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EXISTENCE AND BOUNDEDNESS OF SOLUTIONS FOR EVOLUTION VARIATIONAL INEQUALITIES WITH $p(x,t)$-GROWTH

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Abstract. In this article, we study a class of evolution variational inequalities with $p(x,t)$-growth conditions on bounded domains. By means of the penalty method and Galerkin's approximation, we obtain the existence of weak solutions. Moreover, the boundedness of weak solutions is also investigated by applying Moser's iterative method.

1. Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary, $0 < T < \infty$ be given and $Q_T = \Omega \times (0,T)$. Denote

$$\begin{aligned}\mathscr{K} = \big\{ & w \in X(Q_T) \cap C(0,T;L^2(\Omega)),\ \frac{\partial w}{\partial t} \in X'(Q_T) : 0 \leq w(x,0) = u_0(x) \in L^2(\Omega), \\ & w(x,t) \geq 0 \text{ a.e. on } Q_T \big\},\end{aligned}$$

where $X(Q_T)$ is a variable exponent Sobolev space and $X'(Q_T)$ is the dual space of $X(Q_T)$. In this paper, we are concerned with the existence of weak solutions for a class of evolution variational inequality. More precisely, we shall find a function $u(x,t) \in \mathscr{K}$ satisfying the inequality

$$\begin{aligned}&\int_0^T \int_\Omega \frac{\partial u}{\partial t}(v-u) + a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u \nabla(v-u) \\ &+ b(x,t)|u|^{p(x,t)-2}u(v-u)\,dx\,dt \\ &\geq \int_0^T \int_\Omega f(x,t,u)(v-u)\,dx\,dt,\end{aligned} \tag{1.1}$$

for all $v \in X(Q_T)$ with $v \geq 0$ a.e. on Q_T, where the functions a, b, p, f satisfy the following conditions.

(H1) $p : Q_T \to \mathbb{R}^+$ is a global log-Hölder continuous function satisfying

$$\frac{2N}{N+2} < p^- = \inf_{\overline{Q}_T} p(x,t) \leq \sup_{\overline{Q}_T} p(x,t) = p^+ < \infty,$$

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where $\overline{Q}_T$ is the closure of Q_T.

(H2) $a : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^+$ is continuous and there exist constants $a_0,\ a_1 > 0$ such that $0 < a_0 \le a(x, t, \xi) \le a_1 < \infty$ for all $(x, t, \xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$. $b : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $0 < b_0 \le b(x, t) \le b_1 < \infty$ for all $(x, t) \in Q_T$.

(H3) $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(x, t, \eta)| \le C_0|\eta|^{q(x,t)-1} \quad \text{for all } (x, t, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R},$$

where $C_0 > 0$ is a constant and q is a bounded continuous function in Q_T with

$$1 \ll q(x, t) \ll p(x, t).$$

Here $q(x, t) \ll p(x, t)$ means that $\inf_{\overline{Q}_T}(p(x, t) - q(x, t)) > 0$.

Throughout this paper, without further mentioning, we always assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $Q_T = \Omega \times (0, T)$, and the exponent p satisfies (H1).

In recent years there is interest in the study on various mathematical problems with variable exponents growth conditions. $p(\cdot)$-growth problems can be regarded as a kind of nonstandard growth problems and possess very complicated nonlinearities, for instance, the $p(\cdot)$-Laplacian operator $-\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u)$ is inhomogeneous. These problems arise in many applications, for example in nonlinear elastic, electrorheological fluids, imaging processing and other physics phenomena (see [1, 8, 30, 33, 34, 35]). Many results have been obtained on this kind of problems, see for example [10, 13, 14, 15, 16, 26, 29]. Especially, in [2, 3, 11], the authors studied the existence and uniqueness of weak solutions for anisotropy parabolic equations under the framework of variable exponent Sobolev spaces. Motivated by their works, we shall study the existence and boundedness of weak solutions to problem (1.1) with sublinear growth. When the variable exponent depend only on space variable x, evolution variational inequality without initial conditions has been studied in [4, 5, 24]. Concerning the existence of solutions for some interesting problems in the more general spaces, for instance, we refer to [6, 7, 25]. For a recent overview of variable exponent spaces with applications to nonlinear elliptic equations we refer to [28]. For the fundamental theory about variable exponent Lebesgue and Sobolev spaces and their various applications, we refer to [17, 19, 27].

Variational inequalities as the development and extension of classic variational problems, are a very useful tool to research PDEs, optimal control and other fields. In the case that $p \equiv \text{const}$, many papers are devoted to the solvability of the different kinds of parabolic variational inequalities, see [18, 20, 23, 31, 32]. The method is based on a time discretion and the semigroup property of the corresponding differential quotient. Another approach is available via a suitable penalization method. In these works, a crucial assumption on the obstacles is monotonicity or regularity conditions. A new method can be found in [21], where the obstacles are only continuous.

This article is organized as follows. In section 2, we give some necessary definitions and properties of variable exponent Lebesgue spaces and Sobolev spaces. Moreover, we introduce the space $X(Q_T)$ and give some necessary properties, which provide a basic framework to solve our problem. At the end of this section, we give a compact embedding theorem to space $X(Q_T)$. In section 3, for $\varepsilon \in (0, 1)$ fixed, under appropriate penalty function, we transform the existence of solutions of evolution variational inequality (1.1) into the existence of weak solutions of the following initial boundary-value problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u) + b(x,t)|u|^{p(x,t)-2}u - |\frac{u^-}{\varepsilon}|^{p(x,t)-2}\frac{u^-}{\varepsilon} \\ = f(x,t,u) \quad \text{in } Q_T, \\ u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \text{in } \Omega,\end{gathered} \tag{1.2}$$

where $u^- = \max\{-u, 0\}$. The existence of weak solutions for the parametric problem (1.2) is proved by applying Galerkin's approximation method. Starting from problem (1.2), in order to obtain the solutions of (1.1), we must obtain a priori estimates for the solutions of problem (1.2), and then let $\varepsilon \to 0$. In section 4, we obtain the existence of weak solutions of parabolic variational inequality (1.1). In section 5, by means of Moser's iterative technique, we study the boundedness of weak solutions to problem (1.1).

2. Preliminaries

In this section, we first recall some necessary properties of variable exponent Lebesgue spaces and Sobolev spaces, see [10, 17, 19, 27] for more details. We define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) = \big\{u | u : \Omega \to \mathbb{R} \text{ is a measurable function, } \rho_{p(\cdot)}(u) = \int_\Omega |u(z)|^{p(z)}dz < \infty\big\}$$

then $L^{p(\cdot)}(\Omega)$ endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(\lambda^{-1}u) \le 1\},$$

becomes a separable, reflexive Banach space. The dual space $(L^{p(\cdot)}(\Omega))'$ can be identified with $L^{p'(\cdot)}(\Omega)$, where the conjugate exponent p' is defined by $p' = \frac{p}{p-1}$. In variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$, we have the following relations

$$\min\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\} \le \rho_{p(\cdot)}(u) \le \max\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\}.$$

Hence the norm convergence is equivalent to convergence with respect to the modular $\rho_{p(\cdot)}$.

In the variable exponent Lebesgue space, Hölder's inequality is still valid. For all $u \in L^{p(\cdot)}(\Omega),\ v \in L^{p'}(\Omega)$ the following inequality holds

$$\int_\Omega |uv|dz \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Definition 2.1 ([10, Definition 4.1]). We say a bounded exponent $p : \Omega \to \mathbb{R}$ is globally log-Hölder continuous if p satisfies the following two conditions

(1) there is a constant $c_1 > 0$ such that

$$|p(y) - p(z)| \le \frac{c_1}{\log(\mathrm{e} + |y-z|^{-1})},$$

for all points $y, z \in \Omega$;

(2) there exist constants $c_2 > 0$ and $p_\infty \in \mathbb{R}$ such that
$$|p(y) - p_\infty| \le \frac{c_2}{\log(\mathrm{e} + |y|)}.$$
for all $y \in \Omega$.

The log-Hölder constant of p is defined by $c_{\log}(p) = \max\{c_1, c_2\}$.

Proposition 2.2 ([10, Proposition 4.1.7]). *If $p : \Omega \to \mathbb{R}$ is globally log-Hölder continuous, then there exists an extension $\bar{p}$ such that $\bar{p}$ is globally log-Hölder continuous on $\mathbb{R}^N$, and $(\bar{p})^- = p^-$, $(\bar{p})^+ = p^+$, $c_{\log}(\bar{p}) = c_{\log}(p)$.*

For $f \in L^1_{\rm loc}(\mathbb{R}^N)$, the Hardy-Littlewood maximal operator is defined as
$$Mf(x) = \sup_{R>0} \frac{1}{|B_R^N(x)|} \int_{B_R^N(x)} |f(y)| dy,$$
where $|B_R^N(x)|$ denotes the Lebesgue measure of N-dimensional ball centered at x with radius R.

