p > \gamma_2/2, \\ s_2 \geq \gamma_2/2. \end{gathered}$$

We also assume that g_1 and g_2 are Mikhlin multipliers (see inequality (5.1)*). Then for any divergence free $u_0 \in B^{s_1}_{p,q}(\mathbb{R}^n)$, there exists a unique local solution u to the generalized Leray-alpha equation* (1.2)*, with*

$$u \in BC([0,T) : B^{s_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s_2,p,q},$$

where $a = (s_2-s_1)/\gamma_1$. T can be chosen to be a non-increasing function of $\|u_0\|_{B^{s_1}_{p,q}}$ with $T = \infty$ if $\|u_0\|_{B^{s_1}_{p,q}}$ is sufficiently small.

We remark that in the first theorem, γ_2 can be arbitrarily large, but $s_1 > -1$, while in the second theorem $\gamma_2 < 2n/p$, but for sufficiently large γ_1 and sufficiently small γ_2, $s_1 > \gamma_2 - n/p - 1$ can be less than -1. Thus the non-standard product estimate allows us to obtain existence results for initial data with lower regularity, but requires γ_2 to be small and requires the use of Besov spaces.

We also note that if we set $\gamma_2 = 0$ and $g_2(|\xi|) = 1$ (and thus are back in the case of the generalized Navier-Stokes equation), then these techniques would recover the results of Wu in [20] for the generalized Navier-Stokes equation.

As was stated above, these results will hold if the g_i are Mikhlin multipliers. However, there are interesting choices of g_i (specifically g_i being, essentially, a logarithm) which are not Mikhlin multipliers. For this case we have analogous, but slightly weaker, results. In what follows, we let r^- indicate a number arbitrarily close to, but strictly less than, r (and similarly let r^+ be a number arbitrarily close to, but strictly greater than, r).

Theorem 1.3. *Let $\gamma_1 > 1$, $\gamma_2 > 0$, $q \geq 1$, $p \geq 2$, and s_1, s_2 be real numbers such that $s_2 > \gamma_2$, $0 < s_2 - s_1 < \min\{\gamma_1^-/2, 1\}$ and $\gamma_1^- \geq s_2 - s_1 + 1 + n/p$. We also assume that, for $i = 1, 2$, $|g_i(r)| \leq Cr^\delta$ for any $\delta > 0$ and $|g_i^{(k)}(r)| \leq Cr^{-k}$ for $1 \leq k \leq n/2 + 1$. Then for any divergence free $u_0 \in B^{s_1}_{p,q}(\mathbb{R}^n)$, there exists a unique local solution u to the generalized Leray-alpha equation* (1.2)*, with*

$$u \in BC([0,T) : B^{s_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s_2,p,q},$$

where $a = (s_2 - s_1)/\gamma_1^-$ for arbitrarily small $\varepsilon > 0$. T can be chosen to be a non-increasing function of $\|u_0\|_{B^{s_1}_{p,q}}$ with $T = \infty$ if $\|u_0\|_{B^{s_1}_{p,q}}$ is sufficiently small.

Theorem 1.4. *Let $\gamma_1 > 1$, $\gamma_2 > 0$, $q \geq 1$, $p \geq 2$, s_1 and s_2 satisfy*

$$\begin{gathered}
0 < s_2 - s_1 < \gamma_1^-/2, \\
s_1 > \gamma_2^- - n/p - 1, \\
\gamma_1^- \geq 2s_2 - s_1 - \gamma_2^- + n/p + 1, \\
n/p > \gamma_2^-/2, \\
s_2 \geq \gamma_2^-/2.
\end{gathered}$$

We also assume that, for $i = 1, 2$, $|g_i(r)| \leq Cr^\delta$ for any $\delta > 0$ and $|g_i^{(k)}(r)| \leq Cr^{-k}$ for $1 \leq k \leq n/2 + 1$. Then for any divergence free $u_0 \in B^{s_1}_{p,q}(\mathbb{R}^n)$, there exists a unique local solution u to the generalized Leray-alpha equation (1.2)*, with*

$$u \in BC([0,T) : B^{s_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s_2,p,q},$$

where $a = (s_2 - s_1)/\gamma_1^-$. T can be chosen to be a non-increasing function of $\|u_0\|_{B^{s_1}_{p,q}}$ with $T = \infty$ if $\|u_0\|_{B^{s_1}_{p,q}}$ is sufficiently small.

Incorporating the additional constraints from the energy bound in [22], we can now state the global existence result.

Corollary 1.5. *Let $p = 2$ and let $n = 3$. Then, for any of our local existence results, if we additionally assume that*

$$\begin{gathered}
2\gamma_1 + \gamma_2 \geq 5, \\
\int_1^\infty \frac{ds}{s g_1^2(s) g_2(s)} = \infty,
\end{gathered}$$

then the local solutions can be extended to global solutions.

Note that if g_1 and g_2 are Mikhlin multipliers, then all of the constraints on g_1 and g_2 are satisfied. Also, if g_1 and g_2 are logarithms, then the corollary extends the appropriate local solutions from Theorem 1.3 and 1.4 to global solutions.

The corollary follows directly from the smoothing effect of the operator $e^{t\mathcal{L}_1}$, which ensures that, for any $t > 0$, our local solution $u(t,\cdot) \in B^r_{2,q}(\mathbb{R}^3)$ for any $r \in \mathbb{R}$. This provides the smoothness necessary to use the energy bound from [22] to obtain a uniform-in-time bound on the $B^{s_1}_{2,q}(\mathbb{R}^3)$ norm of the solution, and then a standard bootstrapping argument completes the proof of global existence. In Section 7, we include an argument detailing this smoothing effect for the solution to Theorem 1.1.

Finally, we remark that extending the local solutions to global solutions for $p \neq 2$ and $n > 3$ will be the subject of future work. Handling $n > 3$ should follow by tweaking the argument used in [22]. Obtaining global solutions for $p \neq 2$ is significantly more complicated, and the argument will follow the interpolation based argument used by Gallagher and Planchon [4] for the two dimensional Navier-Stokes equation.

2. Besov spaces

We begin by defining the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$. Let ψ_0 be an even, radial, Schwartz function with Fourier transform $\hat{\psi}_0$ that has the following properties:

$$
\begin{gathered}
\hat{\psi}_0(x) \geq 0,\\
\operatorname{support} \hat{\psi}_0 \subset A_0 := \{\xi \in \mathbb{R}^n : 2^{-1} < |\xi| < 2\},\\
\sum_{j\in\mathbb{Z}} \hat{\psi}_0(2^{-j}\xi) = 1, \quad \text{for all } \xi \neq 0.
\end{gathered}
$$

We then define $\hat{\psi}_j(\xi) = \hat{\psi}_0(2^{-j}\xi)$ (from Fourier inversion, this also means $\psi_j(x) = 2^{jn}\psi_0(2^j x)$), and remark that $\hat{\psi}_j$ is supported in $A_j := \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}$. We also define Ψ by

$$
\hat{\Psi}(\xi) = 1 - \sum_{k=0}^{\infty} \hat{\psi}_k(\xi). \tag{2.1}
$$

We define the Littlewood Paley operators Δ_j and S_j by

$$
\Delta_j f = \psi_j * f, \quad S_j f = \sum_{k=-\infty}^{j} \Delta_k f,
$$

and record some properties of these operators. Applying the Fourier Transform and recalling that $\hat{\psi}_j$ is supported on $2^{j-1} \leq |\xi| \leq 2^{j+1}$, it follows that

$$
\begin{gathered}
\Delta_j \Delta_k f = 0, \quad |j-k| \geq 2,\\
\Delta_j(S_{k-3} f \Delta_k g) = 0 \quad |j-k| \geq 4,
\end{gathered} \tag{2.2}
$$

and, if $|i-k| \leq 2$, then

$$
\Delta_j(\Delta_k f \Delta_i g) = 0 \quad j > k+4. \tag{2.3}
$$

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ we define the space $\tilde{B}^s_{p,q}(\mathbb{R}^n)$ to be the set of distributions such that

$$\|u\|_{\tilde{B}^s_{p,q}} = \Big(\sum_{j=0}^{\infty}(2^{js}\|\Delta_j u\|_{L^p})^q\Big)^{1/q} < \infty,$$

with the usual modification when $q = \infty$. Finally, we define the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ by the norm

$$\|f\|_{B^s_{p,q}} = \|\Psi * f\|_p + \|f\|_{\tilde{B}^s_{p,q}},$$

for $s > 0$. For $s > 0$, we define $B^{-s}_{p',q'}$ to be the dual of the space $B^s_{p,q}$, where p', q' are the Holder-conjugates to p, q.

These Littlewood-Paley operators are also used to define Bony's paraproduct. We have

$$fg = \sum_k S_{k-3} f \Delta_k g + \sum_k S_{k-3} g \Delta_k f + \sum_k \Delta_k f \sum_{l=-2}^{2} \Delta_{k+l} g. \tag{2.4}$$

The estimates (2.2) and (2.3) imply that

$$\begin{aligned} \Delta_j(fg) = &\sum_{k=-3}^{3} \Delta_j(S_{j+k-3} f \Delta_{j+k} g) + \sum_{k=-3}^{3} \Delta_j(S_{j+k-3} g \Delta_{j+k} f) \\ &+ \sum_{k>j-4} \Delta_j\Big(\Delta_k f \sum_{l=-2}^{2} \Delta_{k+l} g\Big). \end{aligned} \tag{2.5}$$

Now we turn our attention to establishing some basic Besov space estimates. First, we let $1 \le q_1 \le q_2 \le \infty$, $\beta_1 \le \beta_2$, $1 \le p_1 \le p_2 \le \infty$, $\alpha > 0$, and set $\tilde{p} = np/(n - \alpha p)$ with $\alpha < n/p$. Then we have the following Besov embedding results:

$$\|f\|_{B^{\beta_1}_{p,q_2}} \le C\|f\|_{B^{\beta_2}_{p,q_1}}, \tag{2.6a}$$

$$\|f\|_{B^{\beta_1}_{\tilde{p},q_1}} \le C\|f\|_{B^{\beta_1+\alpha}_{p_1,q_1}}, \tag{2.6b}$$

$$\|f\|_{H^{\beta_1,2}} = \|f\|_{B^{\beta_1}_{2,2}}. \tag{2.6c}$$

The following result is straightforward, but will be used often.