Theorem 2.3 ([10, Theorem 4.3.8]). *Assume that $p : \mathbb{R}^N \to (1, \infty)$ is a bounded globally log-Hölder continuous exponent such that $p^- > 1$, then the Hardy-Littlewood maximal operator M is bounded from $L^{p(\cdot)}(\mathbb{R}^N)$ to $L^{p(\cdot)}(\mathbb{R}^N)$.*

Theorem 2.4 ([10, Theorem 4.6.4]). *Let $p : \mathbb{R}^N \to (1, \infty)$ be a bounded globally log-Hölder continuous exponent with $p^- > 1$ and $f \in L^{p(\cdot)}(\mathbb{R}^N)$. For a standard mollifier ψ we set $\psi_\varepsilon(y) = \varepsilon^{-N} \psi(\frac{y}{\varepsilon})$, then there holds:*

(a) *There is a constant $K > 0$ which depends only on $\|\psi\|_{L^1(\mathbb{R}^N)}$ and the log-Hölder constant of p such that $\|\psi_\varepsilon * f\|_{L^{p(\cdot)}(\mathbb{R}^N)} \le K \|f\|_{L^{p(\cdot)}(\mathbb{R}^N)}$.*
(b) *Denote $A = \|\psi\|_{L^1(\mathbb{R}^N)}$, we have $\sup_{\varepsilon>0} |(\psi_\varepsilon * f)(x)| \le 2AMf(x)$.*
(c) *If $\int_{\mathbb{R}^N} \psi(y) dy = 1$, then $\psi_\varepsilon * f \to f$ almost everywhere and $\psi_\varepsilon * f \to f$ in $L^{p(\cdot)}(\mathbb{R}^N)$ as $\varepsilon \to 0$.*

Note that all previous definitions and results also hold for domains $\Omega \subset \mathbb{R}^{N+1}$, see [11] for more details.

Now we give the definition of variable exponent Sobolev space. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined as
$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$
equipped with the norm
$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$
then $W^{1,p(\cdot)}(\Omega)$ is a separable, reflexive Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$.

Theorem 2.5 ([10, Theorem 8.2.4]). *For all $u \in W_0^{1,p(\cdot)}(\Omega)$, we have*
$$\|u\|_{L^{p(\cdot)}(\Omega)} \le c \,\mathrm{diam}(\Omega) \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$
where the constant c only depends on the dimension N and the log-Hölder constant of p.

Definition 2.6. For any fixed $\tau \in (0,T)$, we define
$$W_\tau(\Omega) = \{u \in W_0^{1,1}(\Omega) : u \in L^{p(\cdot,\tau)}(\Omega), |\nabla u| \in L^{p(\cdot,\tau)}(\Omega)\},$$
and equip $W_\tau(\Omega)$ with the norm
$$\|u\|_{W_\tau(\Omega)} = \|u\|_{L^{p(\cdot,\tau)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot,\tau)}(\Omega)}.$$

Remark 2.7. Similar to the discussion in [11, Lemma 4.2], for every $\tau \in (0,T)$, the space $W_\tau(\Omega)$ is a separable and reflexive Banach space.

Definition 2.8. Set
$$X(Q_T) = \{u \in L^{p(x,t)}(Q_T) : |\nabla u| \in L^{p(x,t)}(Q_T),\ u(\cdot,\tau) \in W_\tau(\Omega) \text{ a.e. } \tau \in (0,T)\},$$
endowed with the norm $\|u\|_{X(Q_T)} = \|u\|_{L^{p(x,t)}(Q_T)} + \|\nabla u\|_{L^{p(x,t)}(Q_T)}$.

Remark 2.9. We can prove that $X(Q_T)$ is a Banach space, and $X(Q_T)$ can be continuously embedded into the space $L^{p^-}(0,T;W_0^{1,p^-}(\Omega))$, see [11]. It is worth mentioning that the space $X(Q_T)$ is defined in a similar way in [2].

Similarly to [11, Theorem 4.6], we obtain the following theorem by using Theorem 2.3 and Theorem 2.4.

Theorem 2.10. *The space $C_0^\infty(Q_T)$ is dense in $X(Q_T)$.*

Since $C_0^\infty(Q_T) \subset C^\infty(0,T;C_0^\infty(\Omega))$, we have

Lemma 2.11. *The space $C^\infty(0,T;C_0^\infty(\Omega))$ is dense in $X(Q_T)$.*

Let $X'(Q_T)$ denote the dual space of $X(Q_T)$. Similarly to [11, Proposition 5.1], we have

Theorem 2.12. *A function $g \in X'(Q_T)$ if and only if there exist $\overline{g} \in L^{p'}(Q_T)$ and $\overline{G} \in (L^{p'(x,t)}(Q_T))^N$ such that*
$$\int_{Q_T} g\varphi \,dx\,dt = \int_{Q_T} \overline{g}\varphi \,dx\,dt + \int_{Q_T} \overline{G}\nabla\varphi \,dx\,dt.$$

Remark 2.13. Similar to [11, Remark 5.6], $X(Q_T)$ is reflexive and
$$X'(Q_T) \hookrightarrow L^{(p^+)'}(0,T;W^{-1,(p^+)'}(\Omega)),$$
where $(p^+)' = \frac{p^+}{p^+-1}$. Note that similar results were obtained in [4, Remarks 3, 4] in the stationary case.

Similarly to [11], we give the following definition.

Definition 2.14. We define the space $W(Q_T) = \{u \in X(Q_T) : \frac{\partial u}{\partial t} \in X'(Q_T)\}$ with the norm
$$\|u\|_{W(Q_T)} = \|u\|_{X(Q_T)} + \|\frac{\partial u}{\partial t}\|_{X'(Q_T)},$$
where $\partial u/\partial t$ is the weak derivative of u in time variable t defined by
$$\int_{Q_T} \frac{\partial u}{\partial t}\varphi \,dx\,dt = -\int_{Q_T} u\frac{\partial u}{\partial t}\,dx\,dt, \quad \text{for all } \varphi \in C_0^\infty(Q_T).$$

Lemma 2.15 ([11, Lemma 6.3])**.** *$W(Q_T)$ is a Banach space.*

By means of the method in [11, Theorem 6.6], we have

Theorem 2.16. *$C^\infty(0,T;C_0^\infty(\Omega))$ is dense in $W(Q_T)$.*

Using a similar discussion to that in [11, Theorem 7.1], we can prove the following theorem.

Theorem 2.17. *$W(Q_T)$ can be continuously embedded into $C(0,T;L^2(\Omega))$. Furthermore, for all $u,v \in W(Q_T)$ and $s,t \in [0,T]$ the following rule for integration by parts is valid*

$$\int_s^t \int_\Omega \frac{\partial u}{\partial t} v dx d\tau = \int_\Omega u(x,t)v(x,t)dx - \int_\Omega u(x,s)v(x,s)dx - \int_s^t \int_\Omega u \frac{\partial v}{\partial t} dx d\tau.$$

Remark 2.18. Note that the formula of integration by parts is also obtained for the stationary case in [24].

The following theorem gives a relation between almost everywhere convergence and weak convergence.

Theorem 2.19 ([13, Lemma 2.5])**.** *If $\{u_n\}_{n=1}^\infty$ is bounded in $L^{p(x,t)}(Q_T)$ and $u_n \to u$ a.e. on Q_T as $n \to \infty$, then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $u_n \to u$ weakly in $L^{p(x,t)}(Q_T)$ as $n \to \infty$.*

Theorem 2.20 ([32, Proposition 1.3])**.** *Let $B_0 \subset B \subset B_1$ be three Banach spaces, where B_0, B_1 are reflexive, and the embedding $B_0 \subset B$ is compact. Denote $W = \{v : v \in L^{p_0}(0,T;B_0),\ \frac{\partial v}{\partial t} \in L^{p_1}(0,T;B_1)\}$, where T is a fixed positive number, $1 < p_i < \infty,\ i = 0,1$. Then W can be compactly embedded into $L^{p_0}(0,T;B)$.*

Now we can give a compact embedding for $X(Q_T)$ as follows.

Theorem 2.21. *Let F be bounded subset in $X(Q_T)$ and $\{\frac{\partial u}{\partial t} : u \in F\}$ be bounded in $X'(Q_T)$, then F is relatively compact in $L^{p^-}(0,T;L^2(\Omega))$.*

Proof. Since $p^- > \frac{2N}{N+2}$ $(N \geq 2)$, the embedding $W_0^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. By Remarks 2.9 and 2.13, the embedding $X(Q_T) \hookrightarrow L^{p^-}(0,T;W_0^{1,p^-}(\Omega))$ and

$$X'(Q_T) \hookrightarrow L^{(p^+)'}(0,T;W^{-1,(p^+)'}(\Omega)) \hookrightarrow L^{(p^+)'}(0,T;W^{-1,\lambda}(\Omega))$$

are continuous, where $\lambda = \min\{2,(p^+)'\}$. As the embedding $L^2(\Omega) \hookrightarrow W^{-1,\lambda}(\Omega)$ is continuous, by Theorem 2.20, F is relatively compact in $L^{p^-}(0,T;L^2(\Omega))$. $\square$

3. Solutions to parameterized parabolic equations

In this section for $\varepsilon \in (0,1)$ fixed, we consider the existence of weak solutions for problem (1.2).

Definition 3.1. A function $u_\varepsilon \in X(Q_T)$ with $\frac{\partial u_\varepsilon}{\partial t} \in X'(Q_T)$ is called a weak solution of (1.2), if for all $\varphi \in X(Q_T)$, there holds

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi \,dx\,dt + \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\varphi \\
&+ b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon\varphi \,dx\,dt - \int_{Q_T} |\frac{u_\varepsilon^-}{\varepsilon}|^{p(x,t)-2}\frac{u_\varepsilon^-}{\varepsilon}\varphi \,dx\,dt \\
&= \int_{Q_T} f(x,t,u_\varepsilon)\varphi \,dx\,dt.
\end{aligned}$$

We choose a sequence $\{w_j\}_{j=1}^{\infty}\subset C_0^{\infty}(\Omega)$ such that $C_0^{\infty}(\Omega)\subset\overline{\cup_{n=1}^{\infty}V_n}^{C^1(\bar\Omega)}$ and $\{w_j\}_{j=1}^{\infty}$ is a standard orthogonal basis in $L^2(\Omega)$, where $V_n=\operatorname{span}\{w_1,w_2,\dots,w_n\}$. The existence of such $\{w_j\}_{j=1}^{\infty}$ can be found in [13] or in [22]. Since $u_0\in L^2(\Omega)$, there exists a sequence $\psi_n\in V_n$ such that $\psi_n\to u_0$ strongly in $L^2(\Omega)$ as $n\to\infty$.

Theorem 3.2. *Suppose that* (H1)–(H3) *are satisfied. Then for* $\varepsilon\in(0,1)$ *fixed, there exists a weak solution of problem* (1.2).

Proof. **(i) Galerkin approximation** For all $n\in\mathbb{N}$, we want to find the approximate solutions (1.2) in the form
$$u_n(x,t)=\sum_{j=1}^{n}\big(\eta_n(t)\big)_j w_j(x).$$
First we define two vector-valued functions $F_n(t,\eta),P_n(t,\eta):[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ as
$$\begin{aligned}
(P_n(t,\eta))_i=\int_\Omega & a\Big(x,t,\sum_{j=1}^{n}\eta_j\nabla w_j\Big)\Big|\sum_{j=1}^{n}\eta_j\nabla w_j\Big|^{p(x,t)-2}\Big(\sum_{j=1}^{n}\eta_j\nabla w_j\Big)\nabla w_i\\
&+b(x,t)\Big|\sum_{j=1}^{n}\eta_j w_j\Big|^{p(x,t)-2}\Big(\sum_{j=1}^{n}\eta_j w_j\Big)w_i\\
&-\big(\frac{1}{\varepsilon}\big)^{p(x,t)-1}\Big|\Big(\sum_{j=1}^{n}\eta_j w_j\Big)^-\Big|^{p(x,t)-2}\Big(\sum_{j=1}^{n}\eta_j w_j\Big)^- w_i dx,\\
(F_n(t,\eta))_i=\int_\Omega & f\big(x,t,\sum_{j=1}^{n}\eta_j w_j\big)w_i dx,
\end{aligned}$$
where $\eta=(\eta_1,\dots,\eta_n)$. Then we consider the ordinary differential systems
$$\begin{gathered}
\eta'+P_n(t,\eta)=F_n(t,\eta),\\
\eta(0)=U_n(0),
\end{gathered}\tag{3.1}$$
where
$$(U_n(0))_i=\int_\Omega\psi_n(x)w_i dx,\ \psi_n(x)\in V_n$$
and $\psi_n(x)\to u_0(x)$ strongly in $L^2(\Omega)$ as $n\to\infty$. Multiplying (3.1) by $\eta(t)$, we arrive at the equality
$$\eta'(t)\eta(t)+P_n(t,\eta(t))\eta(t)=F_n(t,\eta(t))\eta(t).\tag{3.2}$$
By (H2), we obtain
$$\begin{aligned}
P_n(t,\eta)\eta=\int_\Omega & a\Big(x,t,\sum_{j=1}^{n}\eta_j\nabla w_j\Big)\Big|\sum_{j=1}^{n}\eta_j\nabla w_j\Big|^{p(x,t)-2}\Big(\sum_{j=1}^{n}\eta_j\nabla w_j\Big)\Big(\sum_{i=1}^{n}\eta_i\nabla w_i\Big)\\
&+b(x,t)\Big|\sum_{j=1}^{n}\eta_j w_j\Big|^{p(x,t)-2}\Big(\sum_{j=1}^{n}\eta_j w_j\Big)\Big(\sum_{i=1}^{n}\eta_i w_i\Big)\\
&-\big(\frac{1}{\varepsilon}\big)^{p(x,t)-1}\Big|(\sum_{j=1}^{n}\eta_j w_j)^-\Big|^{p(x,t)-2}\Big(\sum_{i=1}^{n}\eta_i w_i\Big)^-(\sum_{i=1}^{n}\eta_i w_i)dx\\
\ge a_0&\int_\Omega\Big|\sum_{j=1}^{n}\eta_j\nabla w_j\Big|^{p(x,t)}dx+b_0\int_\Omega\Big|\sum_{j=1}^{n}\eta_j w_j\Big|^{p(x,t)}dx
\end{aligned}$$

$$+\int_\Omega (\frac{1}{\varepsilon})^{p(x,t)-1}\Big|\Big(\sum_{i=1}^n \eta_i w_i\Big)^-\Big|^{p(x,t)}dx.$$

Since $q(x,t) \ll p(x,t)$, by (H3) and Young's inequality we have

$$\begin{aligned}
F_n(t,\eta)\eta &\le C_0\int_\Omega \Big|\sum_{j=1}^n \eta_j w_j\Big|^{q(x,t)}dx\\
&\le \frac{b_0}{2}\int_\Omega \Big|\sum_{j=1}^n \eta_j w_j\Big|^{p(x,t)}dx + C_0^{\frac{p(x,t)}{p(x,t)-q(x,t)}}(\frac{2}{b_0})^{\frac{q(x,t)}{p(x,t)-q(x,t)}}|\Omega|,\\
&\le \frac{b_0}{2}\int_\Omega \Big|\sum_{j=1}^n \eta_j w_j\Big|^{p(x,t)}dx + C_1,
\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of set Ω and

$$C_1 = (C_0+1)^{\frac{p^+}{\inf_{Q_T}(p(x,t)-q(x,t))}}\big(\frac{2}{b_0}+1\big)^{\frac{\sup_{Q_T} q(x,t)}{\inf_{Q_T}(p(x,t)-q(x,t))}}|\Omega|.$$

Combining the above two inequalities with (3.2), we arrive at the inequality

$$\begin{aligned}
&\eta'\eta + a_0\int_\Omega \Big|\sum_{j=1}^n \eta_j \nabla w_j\Big|^{p(x,t)}dx + \frac{b_0}{2}\int_\Omega \Big|\sum_{j=1}^n \eta_j w_j\Big|^{p(x,t)}dx\\
&+\int_\Omega (\frac{1}{\varepsilon})^{p(x,t)-1}\Big|(\sum_{i=1}^n \eta_i w_i)^-\Big|^{p(x,t)}dx \le C_1
\end{aligned}$$

Integrating this inequality with respect to t from 0 to t, we obtain

$$|\eta(t)| \le \int_\Omega |\psi_n(x)|^2dx + 2C_1T \le C(T) \quad \text{for all } t\in[0,T],$$

where $C(T)>0$ is a constant only depending on T.

Denote

$$L_n = \max_{(t,\eta)\in[0,T]\times B(\eta(0),2C(T))}\big|F_n(t,\eta)-P_n(t,\eta)\big|, \quad T_n=\min\Big\{T,\frac{2C(T)}{L_n}\Big\},$$

where $B(\eta(0),2C(T))$ is the ball of the radius $2C(T)$ with center at $\eta(0)$ in $\mathbb{R}^n$. Since a,b,f are continuous with respect to t, from the definition of $P_n(t,\eta)$ and $F_n(t,\eta)$, we obtain that $P_n(t,\eta)$ and $F_n(t,\eta)$ are continuous with respect to t and η. The Peano Theorem gives that (3.1) admits a C^1 solution locally in $[0,T_n]$. Without loss of generality, we assume that $T=[\frac{T}{T_n}]T_n+(\frac{T}{T_n})T_n,\ 0<(\frac{T}{T_n})<1$, where $[\frac{T}{T_n}]$ is the integer part of $\frac{T}{T_n}$ and $(\frac{T}{T_n})$ is the decimal part of $\frac{T}{T_n}$. Let $\eta(T_n)$ be a initial value of problem (3.1), then we can repeat the above process and get a C^1 solution on $[T_n,2T_n]$. We can divide $[0,T]$ into $[(i-1)T_n,iT_n]$ and $[mT_n,T]$, where $i=1,\dots,m$ and $m=[\frac{T}{T_n}]$. Then there exist C^1 solution $\eta_n^i(t)$ in $[(i-1)T_n,iT_n], i=1,\dots,m$, and $\eta_n^{m+1}(t)$ in $[mT_n,T]$. Hence, we obtain a solution

$\eta_n(t) \in C^1[0,T]$ defined by

$$\eta_n(t) = \begin{cases} \eta_n^1(t), & \text{if } t \in [0, T_n], \\ \eta_n^2(t), & \text{if } t \in (T_n, 2T_n], \\ \dots \\ \eta_n^m(t), & \text{if } t \in ((m-1)T_n, mT_n] \\ \eta_n^{m+1}(t), & \text{if } t \in (mT_n, T]. \end{cases}$$

Therefore, we obtain the approximation solutions $u_n(x,t) = \sum_{j=1}^n (\eta_n(t))_j w_j(x)$. Notice that by (3.1) $u_n(x,t)$ should be dependent on ε. For convenience we omit ε. It follows from (3.1) that for all $\varphi \in C^1(0,\tau;V_k)$, with $k \le n$ and $\tau \in (0,T]$, we have

$$\begin{aligned} &\int_0^\tau \int_\Omega \frac{\partial u_n}{\partial t}\varphi + a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n \nabla\varphi \\ &+ b(x,t)|u_n|^{p(x,t)-2}u_n\varphi - (\frac{1}{\varepsilon})^{p(x,t)-1}|u_n^-|^{p(x,t)-2}u_n^-\varphi \,dx\,dt \\ &= \int_0^\tau \int_\Omega f(x,t,u_n)\varphi \,dx\,dt. \end{aligned} \tag{3.3}$$

(ii) Passage to the limit Taking $\varphi = u_n$ in (3.3), we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u_n(x,\tau)|^2 dx + \int_0^\tau \int_\Omega a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)} + b(x,t)|u_n|^{p(x,t)} \\ &+ (\frac{1}{\varepsilon})^{p(x,t)-1}|u_n^-|^{p(x,t)}\,dx\,dt \\ &\le \int_0^\tau \int_\Omega |f(x,t,u_n)u_n|\,dx\,dt + \frac{1}{2}\int_\Omega |u_n(x,0)|^2 dx. \end{aligned}$$