Proposition 2.1. *Let $1 \le p < \infty$, $0 < \alpha < n/p$ and set $\tilde{p} = np/(n - \alpha p)$. Then*

$$\|f\|_{L^{\tilde{p}}} \le C\|f\|_{B^{\alpha+}_{p,q}}, \tag{2.7}$$

for any $1 \le q \le \infty$.

For any $\varepsilon > 0$, we have

$$\|f\|_{L^{\tilde{p}}} \le \|f\|_{B^{\varepsilon}_{\tilde{p},q}} \le \|f\|_{B^{\alpha+\varepsilon}_{p,q}}, \tag{2.8}$$

where we used the definition of Besov spaces for the first inequality and (2.6b) for the second.

Next we record our two Leibnitz-rule type estimate. The first is the standard estimate, which can be found in (among many other places) [2, Lemma 2.2]. See also [18, Proposition 1.1].

Proposition 2.2. *Let $s > 0$ and $q \in [1, \infty]$. Then*

$$\|fg\|_{B^s_{p,q}} \leq C\big(\|f\|_{L^{p_1}}\|g\|_{B^s_{p_2,q}} + \|f\|_{B^s_{q_1,q}}\|g\|_{L^{q_2}}\big),$$

where $1/p = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$ and $p_i, q_i \in [1, \infty]$ for $i = 1, 2$.

Our second product estimate is less common. The estimate originated in [3]; another proof can be found in [14].

Proposition 2.3. *Let $f \in B^{s_1}_{p_1,q}(\mathbb{R}^n)$ and let $g \in B^{s_2}_{p_2,q}(\mathbb{R}^n)$. Then, for any p such that $1/p \leq 1/p_1 + 1/p_2$ and with $s = s_1 + s_2 - n(1/p_1 + 1/p_2 - 1/p)$, we have*

$$\|fg\|_{B^s_{p,q}} \leq C\|f\|_{B^{s_1}_{p_1,q}}\|g\|_{B^{s_2}_{p_2,q}},$$

provided $s_1 < n/p_1$, $s_2 < n/p_2$, and $s_1 + s_2 > 0$.

3. Local existence by Proposition 2.2

Theorem 1.1 and Theorem 1.3 are both proven using the standard product estimate (Proposition 2.2). These theorems are both special cases of more general theorems, and the primary task of this section is to prove the theorem which implies Theorem 1.3. There is a similar result associated with Theorem 1.1, and it will be discussed at the end of the section.

Theorem 3.1. *Let $\gamma_1 > 1$, $\gamma_2 > 0$, $q \geq 1$, and $p \geq 2$. Assume g_1 and g_2 satisfy $|g_i(r)| \leq C(1 + r)^\delta$ for any $\delta > 0$ and $|g_i^{(k)}(r)| \leq Cr^{-k}$ for $1 \leq k \leq n/2 + 1$. Let $u_0 \in B^{s_1}_{p,q}(\mathbb{R}^n)$ be divergence-free. Then there exists a unique local solution u to the generalized Leray-alpha equation* (1.2)*, with*

$$u \in BC([0, T) : B^{s_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s_2,p,q},$$

where $a = (s_2 - s_1)/(\gamma_1 - \varepsilon)$ for any sufficiently small $\varepsilon > 0$ if there exists $k > 0$ such that the parameters satisfy (3.11)*. T can be chosen to be a non-increasing function of $\|u_0\|_{B^{s_1}_{p,q}}$ with $T = \infty$ if $\|u_0\|_{B^{s_1}_{p,q}}$ is sufficiently small.*

We begin by re-writing equation (1.2) as

$$\begin{gathered}\partial_t u + P(1 - \alpha^2\mathcal{L}_2)^{-1}\operatorname{div}(u \otimes (1 - \alpha^2\mathcal{L}_2)u) - \nu\mathcal{L}_1 u = 0,\\ u(0, x) = u_0(x), \quad \operatorname{div} u_0 = \operatorname{div} u = 0,\end{gathered} \tag{3.1}$$

where P is the Hodge projection onto divergence free vector fields and an application of the divergence free condition shows $\nabla_u(1-\alpha^2\mathcal{L}_2)u = \operatorname{div}(u\otimes(1-\alpha^2\mathcal{L}_2)u)$, where $v \otimes w$ is the matrix with ij entry equal to the product of the i^{th} coordinate of v and the j^{th} coordinate of w.

Setting $\alpha = 1$ and $\nu = 1$ for notational simplicity and applying Duhamel's principle, we obtain that u is a solution to the equation if and only if u is a fixed point of the map Φ given by

$$\Phi(u) = e^{t\mathcal{L}_1}u_0 + \int_0^t e^{(t-s)\mathcal{L}_1}(W(u(s)))ds,$$

where $W(u, v) = -P(1 - \mathcal{L}_2)^{-1}\operatorname{div}(u(s) \otimes (1 - \mathcal{L}_2)v(s))$. To simplify notation, we will also set $W(u, u) = W(u)$. Our goal is to show that Φ is a contraction in the space

$$X_{T,M} = \Big\{f \in BC([0, T) : B^{s_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}_{a;s_2,p,q}\text{and}$$

$$\sup_t \|f(t) - e^{t\mathcal{L}_1} u_0\|_{B^{s_1}_{p,q}} + \sup_t t^a \|u(t)\|_{B^{s_2}_{p,q}} < M \Big\},$$

where $a = (s_2 - s_1)/(\gamma_1^-)$, for $0 < T \le 1$ and $M > 0$ to be chosen later.

Following the arguments outlined in [9] and [15], Φ will be a contraction if we can show that

$$\begin{gathered}
\sup_t t^a \|e^{t\mathcal{L}_1} u_0\|_{B^{s_2}_{p,q}} < M/3, \\
\sup_t \| \int_0^t e^{(t-s)\mathcal{L}_1} W(u(s)) ds\|_{B^{s_1}_{p,q}} < M/3, \\
\sup_t t^a \| \int_0^t e^{(t-s)\mathcal{L}_1} W(u(s)) ds\|_{B^{s_2}_{p,q}} < M/3,
\end{gathered} \tag{3.2}$$

for $u \in X_{T,M}$.

For the first of these terms, we let φ be in the Schwartz space. Then using Proposition 6.1 and Proposition 6.9 we have

$$\begin{aligned}
&\sup_t t^a \|e^{t\mathcal{L}_1}(u_0 - \varphi + \varphi)\|_{B^{s_2}_{p,q}} \\
&\le \sup_t t^a \|e^{t\mathcal{L}_1}(u_0 - \varphi)\|_{B^{s_2}_{p,q}} + \sup_t t^a \|e^{t\mathcal{L}_1}\varphi\|_{B^{s_2}_{p,q}} \\
&\le \sup_t t^a t^{-a} \|u_0 - \varphi\|_{B^{s_1}_{p,q}} + \sup_t t^a \|\varphi\|_{B^{s_2}_{p,q}} \\
&\le \|u_0 - \varphi\|_{B^{s_1}_{p,q}} + T^a \|\varphi\|_{B^{s_2}_{p,q}}.
\end{aligned}$$

Since the Schwartz space is dense in $B^{s_1}_{p,q}(\mathbb{R}^n)$, we can choose φ so that the first term is arbitrarily small. Then we choose T to be small enough so that the sum is bounded by $M/3$.

Turning to the second inequality, applying Minkowski's inequality and Proposition 6.9, we have

$$\begin{aligned}
&\sup_t \| \int_0^t e^{(t-s)\mathcal{L}_1} W(u(s)) ds\|_{B^{s_1}_{p,q}} \\
&\le \sup_t \int_0^t \|e^{(t-s)\mathcal{L}_1} W(u(s))\|_{B^{s_1}_{p,q}} ds \\
&\le \sup_t \int_0^t |t-s|^{-(s_1 - r + n/p^* - n/p)/(\gamma_1^-)} \|W(u(s))\|_{B^r_{p^*,q}} ds,
\end{aligned} \tag{3.3}$$

where $p^* \le p$ will be specified later. Using Proposition 5.3, then Proposition 2.2, and finally Propositions 2.1 and 5.2, we have

$$\begin{aligned}
\|W(u(s))\|_{B^r_{p^*,q}} &\le C\|u \otimes (1 - \mathcal{L}_2)u\|_{B^{r+1-\gamma_2^-}_{p^*,q}} \\
&\le C\|u\|_{L^{p_1}} \|(1-\mathcal{L}_2)u\|_{B^{r+1-\gamma_2^-}_{p_2,q}} + C\|u\|_{B^{r+1-\gamma_2^-}_{q_1,q}} \|(1-\mathcal{L}_2)u\|_{L^{q_2}} \\
&\le C\|u\|_{L^{p_1}} \|u\|_{B^{r+1+\gamma_2-\gamma_2^-}_{p_2,q}} + C\|u\|_{B^{r+1-\gamma_2^-}_{q_1,q}} \|(1-\mathcal{L}_2)u\|_{B^{0+}_{q_2,q}} \\
&\le C\|u\|_{L^{p_1}} \|u\|_{B^{r+1+\gamma_2-\gamma_2^-}_{p_2,q}} + C\|u\|_{B^{r+1-\gamma_2^-}_{q_1,q}} \|u\|_{B^{\gamma_2^+}_{q_2,q}},
\end{aligned} \tag{3.4}$$

where $1/p^* = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$, provided $r + 1 - \gamma_2^- > 0$. To complete the argument, we need to bound this by $\|u\|^2_{B^{s_2}_{p,q}}$. To facilitate this, we define ε by setting $\gamma_2 - \gamma_2^- = \varepsilon$, choose $r + 1 + \varepsilon = s_2$ (which forces $s_2 > \gamma_2$), and $q_2 = p_2 = p$

(which forces $p_1 = q_1$). Applying these choices and using the Besov embedding (2.6a), inequality (3.4) becomes

$$\begin{aligned} \|W(u(s))\|_{B^r_{p^*,q}} &\le C\|u\|_{L^{p_1}}\|u\|_{B^{s_2}_{p_2,q}} + C\|u\|_{B^{s_2-\gamma_2}_{q_1,q}}\|u\|_{B^{\gamma_2^+}_{q_2,q}} \\ &\le C\|u\|_{B^{s_2}_{p,q}}\big(\|u\|_{L^{p_1}} + \|u\|_{B^{s_2-\gamma_2}_{p_1,q}}\big). \end{aligned} \tag{3.5}$$

Finally, choosing $p_1 = np/(n-kp)$ for some $k < n/p$ and $k \le \gamma_2 < s_2$, we use Proposition 2.1 to obtain

$$\begin{aligned} \|W(u(s))\|_{B^r_{p^*,q}} &\le C\|u\|_{B^{s_2}_{p,q}}\big(\|u\|_{L^{p_1}} + \|u\|_{B^{s_2-\gamma_2}_{p_1,q}}\big) \\ &leqC\|u\|_{B^{s_2}_{p,q}}\big(\|u\|_{B^{k+}_{p,q}} + \|u\|_{B^{s_2-\gamma_2+k}_{p,q}}\big) \le C\|u\|^2_{B^{s_2}_{p,q}}, \end{aligned} \tag{3.6}$$

provided

$$\begin{gathered} 1/p^* - 1/p = 1/p_1 = (n-kp)/np, \\ s_2 > \gamma_2, \quad s_2 = r + 1 + \varepsilon, \\ k \le \gamma_2, \quad kp < n. \end{gathered} \tag{3.7}$$

Returning to the estimate begun in (3.3), using (3.6), we have

$$\begin{aligned} &\sup_t \|\int_0^t e^{(t-s)\mathcal{L}_1} W(u(s))ds\|_{B^{s_1}_{p,q}} \\ &\le C\sup_t \int_0^t |t-s|^{-(s_1-r+n/p^*-n/p)/(\gamma_1^-)} s^{-2a} s^{2a}\|u(s)\|^2_{B^{s_2}_{p,q}} ds \\ &\le C\sup_t \|u\|^2_{a;s_2,p,q} t^{-(s_1-r+n/p^*-n/p)/(\gamma_1^-)-2(s_2-s_1)/(\gamma_1^-)+1} \\ &\le CM^2 T^{-(s_1-r+n/p^*-n/p)/(\gamma_1^-)-2(s_2-s_1)/(\gamma_1^-)+1} < M/3, \end{aligned}$$

provided

$$\begin{gathered} 0 \le (s_1 - r + n/p^* - n/p)/(\gamma_1^-) < 1 \\ 1 > 2(s_2 - s_1)/(\gamma_1^-) \\ 0 \le -(s_1 - r + n/p^* - n/p)/(\gamma_1^-) - 2(s_2 - s_1)/(\gamma_1^-) + 1. \end{gathered} \tag{3.8}$$

The first inequality in this list ensures that the $|t-s|$ term is integrable as s goes to t, the second inequality does the same for the s^{-2a} term as s goes to 0, and the last inequality makes the power on the post-integration t positive. The last inequality follows by recalling that $T \le 1$ and by choosing a sufficiently small M.

For the last term in (3.2), a similar argument gives

$$\begin{aligned} &\sup_t t^a\|\int_0^t e^{(t-s)\mathcal{L}_1} W(u(s))ds\|_{B^{s_2}_{p,q}} \\ &\le \sup_t t^a \int_0^t |t-s|^{-(s_2-r+n/p^*-n/p)/(\gamma_1^-)}\|W(u(s))\|_{B^r_{p^*,q}} ds \\ &\le \sup_t t^a \int_0^t |t-s|^{-(s_2-r+n/p^*-n/p)/(\gamma_1^-)} s^{-2a} s^{2a}\|u(s)\|^2_{B^{s_2}_{p,q}} ds \\ &\le C\|u\|^2_{a;s_2,p,q} \sup_t t^a t^{-(s_2-r+n/p^*-n/p)/(\gamma_1^-)-2(s_2-s_1)/(\gamma_1^-)+1} \\ &\le CM^2 T^{-(s_2-r)/(\gamma_1^-)-(s_2-s_1)/(\gamma_1^-)+1} < M/3, \end{aligned}$$

provided

$$\begin{gathered} 0 \leq (s_2 - r + n/p^* - n/p)/(\gamma_1^-) < 1, \\ 1 > 2(s_2 - s_1)/(\gamma_1^-), \\ 0 \leq -(s_2 - r + n/p^* - n/p)/(\gamma_1^-) - (s_2 - s_1)/(\gamma_1^-) + 1. \end{gathered} \tag{3.9}$$

Combining (3.8), and (3.9) (and removing redundancies) gives

$$\begin{gathered} s_1 > r, \\ (\gamma_1^-)/2 > s_2 - s_1, \\ 0 \leq s_2 - r + n/p^* - n/p < (\gamma_1^-), \\ (\gamma_1^-) \geq 2s_2 - r + n/p^* - n/p - s_1. \end{gathered} \tag{3.10}$$

Incorporating (3.7), and observing that, since $s_2 > s_1$, the last inequality in (3.10) implies the third inequality, we obtain

$$\begin{gathered} s_2 > \gamma_2 \geq k, \\ kp < n, \\ s_2 - s_1 < \min\{(\gamma_1^-)/2, 1\}, \\ \gamma_1^- \geq s_2 - s_1 + 1 + n/p + \varepsilon - k. \end{gathered} \tag{3.11}$$

This completes Theorem 3.1. Replacing γ_1^- with γ_1 and setting $\varepsilon = 0$ recovers the result for the case where the g_i are Mikhlin multipliers. In that case, note that for $\gamma_1 = 2$ and $\gamma_2 = 0$, this recovers, up to a slight modification in the argument, the result from [9] for the Navier-Stokes equation.

In comparison with the existence result for the next section, this existence result requires a larger initial regularity, but imposes no restrictions on the value of γ_2 (beyond the requirement that $\gamma_2 > 0$). To get Theorem 1.1 or Theorem 1.3, choose k to be an arbitrarily small positive number, which removes k from the last inequality and forces $\gamma_2 > 0$.

4. Local existence using Proposition 2.3

In this section we prove the following local existence result, which implies Theorem 1.2. We address Theorem 1.4 at the end of the section.

Theorem 4.1. *Let $\gamma_1 > 1$, $\gamma_2 > 0$, $q \geq 1$, $p \geq 2$, and assume g_1 and g_2 satisfy the Mikhlin condition (see inequality* (5.1)*). Let $u_0 \in B^{s_1}_{p,q}(\mathbb{R}^n)$ be divergence-free. Then there exists a unique local solution u to the generalized Leray-alpha equation* (1.2)*, with*

$$u \in BC([0,T) : B^{s_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s_2,p,q},$$

where $a = (s_2 - s_1)/\gamma_1$, if there exists r, r_1 and r_2 such that all the parameters satisfy (4.12)*. T can be chosen to be a non-increasing function of $\|u_0\|_{B^{s_1}_{p,q}}$ with $T = \infty$ if $\|u_0\|_{B^{s_1}_{p,q}}$ is sufficiently small.*

With the same set-up as the previous section, our goal is to show that

$$\begin{gathered}
\sup_t t^a \|e^{t\mathcal{L}_1} u_0\|_{B^{s_2}_{p,q}} < M/3, \\
\sup_t \| \int_0^t e^{(t-s)\mathcal{L}_1} W(u(s)) ds \|_{B^{s_1}_{p,q}} < M/3, \\
\sup_t t^a \| \int_0^t e^{(t-s)\mathcal{L}_1} W(u(s)) ds \|_{B^{s_2}_{p,q}} < M/3.
\end{gathered} \tag{4.1}$$