In what follows, we denote by C various positive constants. Since $u_n(x,0) = \psi_n(x) \to u_0$ in $L^2(\Omega)$, $\int_\Omega u_n^2(x,0)dx \le C$, where $C > 0$ does not depend on ε and n. By $q(x,t) \ll p(x,t)$, (H2), (H3) and Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_n^2(x,\tau)dx + \int_0^\tau \int_\Omega a_0|\nabla u_n|^{p(x,t)} \\ &+ \frac{b_0}{2}|u_n|^{p(x,t)} + (\frac{1}{\varepsilon})^{p(x,t)-1}|u_n^-|^{p(x,t)}\,dx\,dt \le C. \end{aligned} \tag{3.4}$$

Therefore,

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|u_n\|_{L^{p(x,t)}(Q_T)} + \|\nabla u_n\|_{L^{p(x,t)}(Q_T)} \le C. \tag{3.5}$$

Furthermore, $\|f(x,t,u_n)\|_{L^{p'(x,t)}(Q_T)} \le C$. From (H2), we have

$$\int_{Q_T} \left|a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n\right|^{p'(x,t)} dx\,dt \le C\int_{Q_T} |\nabla u_n|^{p(x,t)}\,dx\,dt \le C,$$

$$\int_{Q_T} \left|b(x,t)|u_n|^{p(x,t)-2}u_n\right|^{p'(x,t)} dx\,dt \le C\int_{Q_T} |u_n|^{p(x,t)}\,dx\,dt \le C.$$

Thus,

$$\begin{aligned} &\left\|a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n\right\|_{L^{p'(x,t)}(Q_T)} \\ &+ \left\|b(x,t)|u_n|^{p(x,t)-2}u_n\right\|_{L^{p'(x,t)}(Q_T)} \le C. \end{aligned} \tag{3.6}$$

Similarly, $\||u_n^-|^{p(x,t)-2}u_n^-\|_{L^{p'(x,t)}(Q_T)} \le C\varepsilon^{\frac{(p^- -1)(p^+ -1)}{p^+}}$.

For each $\varphi \in X(Q_T)$, by Lemma 2.11, there exists a sequence $\varphi_n \in C^1(0,T;V_n)$ such that $\varphi_n \to \varphi$ strongly in $X(Q_T)$. From (3.3), we have

$$
\begin{aligned}
&\Big|\int_{Q_T}\frac{\partial u_n}{\partial t}\varphi_n\,dx\,dt\Big|\\
&=\Big|-\int_{Q_T} a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n\nabla\varphi_n + b(x,t)|u_n|^{p(x,t)-2}u_n\varphi_n\\
&\quad -(\frac{1}{\varepsilon})^{p(x,t)-1}|u_n^-|^{p(x,t)-2}u_n^-\varphi_n\,dx\,dt + \int_{Q_T} f(x,t,u_n)\varphi_n\,dx\,dt\Big|\\
&\le C(\varepsilon)\big\|\varphi_n\big\|_{X(Q_T)},
\end{aligned}
$$

where $C(\varepsilon)$ is a constant only depending on ε. We obtain $\|\frac{\partial u_n}{\partial t}\|_{X'(Q_T)} \le C(\varepsilon)$. It follows that there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\frac{\partial u_n}{\partial t} \rightharpoonup \theta$ in $X'(Q_T)$ as $n\to\infty$. For all $\phi \in C_0^\infty(Q_T)$, we have

$$
-\int_{Q_T} u_n\frac{\partial\phi}{\partial t}\,dx\,dt = \int_{Q_T}\frac{\partial u_n}{\partial t}\phi\,dx\,dt.
$$

Letting $n\to\infty$, then $-\int_{Q_T} u_\varepsilon\frac{\partial\phi}{\partial t}\,dx\,dt = \int_{Q_T}\theta\phi\,dx\,dt$. Thus, $\theta = \frac{\partial u_\varepsilon}{\partial t}$.

Gathering (3.5) with (3.6), we obtain a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$
\begin{gathered}
u_n \rightharpoonup u_\varepsilon \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)),\\
u_n \rightharpoonup u_\varepsilon \quad \text{weakly in } X(Q_T),\\
a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n \rightharpoonup \xi \quad \text{weakly in } (L^{p'(x,t)}(Q_T))^N,\\
b(x,t)|u_n|^{p(x,t)-2}u_n \rightharpoonup \eta \quad \text{weakly in } L^{p'(x,t)}(Q_T),\\
|u_n^-|^{p(x,t)-2}u_n^- \rightharpoonup \alpha \quad \text{weakly in } L^{p'(x,t)}(Q_T).
\end{gathered}
$$

Since $u_n \in X(Q_T)$ and $\frac{\partial u_n}{\partial t} \in X'(Q_T)$, by Theorem 2.21 there exists a subsequence of u_n (still denoted by u_n) such that $u_n \to u_\varepsilon$ strongly in $L^{p^-}(0,T;L^2(\Omega))$ and $u_n \to u_\varepsilon$ a.e. on Q_T. Thus we have

$$
\begin{gathered}
b(x,t)|u_n|^{p(x,t)-2}u_n \to b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon \quad \text{a.e. on } Q_T,\\
|u_n^-|^{p(x,t)-2}u_n^- \to |u_\varepsilon^-|^{p(x,t)-2}u_\varepsilon^- \quad \text{a.e. on } Q_T.
\end{gathered}
$$

By Theorem 2.19, we obtain that $\eta = b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon$, $\alpha = |u_\varepsilon^-|^{p(x,t)-2}u_\varepsilon^-$.

Next we prove that $\xi = a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon$. Since $q(x,t) \ll p(x,t)$, we have $p'(x,t)(q(x,t)-1) \ll p(x,t)$. Thus, for any measurable subset $e \subset Q_T$, we obtain

$$
\begin{aligned}
\int_e |f(x,t,u_n)|^{p'(x,t)}\,dx\,dt &\le C_0\big\||1|\big\|_{L^{\frac{p(x,t)-1}{p(x,t)-q(x,t)}}(e)}\big\||u_n|^{p'(x,t)(q(x,t)-1)}\big\|_{L^{\frac{p(x,t)-1}{q(x,t)-1}}(Q_T)}\\
&\le C_0\big\|1\big\|_{L^{\frac{p(x,t)-1}{p(x,t)-q(x,t)}}(e)}.
\end{aligned}
$$

This implies that the sequence $\{|f(x,t,u_n)-f(x,t,u_\varepsilon)|^{p'(x,t)}\}_{n=1}^\infty$ is uniformly bounded and equi-integrable in $L^1(Q_T)$. As $f(x,t,u_n)\to f(x,t,u_\varepsilon)$ a.e. on Q_T, by Vitali's convergence theorem we obtain that $f(x,t,u_n)\to f(x,t,u_\varepsilon)$ strongly in

$L^{p'(x,t)}(Q_T)$ as $n \to \infty$. Furthermore, we obtain that

$$\lim_{n\to\infty} \int_{Q_T} f(x,t,u_n)u_n \,dx\,dt = \int_{Q_T} f(x,t,u_\varepsilon)u_\varepsilon \,dx\,dt. \tag{3.7}$$

As $a(x,t,\nabla u_n)$ is uniformly bounded and equi-integrable in $L^1(Q_T)$, there exist a subsequence of $\{u_n\}$ (still labeled by $\{u_n\}$) and $\bar{a}$ such that $a(x,t,\nabla u_n) \to \bar{a}$ a.e. on Q_T. Since

$$\big|(a(x,t,u_n) - \bar{a})|\nabla u_\varepsilon|^{p(x,t)-2}u_\varepsilon\big|^{p'(x,t)} \le C\big|\nabla u_\varepsilon\big|^{p(x,t)} \in L^1(Q_T),$$

by Lebesgue's dominated convergence theorem we obtain

$$a(x,t,\nabla u_n)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \to \bar{a}|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \quad \text{strongly in } L^{p'(x,t)}(Q_T). \tag{3.8}$$

Since $\int_\Omega u_n^2(x,T)dx \le C$, there exist a subsequence of $\{u_n(x,T)\}$ (still denoted by $\{u_n(x,T)\}$) and a function $\tilde{u}$ in $L^2(\Omega)$ such that $u_n(x,T) \rightharpoonup \tilde{u}$ weakly in $L^2(\Omega)$, then for all ω_i and $\eta(t) \in C^1[0,T]$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u_n}{\partial t}\omega_i\eta(t)\,dx\,dt \\
&= \int_\Omega u_n(x,T)\omega_i\eta(T)dx - \int_\Omega u_n(x,0)\omega_i\eta(0)dx - \int_0^T\int_\Omega u_n\omega_i\eta'(t)\,dx\,dt.
\end{aligned}$$

Letting $n \to \infty$, by integration by parts we obtain

$$\int_\Omega (\tilde{u} - u_\varepsilon(x,T))\eta(T)\omega_i - (u_\varepsilon(x,0) - u_0(x))\eta(0)\omega_i dx = 0.$$

Choosing $\eta(T) = 1$, $\eta(0) = 0$ or $\eta(T) = 0$, $\eta(T) = 1$, by the completeness of $\{\omega_i\}_{i=1}^\infty$ in $L^2(\Omega)$ we have $\tilde{u} = u(x,T)$ and $u_\varepsilon(x,0) = u_0(x)$, that is $u_n(x,T) \rightharpoonup u_\varepsilon(x,T)$ weakly in $L^2(\Omega)$. Thus we obtain

$$\int_\Omega u_\varepsilon^2(x,T)dx \le \liminf_{n\to\infty} \int_\Omega u_n^2(x,T)dx. \tag{3.9}$$