The first inequality follows exactly as it did in the previous section, and for the second, using Minkowski's inequality and Proposition 6.8, we have

$$\begin{aligned}
&\sup_t \| \int_0^t e^{(t-s)\mathcal{L}_1} W(u(s)) ds \|_{B^{s_1}_{p,q}} \\
&\le \sup_t \int_0^t \|e^{(t-s)\mathcal{L}_1} W(u(s))\|_{B^{s_1}_{p,q}} ds \\
&\le \sup_t \int_0^t |t-s|^{-(s_1-r)/\gamma_1} \|W(u(s))\|_{B^r_{p,q}} ds,
\end{aligned} \tag{4.2}$$

where $r \le s_1$ and will be specified later. Using Proposition 2.3, then Proposition 5.3, and finally Proposition 5.2, we have

$$\begin{aligned}
\|W(u(s))\|_{B^r_{p,q}} &\le \|u \otimes (1-\mathcal{L}_2)u\|_{B^{r+1-\gamma_2}_{p,q}} \\
&\le \|u\|_{B^{r_1}_{p,q}} \|(1-\mathcal{L}_2)u\|_{B^{r_2}_{p,q}} \\
&\le \|u\|_{B^{r_1}_{p,q}} \|u\|_{B^{r_2+\gamma_2}_{p,q}} \le \|u\|^2_{B^{s_2}_{p,q}},
\end{aligned} \tag{4.3}$$

provided

$$\begin{gathered}
r + 1 - \gamma_2 \le r_1 + r_2 - n/p, \\
r_1 + r_2 > 0, \\
r_1, r_2 < n/p, \\
s_2 \ge \max\{r_1, r_2 + \gamma_2\}.
\end{gathered} \tag{4.4}$$

Returning to equation (4.2), we have

$$\begin{aligned}
&\sup_t \| \int_0^t e^{(t-s)\mathcal{L}_1} W(u(s)) ds \|_{B^{s_1}_{p,q}} \\
&\le \sup_t \int_0^t |t-s|^{-(s_1-r)/\gamma_1} s^{-2a} s^{2a} \|u(s)\|^2_{B^{s_2}_{p,q}} ds \\
&\le C \sup_t \|u\|^2_{a;s_2,p,q} t^{-(s_1-r)/\gamma_1 - 2(s_2-s_1)/\gamma_1 + 1} \\
&\le C M^2 T^{-(s_1-r)/\gamma_1 - 2(s_2-s_1)/\gamma_1 + 1} < M/3,
\end{aligned} \tag{4.5}$$

provided

$$\begin{gathered}
0 \le (s_1 - r)/\gamma_1 < 1, \\
1 > 2(s_2 - s_1)/\gamma_1, \\
0 \le -(s_1 - r)/\gamma_1 - 2(s_2 - s_1)/\gamma_1 + 1.
\end{gathered} \tag{4.6}$$

Estimating the last term of (3.2) in a similar fashion, we have

$$
\begin{aligned}
&\sup_t t^a \|\int_0^t e^{(t-s)\mathcal{L}_1} W(u(s))ds\|_{B^{s_2}_{p,q}} \\
&\le \sup_t t^a \int_0^t |t-s|^{-(s_2-r)/\gamma_1} \|W(u(s))\|_{B^r_{p,q}} ds \\
&\le \sup_t t^a \int_0^t |t-s|^{-(s_2-r)/\gamma_1} s^{-2a} s^{2a} \|u(s)\|^2_{B^{s_2}_{p,q}} ds \\
&\le C\|u\|^2_{a;s_2,p,q} \sup_t t^a t^{-(s_2-r)/\gamma_1 - 2(s_2-s_1)/\gamma_1 + 1} \\
&\le CM^2 T^{-(s_2-r)/\gamma_1 - (s_2-s_1)/\gamma_1 + 1} < M/3,
\end{aligned}
\tag{4.7}
$$

provided

$$
\begin{gathered}
0 \le (s_2 - r)/\gamma_1 < 1, \\
1 > 2(s_2 - s_1)/\gamma_1, \\
0 \le -(s_2 - r)/\gamma_1 - (s_2 - s_1)/\gamma_1 + 1.
\end{gathered}
\tag{4.8}
$$

Our final task is to unify the conditions on the parameters. The sets of inequalities from equations (4.6) and (4.8) can be simplified to

$$
\begin{gathered}
0 < s_2 - s_1 < \gamma_1/2, \\
s_1 \ge r > s_2 - \gamma_1, \\
\gamma_1 \ge (s_2 - s_1) + (s_2 - r).
\end{gathered}
\tag{4.9}
$$

Incorporating the inequalities from (4.4), we have

$$
\begin{gathered}
0 < s_2 - s_1 < \gamma_1/2, \\
s_1 \ge r > s_2 - \gamma_1, \\
\gamma_1 \ge (s_2 - s_1) + (s_2 - r), \\
r + 1 - \gamma_2 \le r_1 + r_2 - n/p, \\
r_1 + r_2 > 0, \\
r_1, r_2 < n/p, \\
s_2 \ge \max\{r_1, r_2 + \gamma_2\},
\end{gathered}
\tag{4.10}
$$

and this completes the proof of Theorem 4.1. To obtain the results in Theorem 1.2, we fix the values of the parameters r_1, r_2, and r in the following way. First, since our primary interest is in minimizing s_1 and s_2, we see from the last inequality that this is helped by minimizing $\max\{r_1, r_2+\gamma_2\}$, subject to the constraints $r_1+r_2 > 0$ and $r_1, r_2 < n/p$. This is accomplished by choosing $r_2 = -\gamma_2/2$ and $r_1 = \gamma_2/2+R$, where R is some positive number. Choosing the fourth inequality in the list (4.10) to be an equality, the list (4.10) becomes

$$
\begin{gathered}
0 < s_2 - s_1 < \gamma_1/2, \\
s_1 \ge r > s_2 - \gamma_1, \\
\gamma_1 \ge (s_2 - s_1) + (s_2 - r), \\
r = -1 + \gamma_2 + R - n/p, \\
n/p > \gamma_2/2 + R, \\
s_2 \ge \gamma_2/2 + R.
\end{gathered}
\tag{4.11}
$$

Using the fourth line to eliminate r from the other inequalities, and then removing extraneous inequalities, we finally get

$$\begin{gathered} 0 < s_2 - s_1 < \gamma_1/2, \\ s_1 \geq \gamma_2 + R - n/p - 1, \\ \gamma_1 \geq 2s_2 - s_1 - \gamma_2 - R + n/p + 1, \\ n/p > \gamma_2/2 + R, \\ s_2 \geq \gamma_2/2 + R. \end{gathered} \tag{4.12}$$

Eliminating the free parameter R weakens this to

$$\begin{gathered} 0 < s_2 - s_1 < \gamma_1/2, \\ s_1 > \gamma_2 - n/p - 1, \\ \gamma_1 \geq 2s_2 - s_1 - \gamma_2 + n/p + 1, \\ n/p > \gamma_2/2, \\ s_2 \geq \gamma_2/2, \end{gathered} \tag{4.13}$$

which finishes Theorem 1.2. The analog of Theorem 1.2 for logarithmic g is obtained by replacing γ_1 and γ_2 with γ_1^- and γ_2^-.

5. Operator estimates for $\mathcal{L}_\gamma^g$

In this section, we define the Fourier multiplier $\mathcal{L}_\gamma^g$ by

$$\mathcal{L}_\gamma^g u(x) = \int -\frac{|\xi|^\gamma}{g(|\xi|)} \hat{u}(\xi) e^{ix\cdot\xi} d\xi,$$

where $\gamma \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is radial, nondecreasing, and bounded below by 1. Note that if we define G to be the Fourier multiplier with symbol $1/g$, then $L_\gamma^g = -G(-\Delta)^{\gamma/2}$. The goal here is to prove operator estimates for $\mathcal{L}_\gamma^g$, and we begin by stating the Mikhlin multiplier theorem, which will be referenced often in this section.

Theorem 5.1 (Mikhlin multiplier theorem)**.** *Let M be an operator with symbol $m : \mathbb{R}^n \to \mathbb{R}^n$. If $|x|^k |\nabla^k m(x)|$ is bounded for all $0 \leq k \leq n/2 + 1$, then M is an $L^p(\mathbb{R}^n)$ multiplier for all $1 < p < \infty$.*

The multipliers we are working with will be radial. In this context, the Mikhlin conditions is

$$|m^{(k)}(r)| \leq C r^{-k}, \tag{5.1}$$

for $0 \leq k \leq n/2 + 1$. Now we are ready to prove our fist result.

Proposition 5.2. *Let $1 < p < \infty$ and let $|g^{(k)}(r)| \leq C r^{-k}$ for $1 \leq k \leq n/2 + 1$. Then $\mathcal{L}_\gamma^g : B_{p,q}^{s_1+\gamma}(\mathbb{R}^n) \to B_{p,q}^{s_1}(\mathbb{R}^n)$, with*

$$\|\mathcal{L}_\gamma^g f\|_{B_{p,q}^{s_1}} \leq C \|f\|_{B_{p,q}^{s_1+\gamma}}.$$

Proof. Without loss of generality, we assume $s_1 = 0$. Then we have

$$\begin{aligned} \|(1 - \mathcal{L}_\gamma^g)^{-1} f\|_{B_{p,q}^0} &= \|(1 - \mathcal{L}_\gamma^g)^{-1} (1 - \Delta)^{\gamma/2} (1 - \Delta)^{-\gamma/2} f\|_{B_{p,q}^0} \\ &= \|(1 - \Delta)^{\gamma/2} (1 - \mathcal{L}_\gamma^g)^{-1} f\|_{B_{p,q}^{-\gamma}}. \end{aligned}$$

We finish the proof by showing that the operator $(1 - \Delta)^\gamma (1 - \mathcal{L}_\gamma^g)^{-1}$, with symbol $\frac{(1+r^2)^{\gamma/2}}{(1+r^\gamma/g(r))}$, is a Mikhlin multiplier. Note that the symbol can be written as

$\frac{g(r)(1+r^2)^{\gamma/2}}{(g(r)+r^\gamma)}$, and since g is already known to be a Mikhlin multiplier, this is equivalent to showing that the symbol $\frac{(1+r^2)^{\gamma/2}}{(g(r)+r^\gamma)}$ satisfies the Mikhlin condition, which follows from a straightforward (though lengthy) computation. □

If g is a logarithm, then g is not a Mikhlin multiplier, since g is not bounded. However, g would satisfy $g(r) \le C(1+r)^\delta$ for any $\delta > 0$ and $|g^{(k)}(r)| \le Cr^{-k}$ for all $1 \le k$. These observations inform the next proposition.

Proposition 5.3. *Let $1 < p < \infty$, let $g(r) \le C(1+r)^\delta$ for any $\delta > 0$, and assume $|g^{(k)}(r)| \le Cr^{-k}$ for all $1 \le k \le n/2+1$. Then $(1-\mathcal{L}_\gamma^g)^{-1} : B_{p,q}^{s_1-(\gamma-\varepsilon)}(\mathbb{R}^n) \to B_{p,q}^{s_1}(\mathbb{R}^n)$ for any small $\varepsilon > 0$, with*

$$\|(1-\mathcal{L}_\gamma^g)^{-1} f\|_{B_{p,q}^{s_1}} \le C\|f\|_{B_{p,q}^{s_1-(\gamma-\varepsilon)}}.$$

Proof. As in the previous proposition, this follows by showing that the symbol $\frac{(1+r^2)^{(\gamma-\varepsilon)/2}}{1+r^\gamma/g(r)}$ is a Mikhlin multiplier. First, we re-write this as $\frac{g(r)}{(1+r^2)^{\varepsilon/2}}\frac{(1+r^2)^{\gamma/2}}{g(r)+r^\gamma}$. That each of these terms individually satisfies the Mikhlin condition follows directly from the assumptions on g. □

We remark that the ε loss between these two results is due to the necessity of controlling the growth of the $g(r)$ term.

6. Operator estimates for $e^{t\mathcal{L}}$

As in the previous section, we define the Fourier multiplier $\mathcal{L}_\gamma^g$ by

$$\mathcal{L}_\gamma^g u(x) = \int -\frac{|\xi|^\gamma}{g(|\xi|)}\hat{u}(\xi)e^{ix\cdot\xi}d\xi,$$

where $\gamma \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing and bounded below by 1. We define the operator $e^{t\mathcal{L}_\gamma^g}$ to be the Fourier multiplier with symbol $e^{-t|\xi|^\gamma/g(|\xi|)}$. The goal of this section is to establish operator bounds for $e^{t\mathcal{L}_\gamma^g}$ in the case where $\gamma > 1$, and following the general outline of the same task for $e^{t\Delta}$, we need to first establish $L^p - L^q$ boundedness for the operator. Also note that, throughout the section, we will assume $0 < t < 1$.

We start with the special case of $p = q$.

Proposition 6.1. *Let $1 < p < \infty$ and $\gamma > 1$. Assume $|g(r)| \le C(1+r)^\delta$ for any $\delta > 0$ and assume*

$$|g^{(k)}(r)| \le Cr^{-k} \tag{6.1}$$

holds for $1 \le k \le n/2+1$. Then

$$e^{t\mathcal{L}_\gamma^g} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

and we have the bound

$$\|e^{t\mathcal{L}_\gamma^g} f\|_{L^p} \le C\|f\|_{L^p}. \tag{6.2}$$

Proof. We will show that $e^{-r^\gamma/g(r)}$ satisfies the Mikhlin condition, and then the result follows by the Mikhlin multiplier theorem. First, we observe that the multiplier is clearly bounded. Then

$$\Big|\frac{d}{dr}e^{-r^\gamma/g(r)}\Big| \le C\Big(\frac{r^{\gamma-1}}{g(r)} + \frac{r^\gamma g'(r)}{g(r)^2}\Big)e^{-r^\gamma/g(r)} \le Cr^{\gamma-1}e^{-Cr^{\gamma-\delta}} \le Cr^{-1},$$

where in the last inequality we used that $\gamma > 1$ and we have chosen δ to be a small positive number. Similar calculations hold for the remaining derivatives. $\square$

Now we consider the case where $p \neq q$.

Proposition 6.2. *Let $1 \leq p < q \leq \infty$, and assume $\gamma > 1$. Then*

$$e^{t\mathcal{L}_\gamma^g} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n),$$

and we have the bound

$$\|e^{t\mathcal{L}_\gamma^g} f\|_{L^q} \leq Ct^{-(n/p-n/q)/\gamma} \|f\|_{L^p}, \tag{6.3}$$

provided

$$\sup_{t\in(0,1)} \int_0^\infty r^{n-1} |\partial_r^{(n)} (e^{-r^\gamma/g(rt^{-1/\gamma})})| dr < \infty. \tag{6.4}$$

Proof. For notational convenience, we will suppress the subscript γ and the superscript g on the operator for the duration of the proof. Setting $e^{t\mathcal{L}} f = e^{t\mathcal{L}}\delta * f$, where the Fourier Transform of $e^{t\mathcal{L}}\delta(x)$ is equal to $e^{-t|\xi|^\gamma/g(|x|)}$, and applying Young's inequality, we obtain that

$$\|e^{t\mathcal{L}} f\|_{L^q} \leq \|e^{t\mathcal{L}}\delta\|_{L^r} \|f\|_{L^p},$$

where $1 + 1/q = 1/r + 1/p$. Formally, we have that

$$e^{t\mathcal{L}}\delta(\xi) = C \int_{\mathbb{R}^n} e^{-t|x|^\gamma/g(|x|)} e^{ix\cdot\xi} dx.$$

Making the variable change $x \to t^{-1/\gamma}x$, we obtain

$$e^{t\mathcal{L}}\delta(\xi) = Ct^{-n/\gamma} \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} e^{it^{-1/\gamma}x\cdot\xi} dx.$$

Taking the $L^r(\mathbb{R}^n)$ norm gives

$$\|e^{t\mathcal{L}}\delta\|_{L^r} = Ct^{-n/\gamma} \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} e^{it^{-1/\gamma}x\cdot\xi} dx \Big|^r d\xi \Big)^{1/r}.$$

Making the variable change $\xi \to t^{-1/\gamma}\xi$, this finally becomes

$$\|e^{t\mathcal{L}}\delta\|_{L^r} = Ct^{-n/\gamma+n/(r\gamma)} \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} e^{ix\cdot\xi} dx \Big|^r d\xi \Big)^{1/r}.$$

Since $1 - 1/r = 1/p - 1/q$, it only remains to obtain a t-independent bound for the integral. First, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} e^{ix\cdot\xi} dx \Big|^r d\xi \\
&\leq \int_{|\xi|<1} \Big| \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} e^{ix\cdot\xi} dx \Big|^r d\xi \\
&\quad + \int_{|\xi|>1} \Big| \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} e^{ix\cdot\xi} dx \Big|^r d\xi \\
&\leq C \int_{|\xi|>1} \Big| \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} dx \Big|^r \\
&\quad + \int_{|\xi|>1} \Big| \int_{\mathbb{R}^n} e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)} \frac{\partial_{x_1}\partial_{x_2}\dots\partial_{x_n}}{\xi_1\xi_2\dots\xi_n} e^{ix\cdot\xi} dx \Big|^r d\xi
\end{aligned}$$

$$
\begin{aligned}
&\le C + \int_{|\xi|>1} \frac{1}{|\xi_1|^r|\xi_2|^r\dots|\xi_n|^r}\Big|\int_{\mathbb{R}^n}\partial_{x_1}\partial_{x_2}\dots\partial_{x_n}(e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)})e^{ix\cdot\xi}dx\Big|^r d\xi\\
&\le C + C\Big(\int_{\mathbb{R}^n}\Big|\partial_{x_1}\partial_{x_2}\dots\partial_{x_n}(e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)})\Big|dx\Big)^r,
\end{aligned}
$$

where the last inequality follows from observing that since $p < q$, $r > 1$, and therefore the integral in each ξ coordinate is finite.

To compute the final integral, we convert to polar coordinates, and remark that since $e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)}$ is radial, we can disregard the radial portion of each coordinate derivative's expression in polar form, and we have

$$
\int_{\mathbb{R}^n}\Big|\partial_{x_1}\partial_{x_2}\dots\partial_{x_n}(e^{-|x|^\gamma/g(t^{-1/\gamma}|x|)})\Big|dx \le C\int_0^\infty r^{n-1}\Big|\partial_r^{(n)}(e^{-r^\gamma/g(rt^{-1/\gamma})})\Big|dr,
$$

which is (6.4), and thus completes the proposition. □

Our next task is to establish sufficiency criteria for (6.4).

Proposition 6.3. *If*

$$
|g^{(k)}(r)| \le Cr^{-k}, \tag{6.5}
$$

for all $0 \le k \le n$, *then* (6.4) *holds.*

Proof. We begin with the special case where g is a constant function, and without loss of generality assume $g(r) = 1$. Then (6.4) becomes

$$
\int_0^\infty r^{n-1}\big|\partial_r^{(n)}(e^{-r^\gamma})\big|\,dr.
$$

Computing the derivatives through repeated use of the product rule will yield the addition of 2^{n-1} terms of the form $r^\alpha e^{-r^\gamma}$. The decay provided by the exponential term makes each of these integrable for large r. For the region where r is small, the most singular term is of the form $r^{n-1}r^{n(\gamma-n)}e^{-r^\gamma}$, which is integrable for small r because $\gamma > 1$, and this finishes the argument for the special case where g is constant.