Taking $\varphi = u_n$ in (3.3), we have

$$\begin{aligned}
0 &\le \int_{Q_T} a(x,t,\nabla u_n)(|\nabla u_n|^{p(x,t)-2}\nabla u_n - |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon)(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt \\
&= \int_{Q_T} f(x,t,u_n)u_n\,dx\,dt - \frac{1}{2}\int_\Omega u_n^2(x,T)dx + \frac{1}{2}\int_\Omega u_n^2(x,0)dx \\
&\quad - \int_{Q_T} b(x,t)|u_n|^{p(x,t)} + (\frac{1}{\varepsilon})^{p(x,t)-1}|u_n^-|^{p(x,t)}\,dx\,dt \\
&\quad - \int_{Q_T} a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n\nabla u_\varepsilon \\
&\quad + a(x,t,\nabla u_n)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt.
\end{aligned}$$

In view of (3.3), fixing k and letting $n \to \infty$, we obtain

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi + \xi\nabla\varphi + b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon\varphi - (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)-2}u_\varepsilon^-\varphi\,dx\,dt \\
&= \int_{Q_T} f(x,t,u_\varepsilon)\varphi\,dx\,dt,
\end{aligned}$$

for all $\varphi \in C^1(0,T;V_k)$ with $k \in \mathbb{N}$. Since $C^1\big(0,T;\cup_{k=1}^{\infty}V_k\big)$ is dense in the space $C^1(0,T;C^1(\Omega))$, the above equality is valid for all $\varphi \in X(Q_T)$. Taking $\varphi = u_\varepsilon$, by Theorem 2.17 we arrive at the equality

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u_\varepsilon(x,T)|^2 dx - \frac{1}{2}\int_\Omega |u_\varepsilon(x,0)|^2 dx + \int_{Q_T} \xi\nabla u_\varepsilon + b(x,t)|u_\varepsilon|^{p(x,t)} \\
&+ (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)}\,dx\,dt \\
&= \int_{Q_T} f(x,t,u_\varepsilon)u_\varepsilon\,dx\,dt.
\end{aligned} \tag{3.10}$$

Combining (3.7)–(3.10) and Fatou's lemma, we have

$$\begin{aligned}
0 &\le \limsup_{n\to\infty} \int_{Q_T} a(x,t,\nabla u_n)(|\nabla u_n|^{p(x,t)-2}\nabla u_n \\
&\quad - |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon)(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt \\
&\le \int_{Q_T} f(x,t,u_\varepsilon)u_\varepsilon\,dx\,dt - \frac{1}{2}\int_\Omega u_\varepsilon^2(x,T)dx + \frac{1}{2}\int_\Omega u_\varepsilon^2(x,0)dx \\
&\quad - \int_{Q_T} b(x,t)|u_\varepsilon|^{p(x,t)} + (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)}\,dx\,dt - \int_{Q_T}\xi\nabla u_\varepsilon\,dx\,dt = 0;
\end{aligned}$$

that is,

$$\begin{aligned}
&\lim_{n\to\infty} \int_{Q_T} a(x,t,\nabla u_n)(|\nabla u_n|^{p(x,t)-2}\nabla u_n \\
&- |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon)(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt = 0.
\end{aligned} \tag{3.11}$$

Similarly as in [9, Theorem 3.1], we set $Q_1 = \{(x,t) \in Q_T :\ p(x,t) \ge 2\}$ and $Q_2 = \{(x,t) \in Q_T :\ \frac{2N}{N+2} < p(x,t) < 2\}$. By (3.11), we conclude that

$$\begin{aligned}
&\int_{Q_1} |\nabla u_n - \nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \\
&\le C\int_{Q_1} a(x,t,\nabla u_n)(|\nabla u_n|^{p(x,t)-2}\nabla u_n \\
&\quad - |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon)(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt \to 0
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\int_{Q_2} |\nabla u_n - \nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \\
&\le C\big\|[a(x,t,\nabla u_n)(|\nabla u_n|^{p(x,t)-2}\nabla u_n \\
&\quad - |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon)(\nabla u_n - \nabla u_\varepsilon)]^{\frac{p(x,t)}{2}}\big\|_{L^{\frac{2}{p(x,t)}}(Q_T)} \\
&\quad \times \big\|(|\nabla u_n|^{p(x,t)} + |\nabla u_\varepsilon|^{p(x,t)})^{\frac{2-p(x,t)}{2}}\big\|_{L^{\frac{2}{2-p(x,t)}}(Q_T)} \to 0.
\end{aligned} \tag{3.13}$$

Combining (3.12) with (3.13), we obtain that $\nabla u_n \to \nabla u_\varepsilon$ in $(L^{p(x,t)}(Q_T))^N$. Thus there exists a subsequence of $\{u_n\}$ (still labeled by $\{u_n\}$) such that $\nabla u_n \to \nabla u_\varepsilon$ a.e. on Q_T. Furthermore,

$$a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n \to a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon$$

a.e. on Q_T. Theorem 2.19 implies that $\xi = a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon$. It follows from (3.3) that

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt + \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\varphi + b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon \\
&- (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)-2}u_\varepsilon^-\varphi\,dx\,dt = \int_{Q_T} f(x,t,u_\varepsilon)\varphi\,dx\,dt,
\end{aligned}$$

for all $\varphi \in X(Q_T)$. □

4. Existence of solutions for the variational inequality

In this section, we prove the main theorem of this article.

Theorem 4.1. *Under the assumptions* (H1)–(H3)*, there exists a function* $u(x,t) \in \mathscr{K}$ *such that*

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u}{\partial t}(v-u)\,dx\,dt \\
&+ \int_{Q_T} a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u\nabla(v-u) + b(x,t)|u|^{p(x,t)-2}u(v-u)\,dx\,dt \\
&\geq \int_{Q_T} f(x,t,u)(v-u)\,dx\,dt.
\end{aligned}$$

for all $v \in X(Q_T)$ with $v(x,t) \geq 0$ *a.e. on* Q_T*.*

Proof. We divide the proof into three steps.

(i) A priori estimates Taking $\varphi = u_\varepsilon\chi_{(0,\tau)}$ as a test function in Definition 3.1, where $\chi_{(0,\tau)}$ is defined as the characteristic function of $(0,\tau)$, $\tau \in (0,T]$, we have

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial u_\varepsilon}{\partial t}u_\varepsilon\,dx\,dt + \int_{Q_\tau} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)} + b(x,t)|u_\varepsilon|^{p(x,t)} \\
&\quad + (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)}\,dx\,dt \\
&= \int_{Q_\tau} f(x,t,u_\varepsilon)u_\varepsilon\,dx\,dt,
\end{aligned}$$

where $Q_\tau = \Omega \times (0,\tau)$. By $q(x,t) \ll p(x,t)$, Theorem 2.17, (H2)–(H3) and Young's inequality, we obtain

$$\begin{aligned}
&\int_\Omega |u_\varepsilon(x,\tau)|^2 dx + \int_{Q_\tau} a_0|\nabla u_\varepsilon|^{p(x,t)} + b_0|u_\varepsilon|^{p(x,t)} \\
&+ (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)}\,dx\,dt \leq C,
\end{aligned}$$

where C is a constant independent of ε and τ. Obviously, $u_\varepsilon^- \in X(Q_T)$. Thus we can take $\varphi = \frac{-u_\varepsilon^-}{\varepsilon}$ in Definition 3.1, then

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}(-u_\varepsilon^-)\,dx\,dt + \frac{1}{\varepsilon}\int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(-u_\varepsilon^-) \\
&+ b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon(-u_\varepsilon^-) - (\frac{1}{\varepsilon})^{p(x,t)}|u_\varepsilon^-|^{p(x,t)-2}u_\varepsilon^-(-u_\varepsilon^-)\,dx\,dt \\
&= \int_{Q_T} f(x,t,u_\varepsilon)\frac{-u_\varepsilon^-}{\varepsilon}\,dx\,dt.
\end{aligned}$$

Combining (H3) with Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}(-u_\varepsilon^-)\,dx\,dt+\frac{1}{\varepsilon}\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(-u_\varepsilon^-)\\
&+b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon(-u_\varepsilon^-)\,dx\,dt+\int_{Q_T}|\frac{u_\varepsilon^-}{\varepsilon}|^{p(x,t)}\,dx\,dt\\
&\le C+\frac{1}{2}\int_{Q_T}|\frac{u_\varepsilon^-}{\varepsilon}|^{p(x,t)}\,dx\,dt.
\end{aligned}$$

Since

$$\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}(-u_\varepsilon^-)=\frac{1}{2}\int_\Omega|u_\varepsilon^-(x,T)|^2-|u_\varepsilon^-(x,0)|^2dx\ge 0,$$

and

$$\begin{aligned}
&\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(-u_\varepsilon^-)+b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon(-u_\varepsilon^-)\,dx\,dt\\
&=\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon^-|^{p(x,t)}+b(x,t)|u_\varepsilon^-|^{p(x,t)}\,dx\,dt\ge 0,
\end{aligned}$$

we obtain that $\int_{Q_T}|\frac{u_\varepsilon^-}{\varepsilon}|^{p(x,t)}\,dx\,dt\le C$. Therefore,

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}+\|u_\varepsilon\|_{X(Q_T)}+\|\frac{u_\varepsilon^-}{\varepsilon}\|_{L^{p(x,t)}(Q_T)}\le C. \tag{4.1}$$

Since

$$\int_{Q_T}\left|a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\right|^{p'(x,t)}\,dx\,dt\le C\int_{Q_T}|\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt\le C,$$

the following inequality holds

$$\|\ a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\ \|_{L^{p'(x,t)}(Q_T)}\le C. \tag{4.2}$$

Similarly,

$$\|b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon\|_{L^{p'(x,t)}(Q_T)}+\||\frac{u_\varepsilon^-}{\varepsilon}|^{p(x,t)-2}\frac{u_\varepsilon^-}{\varepsilon}\|_{L^{p'(x,t)}(Q_T)}\le C. \tag{4.3}$$

From $q(x,t)\ll p(x,t)$, (4.1) and Young's inequality, we have

$$\begin{aligned}
\int_{Q_T}|f(x,t,u_\varepsilon)|^{q'(x,t)}\,dx\,dt&\le C\int_{Q_T}|u_\varepsilon|^{q(x,t)}\,dx\,dt\\
&\le C(\int_{Q_T}|u_\varepsilon|^{p(x,t)}\,dx\,dt+1)\le C.
\end{aligned}$$

Thus, $\|f(x,t,u_\varepsilon)\|_{L^{p'(x,t)}(Q_T)}\le C$. Here C denotes various positive constants.