If $\gamma > 1$ and g is not constant, the proof is similar, but more combinatorially intense. To see this, we first consider the case where $n = 1$. Then we have

$$
\begin{aligned}
&\int_0^\infty \big|\partial_r(e^{-r^\gamma/g(rt^{-1/\gamma})})\big|\,dr\\
&= \int_0^\infty\Big(\frac{Cr^{\gamma-1}}{g(rt^{-1/\gamma})} + C\frac{r^\gamma g'(rt^{-1/\gamma})t^{-1/\gamma}}{(g(rt^{-1/\gamma}))^2}\Big)e^{-r^\gamma/g(rt^{-1/\gamma})}dr.
\end{aligned}
$$

Using our assumption on g, we have

$$
\begin{aligned}
&\int_0^\infty\Big(\frac{Cr^{\gamma-1}}{g(rt^{-1/\gamma})} + C\frac{r^\gamma g'(rt^{-1/\gamma})t^{-1/\gamma}}{(g(rt^{-1/\gamma}))^2}\Big)e^{-r^\gamma/g(rt^{-1/\gamma})}dr.\\
&\le C\int_0^\infty\Big(\frac{Cr^{\gamma-1}}{g(rt^{-1/\gamma})} + C\frac{r^\gamma t^{-1/\gamma}}{rt^{-1/\gamma}(g(rt^{-1/\gamma}))^2}\Big)e^{-r^\gamma/g(rt^{-1/\gamma})}dr.
\end{aligned}
$$

Finally, since g is bounded below by one and is bounded above by assumption, we have

$$
\int_0^\infty\Big(\frac{Cr^{\gamma-1}}{g(rt^{-1/\gamma})} + C\frac{r^\gamma t^{-1/\gamma}}{rt^{-1/\gamma}(g(rt^{-1/\gamma}))^2}\Big)e^{-r^\gamma/g(rt^{-1/\gamma})}dr \le C\int_0^\infty r^{\gamma-1}e^{-Cr^\gamma}dr,
$$

which reduces the problem to the constant g case. In general, when computing the derivatives, either the derivatives fall on the r^γ term (and then since g is bounded below, we are back in constant g case) or the derivatives act on the $g(rt^{-1/\gamma})$ term. In that case, because of (6.5), the derivatives of g introduce exactly as much decay in r as occurs when differentiating r^γ, and the t dependent terms cancel out, so this case also reduces to a term from the constant g case. □

Recall from the introduction that we ultimately seek results for g of the form $g(r) = \ln(a + r)$, where $\ln(a) > 1$. Since such a g is unbounded, it does not satisfy the requirements of the previous proposition. The following result adapts that argument to this particular case.

Proposition 6.4. *Let $g(r) \le C(1 + r)^\delta$ for any $\delta > 0$, and assume that*

$$|g^{(k)}(r)| \le C|r|^{-k},$$

for $1 \le k \le n$. Then

$$\|e^{t\mathcal{L}_\gamma^g} f\|_{L^q} \le Ct^{-(n/p-n/q)/(\gamma-\varepsilon)}\|f\|_{L^p}, \tag{6.6}$$

for any small $\varepsilon > 0$, provided $0 < t < 1$.

Before beginning the proof, we note that if $g(r) = \ln(a+r)$, and $\ln(a) \ge 1$, then g satisfies the hypothesis of the proposition.

Proof. First, we observe that when controlling the derivatives, we only required that g be bounded below and that the derivatives of g have sufficient decay. Since that portion of the argument did not require g to be bounded, we can apply that argument here, and we have that

$$\|e^{t\mathcal{L}_\gamma^g} f\|_{L^q} \le Ct^{-(n/p-n/q)/\gamma}\|f\|_{L^p} \int_0^\infty r^\alpha e^{-r^\gamma/g(rt^{-1/\gamma})} dr, \tag{6.7}$$

for some $\alpha > -1$. Since $\alpha > -1$, we have

$$\begin{aligned}
\int_0^\infty r^\alpha e^{-r^\gamma/g(rt^{-1/\gamma})} dr &\le \int_0^1 r^\alpha e^{-r^\gamma/g(rt^{-1/\gamma})} dr + \int_1^\infty r^\alpha e^{-r^\gamma/g(rt^{-1/\gamma})} dr \\
&\le C + \int_1^\infty r^\alpha e^{-r^\gamma/(1+rt^{-1/\gamma})^\delta} dr \\
&\le C + \int_1^\infty r^\alpha e^{-Cr^{\gamma-\delta} t^{\delta/\gamma}} dr,
\end{aligned}$$

where we used the assumption on g and δ is small positive number to be specified later. Making the variable change $r \to rt^{\delta/(\gamma(\gamma-\delta))}$, we finally obtain

$$\int_1^\infty r^\alpha e^{-Cr^\gamma/g(rt^{-1/\gamma})} dr \le C + t^{-\delta\alpha/\gamma(\gamma-\delta)} \int_1^\infty r^\alpha e^{-Cr^{\gamma-\delta}} dr \le C(1 + t^{-\delta\alpha/\gamma(\gamma-\delta)}),$$

provided $\gamma - \delta > 1$. Plugging back into (6.7), we have

$$\|e^{t\mathcal{L}_\gamma^g} f\|_{L^q} \le Ct^{-(n/p-n/q)/\gamma-\delta\alpha/\gamma(\gamma-\delta)}\|f\|_{L^p}.$$

Choosing a sufficiently small δ finishes the proposition. □

Before moving on, we remark that, as in the previous section, adapting to the logarithmic g comes at a cost of additional singularity in the time variable.

Now that we have established $L^p - L^q$ boundedness for $e^{t\mathcal{L}^g_\gamma}$, we turn our attention to incorporating changes in regularity. We will rely heavily on the following, which is [17, Proposition 7.2 Chapter 13].

Proposition 6.5. *Let e^{tA} be a holomorphic semigroup on a Banach space X. Then, for $t > 0$,*

$$\|Ae^{tA}f\|_X \le \frac{C}{t}\|f\|_X,$$

for $0 < t \le 1$.

For our purposes, $A = \mathcal{L}^g_\gamma$ and $X = L^p(\mathbb{R}^n)$. To use this proposition, we need to know that $e^{t\mathcal{L}^g_\gamma}$ is a holomorphic semigroup, and following the proof of [17, Proposition 7.1 Chapter 13], we see that we only need $e^{t\mathcal{L}^g_\gamma}$ to be uniformly bounded from $L^p(\mathbb{R}^n)$ into itself, which is the content of Proposition 6.1. Now we are ready to prove the following result.

Proposition 6.6. *Let $1 < p < \infty$, $s_1 \le s_2$ and assume g satisfies the Mikhlin condition (see inequality (5.1)). Then $e^{t\mathcal{L}^g_\gamma} : B^{s_1}_{p,q}(\mathbb{R}^n) \to B^{s_2}_{p,q}(\mathbb{R}^n)$ and*

$$\|e^{t\mathcal{L}^g_\gamma}f\|_{B^{s_2}_{p,q}} \le t^{-(s_2-s_1)/\gamma}\|f\|_{B^{s_1}_{p,q}}. \tag{6.8}$$

Proof. We first establish this result in the case $s_2 = \gamma$ and $s_1 = 0$. We have

$$\begin{aligned}
\|e^{t\mathcal{L}^g_\gamma}f\|_{B^{\gamma}_{p,q}} &= \|e^{t\mathcal{L}^g_\gamma}(\Psi * f)\|_{L^p} + \Big(\sum_{j=0}^{\infty} 2^{jq\gamma}\|e^{t\mathcal{L}^g_\gamma}\Delta_j f\|^q_{L^p}\Big)^{1/q} \\
&\le C\|\Psi * f\|_{L^p} + \Big(\sum_{j=0}^{\infty} \|(-\Delta)^{\gamma/2}\big(\mathcal{L}^g_\gamma\big)^{-1}\mathcal{L}^g_\gamma e^{t\mathcal{L}^g_\gamma}\Delta_j f\|^q_{L^p}\Big)^{1/q} \\
&\le C\|\Psi * f\|_{L^p} + Ct^{-1}\Big(\sum_{j=0}^{\infty} \|\mathcal{L}^g_\gamma e^{t\mathcal{L}^g_\gamma}\Delta_j f\|^q_{L^p}\Big)^{1/q} \\
&\le Ct^{-1}\|f\|_{B^0_{p,q}},
\end{aligned}$$

where we used (essentially) Proposition 5.3 in the first inequality, Proposition 6.5 in the second, and the fact that $t \le 1$ for the last inequality. Standard interpolation and duality arguments extend this result to the general case of $s_1 \le s_2$. $\square$

We again state the parallel result for the special case where g is, essentially, a logarithm.

Proposition 6.7. *Let $1 < p < \infty$, $s_1 \le s_2$, let $g(r) \le Cr^\varepsilon$ for any $\varepsilon > 0$ and let $|g^{(k)}(r)| \le C|r|^{-k}$ for all $1 \le k \le n/2 + 1$. Then $e^{t\mathcal{L}^g_\gamma} : B^{s_1}_{p,q}(\mathbb{R}^n) \to B^{s_2}_{p,q}(\mathbb{R}^n)$ and*

$$\|e^{t\mathcal{L}^g_\gamma}f\|_{B^{s_2}_{p,q}} \le t^{-(s_2-s_1)/(\gamma-\varepsilon)}\|f\|_{B^{s_1}_{p,q}}, \tag{6.9}$$

for any $\varepsilon > 0$.