For all $\varphi\in X(Q_T)$, from Definition 3.1 and (4.2)-(4.3) there holds

$$\begin{aligned}
&|\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt|\\
&=|-\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\varphi+b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon\varphi\\
&\quad-|\frac{u_\varepsilon^-}{\varepsilon}|^{p(x,t)-2}\frac{u_\varepsilon^-}{\varepsilon}\varphi dxd+\int_{Q_T}f(x,t,u_\varepsilon)\varphi\,dx\,dt|\\
&\le C\|\varphi\|_{X(Q_T)};
\end{aligned}$$

thus we obtain

$$\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|_{X'(Q_T)} = \sup_{\|\varphi\|_{X(Q_T)}\le 1}\Big|\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt\Big| \le C. \tag{4.4}$$

(ii) Passage to the limit By (4.1)–(4.4), there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$, still denoted by $\{u_\varepsilon\}_{\varepsilon>0}$, such that

$$\begin{gathered}
u_\varepsilon \rightharpoonup u \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)),\\
u_\varepsilon \rightharpoonup u \quad \text{weakly in } X(Q_T),\\
a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \rightharpoonup A \quad \text{weakly in } (L^{p'(x,t)}(Q_T))^N,\\
b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon \rightharpoonup B \quad \text{weakly in } L^{p'(x,t)}(Q_T),\\
\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \beta \quad \text{weakly in } X'(Q_T),\\
u_\varepsilon^- \to 0 \quad \text{strongly in } L^{p(x,t)}(Q_T).
\end{gathered} \tag{4.5}$$

Firstly, we prove that $B = b(x,t)|u|^{p(x,t)-2}u$, $\beta = \frac{\partial u}{\partial t}$ and $u \ge 0$ a.e. on Q_T. For all $\varphi \in C_0^\infty(Q_T)$, there holds $\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt = -\int_{Q_T} u_\varepsilon\frac{\partial\varphi}{\partial t}\,dx\,dt$. Thanks to (4.5), the left of the above equality converges to $\int_{Q_T}\beta\varphi\,dx\,dt$ while the right converges to $-\int_{Q_T} u\frac{\partial\varphi}{\partial t}\,dx\,dt$, thus we have $\beta = \frac{\partial u}{\partial t}$. As in Section 3, by Theorem 2.21 there exists a subsequence of u_ε (still denoted by u_ε) such that $u_\varepsilon \to u$ strongly in $L^{p^-}(0,T;L^2(\Omega))$ and $u_\varepsilon \to u$ a.e. on Q_T. Thus we have

$$\begin{gathered}
b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon \to b(x,t)|u|^{p(x,t)-2}u \quad \text{a.e. on } Q_T,\\
u_\varepsilon^- \to u^- \quad \text{a.e. on } Q_T.
\end{gathered}$$

By Theorem 2.19, we obtain $B = b(x,t)|u|^{p(x,t)-2}u$. Moreover, from (4.5) we obtain $u^- = 0$ a.e. on Q_T, that is $u(x,t) \ge 0$ a.e. on Q_T.

Secondly, we prove that $A = a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u$. Since $\int_\Omega |u_\varepsilon(x,T)|^2dx \le C$, we have $u_\varepsilon(x,T) \rightharpoonup \tilde u$ weakly in $L^2(\Omega)$ (up to a subsequence). For all $\eta(t) \in C^1[0,T]$ and $\varphi \in C_0^\infty(\Omega)$, there holds

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}\eta(t)\varphi(x)\,dx\,dt\\
&= \int_\Omega u_\varepsilon(x,T)\eta(T)\varphi(x) - u_0(x)\eta(0)\varphi(x)dx - \int_0^T\int_\Omega u_\varepsilon\frac{\partial\eta}{\partial t}\varphi\,dx\,dt,
\end{aligned}$$

Letting $\varepsilon \to 0$ and using integration by parts, we obtain

$$\int_\Omega (\tilde u - u(x,T))\eta(T)\varphi(x) - (u(x,0) - u_0(x))\eta(0)\varphi(x)dx = 0.$$

Picking $\eta(T) = 1$ and $\eta(0) = 0$, or $\eta(T) = 0$ and $\eta(0) = 1$, we have $\tilde u = u(x,T)$ and $u(x,0) = u_0(x)$ by the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$,

Taking $\varphi = u - u_\varepsilon$ as a test function in Definition 3.1, we obtain

$$\begin{aligned}
&\int_{Q_T}\frac{\partial u}{\partial t}(u-u_\varepsilon) + a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(u-u_\varepsilon)\\
&+ b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon(u-u_\varepsilon) - f(x,t,u_\varepsilon)(u-u_\varepsilon)\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}(u-u_\varepsilon) + a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u-u_\varepsilon) \\
&\quad + b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon(u-u_\varepsilon) - f(x,t,u_\varepsilon)(u-u_\varepsilon)\,dx\,dt \\
&\quad + \int_{Q_T} \frac{\partial(u-u_\varepsilon)}{\partial t}(u-u_\varepsilon)\,dx\,dt \\
&= \int_{Q_T} (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)-2}u_\varepsilon^-(u-u_\varepsilon)\,dx\,dt \\
&\quad + \int_{Q_T} \frac{\partial(u-u_\varepsilon)}{\partial t}(u-u_\varepsilon)\,dx\,dt \geq 0,
\end{aligned}
$$

and furthermore

$$
\begin{aligned}
&\int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \\
&\leq \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla u\,dx\,dt + \int_{Q_T} \frac{\partial u}{\partial t}(u-u_\varepsilon)\,dx\,dt \\
&\quad - \int_{Q_T} f(x,t,u_\varepsilon)(u-u_\varepsilon)\,dx\,dt \\
&\quad + \int_{Q_T} b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon u - b(x,t)|u_\varepsilon|^{p(x,t)}\,dx\,dt.
\end{aligned}
$$

As in Section 3, $f(x,t,u_\varepsilon) \to f(x,t,u)$ strongly in $L^{p'(x,t)}(Q_T)$. Thus we obtain

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \\
&\leq \int_{Q_T} A\nabla u\,dx\,dt = \lim_{\varepsilon\to 0} \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla u\,dx\,dt;
\end{aligned}
$$

that is,

$$
\limsup_{\varepsilon\to 0} \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u)\,dx\,dt \leq 0. \tag{4.6}
$$

As $a(x,t,\nabla u_\varepsilon)$ is uniformly bounded and equi-integrable in $L^1(Q_T)$, there exist a subsequence of $\{u_\varepsilon\}$ (for convenience still relabeled by $\{u_\varepsilon\}$) and a^* such that $a(x,t,\nabla u_\varepsilon) \to a^*$ a.e. on Q_T. In view of

$$
\left|(a(x,t,\nabla u_\varepsilon) - a^*)|\nabla u|^{p(x,t)-2}\nabla u\right|^{p'(x,t)} \leq C|\nabla u|^{p(x,t)} \in L^1(Q_T),
$$

the Lebesgue dominated convergence theorem implies

$$
a(x,t,\nabla u_\varepsilon)|\nabla u|^{p(x,t)-2}\nabla u \to a^*|\nabla u|^{p(x,t)-2}\nabla u \quad \text{strongly in } L^{p'(x,t)}(Q_T).
$$

Since

$$
\begin{aligned}
0 &\leq \int_{Q_T} a(x,t,\nabla u_\varepsilon)(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u)(\nabla u_\varepsilon - \nabla u) \\
&= \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u) \\
&\quad - a(x,t,\nabla u_\varepsilon)|\nabla u|^{p(x,t)-2}\nabla u \nabla(u_\varepsilon - u)\,dx\,dt,
\end{aligned}
$$

we have

$$\liminf_{\varepsilon\to 0}\int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(u_\varepsilon-u)\,dx\,dt\geq 0. \tag{4.7}$$

By (4.6)-(4.7) and $\nabla u_\varepsilon\rightharpoonup\nabla u$ weakly in $(L^{p(x,t)}(Q_T))^N$, we obtain

$$\lim_{\varepsilon\to 0}\int_{Q_T} a(x,t,\nabla u_\varepsilon)(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon-|\nabla u|^{p(x,t)-2}\nabla u)\nabla(u_\varepsilon-u)\,dx\,dt=0.$$

A similar discussion as Section 3 gives that $\nabla u_\varepsilon\to\nabla u$ strongly in $(L^{p(x,t)}(Q_T))^N$ as $\varepsilon\to 0$. Thus there exists a subsequence of $\{u_\varepsilon\}$, still labeled by $\{u_\varepsilon\}$, such that $\nabla u_\varepsilon\to\nabla u$ a.e. on Q_T, from which we obtain that $A=a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u$.