Proof. As in the previous proposition, we let $s_2 = \gamma - \varepsilon$ and set $s_1 = 0$, and the rest of the argument will follow as before provided we show that the Fourier multiplier with symbol

$$m(\xi) = \frac{|\xi|^{\gamma-\varepsilon}g(\xi)}{|\xi|^\gamma},$$

and with support in the annulus $|\xi| \ge 1/2$, is bounded on $L^p(\mathbb{R}^n)$. As this follows from the assumptions on g, the proof is complete. $\square$

The following is direct combination of Proposition 6.3 and Proposition 6.6.

Proposition 6.8. *Let $1 < p < \infty$, $s_1 \leq s_2$, $p_1 \leq p_2$ and let g satisfy the Mikhlin condition. Then $e^{t\mathcal{L}_\gamma^g} : B^{s_1}_{p_1,q}(\mathbb{R}^n) \to B^{s_2}_{p_2,q}(\mathbb{R}^n)$ and*

$$\|e^{t\mathcal{L}_\gamma^g} f\|_{B^{s_2}_{p_2,q}} \leq t^{-(s_2-s_1+n/p_1-n/p_2)/\gamma}\|f\|_{B^{s_1}_{p_1,q}}. \tag{6.10}$$

We also record the analogous result for our special case.

Proposition 6.9. *Let $1 < p < \infty$, $s_1 \leq s_2$, $p_1 \leq p_2$, $g(r) \leq Cr^\varepsilon$ for any $\varepsilon > 0$, and let $|g^{(k)}(r)| \leq C|r|^{-k}$ for all $1 \leq k \leq n/2+1$. Then $e^{t\mathcal{L}_\gamma^g} : B^{s_1}_{p_1,q}(\mathbb{R}^n) \to B^{s_2}_{p_2,q}(\mathbb{R}^n)$ and*

$$\|e^{t\mathcal{L}_\gamma^g} f\|_{B^{s_2}_{p_2,q}} \leq t^{-(s_2-s_1+n/p_1-n/p_2)/(\gamma-\varepsilon)}\|f\|_{B^{s_1}_{p_1,q}}, \tag{6.11}$$

for any small $\varepsilon > 0$.

We remark that these results also apply to Sobolev spaces (see [14, Section 2] for an example of a similar process applied to the standard heat kernel).

7. Higher regularity for the local existence result

As was mentioned in the introduction, the solutions to the generalized Leray-alpha equations constructed here are smooth for all $t > 0$ at which the solution exists. In this section we prove that the solutions to Theorem 1.1 have this additional regularity and quantify the blow-up that occurs in these higher regularity norms as $t \to 0$. We use an induction argument inspired by the results in [8] for the Navier-Stokes equation. We remark that similar results can be proven for the other theorems in this paper, but require different (and in some cases much more involved) arguments.

Proposition 7.1. *Let $u_0 \in B^{s_1}_{p,q}(\mathbb{R}^n)$ be divergence-free. Let u be a solution to the generalized Leray-alpha equation (1.2) given by Theorem 1.1. Then for all $r \geq s_1$ we have that $u \in \dot{C}^T_{(r-s_1)/\gamma_1;r,p,q}$.*

Before starting the proof, recall from Theorem 1.1 that $s_1 > 0$ and that

$$\begin{gathered} \gamma_1 > 1, \quad \gamma_2 > 0, \quad s_2 > \gamma_2, \\ s_2 - s_1 < \min\{\gamma_1/2, 1\}, \\ \gamma_1 \geq s_2 - s_1 + n/p + 1. \end{gathered} \tag{7.1}$$

Proof. We start with the solution u given by Theorem 1.1. Then let $\delta > 0$ be arbitrary and let $v = t^\delta u$. We note that $v(0) = v_0 = 0$. Then

$$\begin{aligned} \partial_t v &= \delta t^{\delta-1}u + t^\delta \partial_t u \\ &= \delta t^{-1}v + t^\delta P(\mathcal{L}_1 u - (1-\mathcal{L}_2)^{-1}\operatorname{div}(u \otimes (1-\mathcal{L}_2)u)) \\ &= \delta t^{-1}v + \mathcal{L}_1 v - t^{-\delta}P(1-\mathcal{L}_2)^{-1}\operatorname{div}(v \otimes (1-\mathcal{L}_2)v)). \end{aligned}$$

Applying Duhamel's principle, we obtain

$$\begin{aligned} v &= e^{t\mathcal{L}_1}v_0 + \delta\int_0^t e^{(t-s)\mathcal{L}_1}s^{-1}v(s)ds + \int_0^t e^{(t-s)\mathcal{L}_1}s^{-\delta}W(v(s))ds \\ &= \delta\int_0^t e^{(t-s)\mathcal{L}_1}s^{-1}v(s)ds + \int_0^t e^{(t-s)\mathcal{L}_1}s^{-\delta}W(v(s))ds, \end{aligned}$$

where we recall $W(f,g) = -P(1-\mathcal{L}_2)^{-1}\operatorname{div}(f(s)\otimes(1-\mathcal{L}_2)g(s))$ (and for notational convenience set $W(f,f) = W(f)$) and in the last line used that $v_0 = 0$. Using $v = t^\delta u$, we obtain

$$u = \delta t^{-\delta}\int_0^t e^{(t-s)\mathcal{L}_1}s^{\delta-1}u(s)ds + t^{-\delta}\int_0^t e^{(t-s)\mathcal{L}_1}s^\delta W(u(s))ds.$$

The key idea here is that we can choose δ to be large enough to cancel any singularities that occur at $s = 0$. Now we are ready to set up the induction. We have by Theorem 1.1 that the local solution u is in $\dot{C}^T_{(s_2-s_1)/\gamma_1;s_2,p,q}$, where $s_2 > \gamma_2$. For induction, we assume this solution u is also in $\dot{C}^T_{(k-s_1)/\gamma_1;k,p,q}$ for some $k \geq s_2$, and seek to show that u is in $\dot{C}^T_{a^*;k+h,p,q}$, where $a^* = (k+h-s_1)/\gamma_1$ and h is a fixed number between 0 and 1 which will be chosen later.

An application of Proposition 6.8 gives

$$\begin{aligned}
&\|u\|_{B^{k+h}_{p,q}} \\
&\leq t^{-\delta}\Big(\delta\int_0^t \|e^{(t-s)\mathcal{L}_1}s^{\delta-1}u(s)\|_{B^{k+h}_{p,q}}ds + \int_0^t \|e^{(t-s)\mathcal{L}_1}s^\delta W(u(s))\|_{B^{k+h}_{p,q}}ds\Big) \\
&\leq Ct^{-\delta}\delta\int_0^t |t-s|^{-h/\gamma_1}s^{\delta-1}\|u(s)\|_{B^k_{p,q}} \\
&\quad + t^{-\delta}\int_0^t |t-s|^{-b_1/\gamma_1}s^\delta\|W(u(s))\|_{B^{k-1}_{\tilde{p},q}}ds,
\end{aligned} \tag{7.2}$$

where $b_1 = h + 1 + n/\tilde{p} - n/p$.

For the first term in the right hand side of (7.2), we have

$$\begin{aligned}
&t^{-\delta}\int_0^t |t-s|^{-h/\gamma_1}s^{\delta-1}\|u(s)\|_{B^k_{p,q}}ds \\
&\leq t^{-\delta}\|u\|_{(k-s_1)/\gamma_1;k,p,q}\int_0^t |t-s|^{-h/\gamma_1}s^{\delta-1-(k-s_1)/\gamma_1}ds \\
&\leq C\|u\|_{(k-s_1)/\gamma_1;k,p,q}t^{-\delta}t^{-h/\gamma_1}t^{\delta-1-(k-s_1)/\gamma_1+1} \\
&\leq Ct^{-(k+h-s_1)/\gamma_1}\|u\|_{(k-s_1)/\gamma_1;k,p,q}
\end{aligned} \tag{7.3}$$

This calculation implicitly assumes that the exponents of $|t-s|$ and s in the integral are both strictly greater than negative 1. For $|t-s|$, this holds provided $h/\gamma_1 < 1$. For s, it works for a sufficiently large choice of δ. We note that without modifying the PDE to include these t^δ terms, we would need $(k-s_1)/\gamma_1$ to be less than 1, which does not hold for large k.

For the second piece, we start by bounding $\|W(u)\|_{B^{k-1}_{\tilde{p},q}}$. Using Proposition 5.3, Proposition 2.2, and finally Proposition 5.2 and Proposition 2.1, we obtain

$$\begin{aligned}
\|W(u(s))\|_{B^{k-1}_{\tilde{p},q}} &\leq \|u\otimes(1-\mathcal{L}_2)u\|_{B^{k-\gamma_2}_{\tilde{p},q}} \\
&\leq \|u\|_{L^{p_1}}\|(1-\mathcal{L}_2)u\|_{B^{k-\gamma_2}_{p,q}} + \|u\|_{B^{k-\gamma_2}_{p,q}}\|(1-\mathcal{L}_2)u\|_{L^{p_3}} \\
&\leq \|u\|_{B^{r_1^+}_{p,q}}\|u\|_{B^k_{p,q}} + \|u\|_{B^{k-\gamma_2}_{p,q}}\|u\|_{B^{\gamma_2+r_2^+}_{p,q}}
\end{aligned} \tag{7.4}$$

where

$$\begin{gathered}
1/\tilde{p} = 1/p_1 + 1/p = 1/p + 1/p_2, \\
p_i = \frac{np}{n-r_ip}, \quad i = 1,2
\end{gathered} \tag{7.5}$$

and

$$\begin{gathered} r_i < n/p, \quad i = 1, 2, \\ r_1 < s_2 \le k, \\ r_2 < s_2 - \gamma_2 \le k - \gamma_2. \end{gathered} \tag{7.6}$$