(iii) Existence of weak solutions By Fatou's lemma, we have

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}+b(x,t)|u_\varepsilon|^{p(x,t)}\,dx\,dt\\ &\geq\int_{Q_T} a(x,t,\nabla u)|\nabla u|^{p(x,t)}+b(x,t)|u|^{p(x,t)}\,dx\,dt.\end{aligned}$$

Since $\|u_\varepsilon(x,t)\|_{L^2(\Omega)}\leq C$ for all $t\in[0,T]$, there exists a subsequence of u_ε (still denoted by u_ε) such that $u_\varepsilon(x,T)\rightharpoonup u(x,T)$ weakly in $L^2(\Omega)$ and $\int_\Omega|u(x,T)|^2dx\leq\liminf_{\varepsilon\to 0}\int_\Omega|u_\varepsilon(x,T)|^2dx$.

For all $v\in X(Q_T)$ with $v\geq 0$ a.e. on Q_T, we take $\varphi=v-u_\varepsilon$ as a test function in Definition 3.1, then

$$\begin{aligned}&\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}v+a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(v-u_\varepsilon)\\ &+b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon(v-u_\varepsilon)-f(x,t,u_\varepsilon)(v-u_\varepsilon)\,dx\,dt\\ &=\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}u_\varepsilon\,dx\,dt+\int_{Q_T}(\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{p(x,t)-2}u_\varepsilon^-(v-u_\varepsilon)\,dx\,dt\\ &\geq\frac{1}{2}\int_\Omega|u_\varepsilon(x,T)|^2dx-\frac{1}{2}\int_\Omega|u_\varepsilon(x,0)|^2dx,\end{aligned}$$

and hence

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}v+a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla v\\ &+b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon v-f(x,t,u_\varepsilon)(v-u_\varepsilon)\,dx\,dt\\ &\geq\int_{Q_T}a(x,t,\nabla u)|\nabla u|^{p(x,t)}+b(x,t)|u|^{p(x,t)}\,dx\,dt+\frac{1}{2}\int_\Omega|u(x,T)|^2dx\\ &\quad-\frac{1}{2}\int_\Omega|u_0(x)|^2dx.\end{aligned}$$

Since

$$\begin{gathered}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\rightharpoonup a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u\\ \text{weakly in } L^{p'(x,t)}(Q_T),\\ b(x,t)|u_\varepsilon|^{p(x,t)-2}u_\varepsilon\rightharpoonup b(x,t)|u|^{p(x,t)-2}u\quad\text{weakly in } L^{p'(x,t)}(Q_T),\\ f(x,t,u_\varepsilon)\to f(x,t,u)\quad\text{strongly in } L^{p'(x,t)}(Q_T),\end{gathered}$$

$$\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in } X'(Q_T),$$

the following inequality holds for all $v \in X(Q_T)$

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u}{\partial t}(v-u)\,dx\,dt + \int_{Q_T} a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u \nabla(v-u) \\
&+ b(x,t)|u|^{p(x,t)-2}u(v-u)\,dx\,dt \\
&\geq \int_{Q_T} f(x,t,u)(v-u)\,dx\,dt.
\end{aligned}$$

As $u \in X(Q_T)$, $\frac{\partial u}{\partial t} \in X'(Q_T)$, by Theorem 2.17 we know that $u \in C(0,T;L^2(\Omega))$. Thus we complete the proof. $\square$

Remark 4.2. For the case that $f(x,t,u) = c(x,t)|u|^{p(x,t)-1}$, that is $q(x,t) = p(x,t)$, if we assume that $c(x,t)$ is small enough, for instance $\sup_{(x,t)\in Q_T} c(x,t) < b_0$, then we can also obtain the existence of weak solutions of problem (1.1).

5. Boundedness of weak solutions

In this section, we give a bounded estimate for the weak solutions to evolution variational inequality (1.1).

Lemma 5.1 ([12, Lemma 4.1])**.** *Let $\{Y_n\}$, $n = 0,1,\dots,$ be a sequence of positive numbers satisfying the recursive inequalities $Y_{n+1} \leq Cb^nY_n^{1+\alpha}$, where $C, b > 1$ and $\alpha > 0$ are given numbers. If $Y_0 \leq C^{-1/\alpha}b^{-1/\alpha^2}$, then $\{Y_n\}$ converges to zero as $n \to \infty$.*

Lemma 5.2 ([12, Proposition 3.1])**.** *There exists a constant C depending only on N, p^- such that for each $v \in L^\infty(0,T;L^2(\Omega)) \cap L^{p^-}(0,T;W_0^{1,p^-}(\Omega))$*

$$\int_{Q_T} |v(x,t)|^q\,dx\,dt \leq C^q\Big(\int_{Q_T} |\nabla v(x,t)|^{p^-}\,dx\,dt\Big)\Big(\operatorname{ess\,sup}_{0<t<T}\int_\Omega |v(x,t)|^2dx\Big)^{\frac{p^-}{N}},$$

where $q = \frac{(N+2)p^-}{N}$.

Theorem 5.3. *Suppose that all conditions in Theorem 4.1 are satisfied. Then for each $t^* \in (0,T]$ and $\sigma \in (0,1)$, there exists a constant $C = C(N,p^-,p^+,|\Omega|) > 0$ such that the weak solution u obtained by Theorem 4.1 has the following estimate*

$$\begin{aligned}
&\operatorname{ess\,sup}_{(x,t)\in\Omega\times(\sigma t^*,t^*)} u(x,t) \\
&\leq \max\Big\{1, C\Big(\frac{\int_{Q_0^{t^*}} u^\delta\,dx\,dt}{|Q_0^{t^*}|}\Big)^{\frac{\delta p^-}{(\delta-2)(N+p^-)}}\Big(t^{*\frac{p^-}{N+p^-}} + \frac{1}{\sigma}t^{*-\frac{N}{N+p^-}}\Big)^{\frac{\delta}{\delta-2}}\Big\},
\end{aligned}$$

where $Q_0^{t^*} = \Omega \times (0,t^*)$, $t^* \in (0,T]$ and $|Q_0^{t^*}|$ is the Lebesgue measure of $Q_0^{t^*}$, $\delta = \frac{(N+2)p^-}{N}$.

Proof. Fix $t^* \in (0,T]$ and introduce the sequence $t_n = \sigma\frac{2^n-1}{2^n}t^*$, $n = 0,1,\dots,$ $\sigma \in (0,1)$. We introduce the cut-off functions

$$\xi_n(t) = \begin{cases} 1 & \text{if } t_{n+1} \leq t \leq t^*, \\ \frac{t-t_n}{t_{n+1}-t_n} & \text{if } t_n < t < t_{n+1}, \\ 0 & \text{if } 0 \leq t \leq t_n. \end{cases}$$

Denote

$$k_n = \frac{2^n - 1}{2^n} k, \quad n = 0, 1, \dots, \ k > 0 \text{ to be chosen.}$$

For $\tau \in (0, t^*]$ and a weak solution $u \in \mathscr{K}$ obtained by Theorem 4.1, it is easy to see that $[u - (u - k_{n+1})_+ \xi_n(t)\chi_{(0,\tau)}] \in X(Q_T)$ and $u - (u - k_{n+1})_+ \xi_n(t)\chi_{(0,\tau)} \geq 0$ a.e. on Q_T. Thus we can take $v = u - (u - k_{n+1})_+ \xi_n(t)\chi_{(0,\tau)}$ as a test function in Theorem 4.1. Then we have

$$\begin{aligned}
&\int_0^\tau \int_\Omega \frac{\partial u}{\partial t}(u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&+ \int_0^\tau \int_\Omega a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u \nabla (u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&+ \int_0^\tau \int_\Omega b(x,t)|u|^{p(x,t)-2} u (u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&\leq \int_0^\tau \int_\Omega f(x,t,u)(u - k_{n+1})_+ \xi_n(t)\,dx\,dt.
\end{aligned} \tag{5.1}$$

By Theorem 2.17, we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega \frac{\partial u}{\partial t}(u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&= \frac{1}{2}\int_\Omega (u - k_{n+1})_+^2 \xi_n(\tau) dx - \frac{1}{2}\int_\Omega (u - k_{n+1})_+^2 \xi_n(0) dx \\
&- \frac{1}{2}\int_0^\tau \int_\Omega (u - k_{n+1})_+^2 \frac{d\xi_n(t)}{dt}\,dx\,dt.
\end{aligned} \tag{5.2}$$

Assumption (H2) and $u \geq 0$ a.e. on Q_T imply that

$$\begin{aligned}
&\int_0^\tau \int_\Omega a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u \nabla (u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&= \int_0^\tau \int_\Omega a(x,t,\nabla u)|\nabla (u - k_{n+1})_+|^{p(x,t)} \xi_n(t)\,dx\,dt \\
&\geq a_0 \int_0^\tau \int_\Omega |\nabla (u - k_{n+1})_+|^{p(x,t)} \xi_n(t)\,dx\,dt.
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\int_0^\tau \int_\Omega b(x,t)|u|^{p(x,t)-2} u (u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&\geq b_0 \int_0^\tau \int_\Omega u^{p(x,t)-1}(u - k_{n+1})_+ \xi_n(t)\,dx\,dt.
\end{aligned} \tag{5.4}$$

Since $q(x,t) \ll p(x,t)$ by (H3), we deduce from Young's inequality that

$$\begin{aligned}
&\int_0^\tau \int_\Omega f(x,t,u)(u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&\leq C_0 \int_0^\tau \int_\Omega u^{q(x,t)-1}(u - k_{n+1})_+ \xi_n(t)\,dx\,dt \\
&\leq \frac{b_0}{2}\int_0^\tau \int_\Omega u^{p(x,t)-1}(u - k_{n+1})_+ \xi_n(t)\,dx\,dt + C\int_0^\tau \int_\Omega (u - k_{n+1})_+ \xi_n(t)\,dx\,dt.
\end{aligned}$$

Furthermore, Young's inequality yields

$$
\begin{aligned}
&\int_0^\tau \int_\Omega f(x,t,u)(u-k_{n+1})_+\xi_n(t)\,dx\,dt \\
&\le \frac{b_0}{2}\int_0^\tau \int_\Omega u^{p(x,t)-1}(u-k_{n+1})_+\xi_n(t)\,dx\,dt \\
&\quad + \frac{b_0}{4}\int_0^\tau \int_\Omega (u-k_{n+1})_+^{p(x,t)}\xi_n(t)\,dx\,dt \\
&\quad + C\int_0^\tau \int_\Omega \xi_n(t)\chi(u>k_{n+1})\,dx\,dt,
\end{aligned} \tag{5.5}
$$

where $\chi(u>k_{n+1})$ is the characteristic function of the set $\{(x,t)\in Q_T : u(x,t) > k_{n+1}\}$ and $C>0$ denotes various constants. Putting (5.2)–(5.5) into (5.1), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (u-k_{n+1})_+^2\xi_n(\tau)dx + a_0\int_0^\tau \int_\Omega |\nabla(u-k_{n+1})_+|^{p(x,t)}\xi_n(t)\,dx\,dt \\
&+ \frac{b_0}{2}\int_0^\tau \int_\Omega u^{p(x,t)-1}(u-k_{n+1})_+\xi_n(t)\,dx\,dt \\
&\le \frac{1}{2}\int_\Omega (u-k_{n+1})_+^2\xi_n(0)dx + \frac{1}{2}\int_0^\tau \int_\Omega (u-k_{n+1})_+^2\frac{d\xi_n(t)}{dt}\,dx\,dt \\
&\quad + \frac{b_0}{4}\int_0^\tau \int_\Omega (u-k_{n+1})_+^{p(x,t)}\xi_n(t)\,dx\,dt + C\int_0^\tau \int_\Omega \xi_n(t)\chi(u>k_{n+1})\,dx\,dt.
\end{aligned}
$$

Since $\xi_n(0)=0$ and $u\ge (u-k_{n+1})_+$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (u-k_{n+1})_+^2\xi_n(\tau)dx + a_0\int_0^\tau \int_\Omega |\nabla(u-k_{n+1})_+|^{p(x,t)}\xi_n(t)\,dx\,dt \\
&+ \frac{b_0}{4}\int_0^\tau \int_\Omega (u-k_{n+1})_+^{p(x,t)}\xi_n(t)\,dx\,dt \\
&\le \frac{1}{2}\int_0^\tau \int_\Omega (u-k_{n+1})_+^2\frac{d\xi_n(t)}{dt}\,dx\,dt + C\int_0^\tau \int_\Omega \xi_n(t)\chi(u>k_{n+1})\,dx\,dt.
\end{aligned}
$$

Using $|\nabla(u-k_{n+1})_+|^{p^-} \le |\nabla(u-k_{n+1})_+|^{p(x,t)}+1, |(u-k_{n+1})_+|^{p^-} \le |(u-k_{n+1})_+|^{p(x,t)}+1$ and $|\frac{d\xi_n(t)}{dt}| \le \frac{1}{t_{n+1}-t_n}$, we obtain

$$
\begin{aligned}
&\sup_{t_{n+1}<\tau<t^*}\int_\Omega (u-k_{n+1})_+^2dx + \int_{t_{n+1}}^{t^*}\int_\Omega |\nabla(u-k_{n+1})_+|^{p^-}\,dx\,dt \\
&+ \int_{t_{n+1}}^{t^*}\int_\Omega (u-k_{n+1})_+^{p^-}\,dx\,dt \\
&\le C\Big(\frac{2^{n+1}}{\sigma t^*}\int_{t_n}^{t^*}\int_\Omega (u-k_{n+1})_+^2\,dx\,dt + \int_{t_n}^{t^*}\int_\Omega \chi(u>k_{n+1})\,dx\,dt\Big).
\end{aligned} \tag{5.6}
$$

Denote $\delta = \frac{(N+2)p^-}{N}$, then $\delta>2$. Since

$$
\begin{aligned}
\int_{t_n}^{t^*}\int_\Omega \chi(u>k_{n+1})\,dx\,dt &= \big(\frac{2^{n+1}}{k}\big)^\delta \int_{t_n}^{t^*}\int_\Omega ((k_{n+1}-k_n))^\delta\chi(u>k_{n+1})\,dx\,dt \\
&\le \big(\frac{2^{n+1}}{k}\big)^\delta \int_{t_n}^{t^*}\int_\Omega (u-k_n)_+^\delta\chi(u>k_{n+1})\,dx\,dt
\end{aligned}
$$

$$\leq \big(\frac{2^{n+1}}{k}\big)^{\delta}\int_{t_n}^{t^*}\int_\Omega (u-k_n)_+^{\delta}\,dx\,dt,$$

Then Hölder's inequality implies

$$\begin{aligned}
&\int_{t_n}^{t^*}\int_\Omega (u-k_{n+1})_+^2\,dx\,dt\\
&\leq \Big(\int_{t_n}^{t^*}\int_\Omega (u-k_{n+1})_+^{\delta}\,dx\,dt\Big)^{2/\delta}\Big(\int_{t_n}^{t^*}\int_\Omega \chi(u>k_{n+1})\,dx\,dt\Big)^{1-\frac{2}{\delta}}\\
&\leq \big(\frac{2^{n+1}}{k}\big)^{\delta-2}\int_{t_n}^{t^*}\int_\Omega (u-k_{n+1})_+^{\delta}\,dx\,dt.
\end{aligned}$$

Combining these two inequalities with (5.6), we obtain

$$\begin{aligned}
&\sup_{t_{n+1}<\tau<t^*}\int_\Omega (u-k_{n+1})_+^2 dx+\int_{t_{n+1}}^{t^*}\int_\Omega |\nabla(u-k_{n+1})_+|^{p^-}\,dx\,dt\\
&+\int_{t_{n+1}}^{t^*}\int_\Omega (u-k_{n+1})_+^{p^-}\,dx\,dt \\
&\leq C\Big(\frac{2^{n\delta}}{\sigma t^* k^{\delta-2}}+\frac{2^{n\delta}}{k^{\delta}}\Big)\int_{t_n}^{t^*}\int_\Omega (u-k_n)_+^{\delta}\,dx\,dt.
\end{aligned}\tag{5.7}$$

Set

$$Y_n=\frac{\int_{t_n}^{t^*}\int_\Omega (u-k_n)_+^{\delta}\,dx\,dt}{|Q_{t_n}^{t^*}|}$$

where $Q_{t_n}^{t^*}=\Omega\times(t_n,t^*)$ and $|Q_{t_n}^{t^*}|$ denotes the Lebesgue measure of $Q_{t_n}^{t^*}$. By (5.7) and Lemma 5.2, we obtain

$$\begin{aligned}
Y_{n+1}&\leq C\Big(\frac{2^{n\delta}}{\sigma t^* k^{1-\frac{2}{\delta}}}+\frac{2^{n\delta}}{k^{\delta}}\Big)^{1+\frac{p^-}{N}}\frac{|Q_{t_n}^{t^*}|^{1+\frac{p^-}{N}}}{|Q_{t_{n+1}}^{t^*}|}Y_n^{1+\frac{p^-}{N}}\\
&\leq C\Big(\frac{2^{n\delta}}{\sigma t^* k^{\delta-2}}+\frac{2^{n\delta}}{k^{\delta}}\Big)^{1+\frac{p^-}{N}}|Q_0^{t^*}|^{\frac{p^-}{N}}Y_n^{1+\frac{p^-}{N}}.
\end{aligned}$$

Assume that $k\geq 1$, then

$$Y_{n+1}\leq C\Big(1+\frac{1}{\sigma t^*}\Big)^{1+\frac{p^-}{N}}\Big(\frac{2^{n\delta}}{k^{\delta-2}}\Big)^{1+\frac{p^-}{N}}|Q_0^{t^*}|^{\frac{p^-}{N}}Y_n^{1+\frac{p^-}{N}}.$$

Choosing k such that

$$\begin{aligned}
Y_0&=\frac{\int_0^{t^*}\int_\Omega u^{\delta}\,dx\,dt}{|Q_0^{t^*}|}\\
&=C^{-\frac{N}{p^-}}(1+\frac{1}{\sigma t^*})^{-1-\frac{N}{p^-}}|Q_0^{t^*}|^{-1}k^{(\delta-2)(1+\frac{N}{p^-})}2^{-\delta(1+\frac{p^-}{N})(\frac{N}{p^-})^2},
\end{aligned}$$

and using Lemma 5.1, we obtain that $Y_n\to 0$ as $n\to\infty$. Since

$$\int_{\sigma t^*}^{t^*}\int_\Omega (u-k_n)_+^{\delta}\,dx\,dt\leq \int_{t_n}^{t^*}\int_\Omega (u-k_n)_+^{\delta}\,dx\,dt,$$

$(u-k_n)_+^{\delta}\leq u^{\delta}$ and $(u-k_n)_+^{\delta}\to(u-k)_+^{\delta}$ as $n\to\infty$, we have $\int_{\sigma t^*}^{t^*}\int_\Omega (u-k)_+^{\delta}\,dx\,dt=0$ by employing the Lebesgue dominated convergence theorem. Hence it immediately follows that

$$\begin{aligned}&\operatorname{ess\,sup}_{(x,t)\in\Omega\times(\sigma t^*,t^*)} u(x,t)\\ &\le \max\Big\{1, C\Big(\frac{\int_{Q_0^{t^*}} u^\delta\,dx\,dt}{|Q_0^{t^*}|}\Big)^{\frac{p^-}{(\delta-2)(N+p^-)}}\Big(t^{*\frac{p^-}{N+p^-}}+\frac{1}{\sigma}t^{*-\frac{N}{N+p^-}}\Big)^{\frac{1}{\delta-2}}\Big\}.\end{aligned}$$

This completes the proof. □

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