Using (7.4) in the last term in (7.2), we obtain

$$t^{-\delta} \int_0^t |t-s|^{-b_1/\gamma_1} s^\delta \|W(u)\|_{B^{k-1}_{\tilde{p},q}} ds \le I_1 + I_2, \tag{7.7}$$

where

$$\begin{aligned} I_1 &= t^{-\delta} \int_0^t |t-s|^{-b_1/\gamma_1} s^\delta \|u(s)\|_{B^{r_1^+}_{p,q}} \|u(s)\|_{B^k_{p,q}} ds, \\ I_2 &= t^{-\delta} \int_0^t |t-s|^{-b_1/\gamma_1} s^\delta \|u(s)\|_{B^{k-\gamma_2}_{p,q}} \|u(s)\|_{B^{\gamma_2+r_2^+}_{p,q}} ds. \end{aligned}$$

Working on I_1, and setting $a_1 = (r_1^+ - s_1)/\gamma_1$ $a_2 = (k - s_1)/\gamma_1$, and recalling that $b_1 = h + 1 + n/\tilde{p} - n/p$, we have

$$\begin{aligned} I_1 &\le Ct^{-\delta} \|u\|_{a_1;r_1^+,p,q} \|u\|_{a_2;k,p,q} \int_0^t |t-s|^{-b_1/\gamma_1} s^{\delta - a_1 - a_2} ds \\ &\le Ct^{-\delta} \|u\|_{a_1;r_1^+p,q} \|u\|_{a_2;k,p,q} t^{-b_1/\gamma_1 + \delta - a_1 - a_2 + 1} \\ &\le Ct^{-(h+1+k+r_1^+ + n/\tilde{p} - n/p - 2s_1)/\gamma_1 + 1} \end{aligned}$$

provided $b_1 < \gamma_1$. For the last inequality, we recall that $\|u\|_{a_2;k,p,q}$ is bounded (by the induction hypothesis) and since $r_1 < k$ (by (7.6)), we have by interpolation that $\|u\|_{a_1;r_1^+,p,q}$ is also bounded. Incorporating the relevant constraints from (7.5) and (7.6) gives

$$\begin{aligned} I_1 &\le Ct^{-(h+1+k+r_1^+ + n/\tilde{p} - n/p - 2s_1)/\gamma_1 + 1} \\ &\le Ct^{-(h+k-s_1)/\gamma_1 - (1+n/p-s_1+\varepsilon)/\gamma_1 + 1}, \end{aligned} \tag{7.8}$$

with the constraint $h + 1 + n/p - r_1^+ < \gamma_1$ and where $\varepsilon = r^+ - r_1$. By the last inequality in (7.1), we know that $\gamma_1 - 1 - n/p > 0$, so the constraint will be satisfied if we choose h to be small enough so that $r_1^+ > h$. Note that $r_1 < \min\{s_2, n/p\}$, so this choice only depends on s_2, n, and p.

Also from (7.1), we have

$$1 - (1 + n/p - s_1 + \varepsilon)/\gamma_1 = 1 - (1 + n/p - s_1 + s_2)/\gamma_1 + (s_2 - \varepsilon)/\gamma_1 \ge s_2/\gamma.$$

Applying this to (7.8), we finally get

$$I_1 \le Ct^{-(h+k-s_1)/\gamma_1 + (s_2 - \varepsilon)/\gamma_1}. \tag{7.9}$$

A similar calculation for I_2 yields

$$\begin{aligned} I_2 &\le C\|u\|_{a_3;k-\gamma_2,p,q} \|u\|_{a_4;\gamma_2+r_2^+,p,q} t^{-(h+k-s_1)/\gamma_1 + s_2/\gamma_1} \\ &\le Ct^{-(h+k-s_1)/\gamma_1 + (s_2-\varepsilon)/\gamma_1}, \end{aligned} \tag{7.10}$$

where we recall that $r_2^+ < s_2 - \gamma_2 < k - \gamma_2$ and we have set

$$a_3 = (k - \gamma_2 + r_2^+ - s_1)/\gamma_1, \quad a_4 = (\gamma_2 + r_3 - s_1)/\gamma_1.$$

This also requires setting $r_2^+ > h$, which now means the choice of h also depends on γ_2.

Using (7.9) and (7.10) in (7.7), we have

$$t^{-\delta}\int_0^t |t-s|^{-b_1/\gamma_1} s^{\delta}\|W(u)\|_{B^{k-1}_{\tilde{p},q}} ds \le Ct^{-(h+k-s_1)/\gamma_1+(s_2-\varepsilon)/\gamma_1}, \tag{7.11}$$

and using (7.3) and (7.11) in (7.2) gives

$$\|u(t)\|_{B^{k+h}_{p,q}} \le C\delta t^{-(k+h-s_1)/\gamma_1}\|u\|_{(k-s_1)/\gamma_1;k,p,q} + Ct^{-(h+k-s_1)/\gamma_1+(s_2-\varepsilon)/\gamma_1},$$

Multiplying both sides by $t^{(k+h-s_1)/\gamma_1}$ and taking the supremem over t, we finally get

$$\|u\|_{(k+h-s_1)/\gamma_1;k+h,p,q} \le C\delta\|u\|_{(k-s_1)/\gamma_1;k,p,q} + C\sup_{t\le T} t^{s_2/\gamma_1},$$

which completes the induction argument. We remark that δ is chosen after beginning the induction step (and thus can be absorbed into the constant), while the appropriate value of h is fixed by the known parameters n, p, s_1, s_2, and γ_2. $\square$

References

[1] D. Barbato, F. Morandin, and M. Romito, *Global regularity for a slightly supercritical hyperdissipative Navier-Stokes system*, arXiv:1407.6734.

[2] D. Chae; *Local existence and blow-up criterion for the Euler equations in the Besov spaces*, Asymptotic Analysis **38** (2004), 339–358.

[3] J. Y. Chemin; *About the navier-stokes system*, Publications du Laboratoire d'analyse numerique (1996).

[4] I. Gallagher, F. Planchon; *On global infinite energy solutions to the Navier-Stokes equations in two dimensions*, Arch. Ration. Mech. Anal. **161** (2002), no. 4, 307–337.

[5] J. D. Gibbon, D. D. Holm; *Estimates for the LANS-α, Leray-α and Bardina models in terms of a Navier-Stokes Reynolds number*, Indiana Univ. Math. J. **57** (2008), no. 6, 2761–2773.

[6] T. Hmidi; *On a maximum principle and its applications to logarithmically critical Boussinesq system*, Analysis and PDE **4** (2001), 247–284.

[7] T. Kato; *Strong L^p-solutions of the Navier Stokes Equation in $\mathbb{R}^m$, with Applications to Weak Solutions*, Mathematische Zeitschrift **187** (1984), 471–480.

[8] T. Kato; *The Navier-Stokes equation for an incompressible fluid in $\mathbf{R}^2$ with a measure as the initial vorticity*, Differential Integral Equations **7** (1994), no. 3-4, 949–966.

[9] T. Kato, G. Ponce; *The Navier-Stokes equation with weak initial data*, Internat. Math. Res. Notices (1994), no. 10, 10 pages (electronic).

[10] H. Koch, D. Tataru; *Well-posedness for the Navier-Stokes equations*, Adv. Math. **157** (2001), no. 1, 22–35.

[11] O. A. Ladyzhenskaya; *The mathematical theory of viscous incompressible flow*, Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2, Gordon and Breach Science Publishers, New York, 1969.

[12] P. G. Lemarie-Rieusset; *Recent developments in the navier-stokes problem*, Chapman and Hall/CRC, 2002.

[13] E. Olson, E. S. Titi; *Viscosity versus vorticity stretching: global well-posedness for a family of Navier–Stokes-alpha-like models*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods **66** (2007), no. 11, 2427–2458.

[14] N. Pennington; *Local and global existence for the Lagrangian Averaged Navier-Stokes equation in Besov spaces*, Electron. J. Diff. Equ. **2012** (2012), no. 89, 1–19.

[15] N. Pennington; *Lagrangian Averaged Navier-Stokes equation with rough data in Sobolev spaces*, Journal of Mathematical Analysis and Applications **403** (2013), 72–88.

[16] T. Tao; *Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation*, Analysis & PDE **2** (2009), no. 3, 361–366.

[17] M. Taylor; *Partial Differential Equations*, Springer-Verlag New York, Inc., 1996.

[18] M. Taylor; *Tools for PDE*, Mathematical Surveys and Monographs, Vol. 81, American Mathematical Society, Providence RI, 2000.

[19] J. Wu; *The generalized MHD equations*, Journal of Differential Equations **195** (2003), 284–312.

[20] J. Wu; *Lower Bounds for an Integral Involving Fractional Laplacians and Generalized Navier-Stokes Equations in Besov Spaces*, Communications in Mathematical Physics **263** (2005), 803–831.

[21] J. Wu; *Global regularity for a class of generalized magnetohydrodynamic equations*, Journal of Mathematical Fluid Mechanics **13** (2011), 295–305.

[22] K. Yamazaki; *On the global regularity of generalized Leray-alpha type models*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods **75** (2012), no. 2, 503–515.

[23] K. Yamazaki; *Global regularity of logarithmically supercritical MHD system with zero diffusivity*, Applied Mathematics Letters **29** (2014), 46–51.

Nathan Pennington

Department of Mathematics, Creighton University, 2500 California Plaza, Omaha, NE 68178, USA

E-mail address: `nathanpennington@creighton.edu`