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OSCILLATORY SOLUTIONS OF THE CAUCHY PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We consider the Cauchy problem for second and third order linear differential equations with constant complex coefficients. We describe necessary and sufficient conditions on the data for the existence of oscillatory solutions. It is known that in the case of real coefficients the oscillatory behavior of solutions does not depend on initial values, but we show that this is no longer true in the complex case: hence in practice it is possible to control oscillatory behavior by varying the initial conditions. Our Proofs are based on asymptotic analysis of the zeros of solutions, represented as linear combinations of exponential functions.

1. Introduction

A solution to a differential equation is said to be *oscillatory* if it has an unbounded infinite sequence of zeros within some interval (t_0, ∞), and *nonoscillatory* otherwise. Since the choice of t_0 does not affect the determination of whether or not a solution is oscillatory, we suppress it in Definition 2.1 below.

In the case where the equation has real coefficients, the theory of oscillatory solutions is well-developed [1, 7, 3], and mostly based on Sturm's famous comparison theorems. However, the case in which the coefficients are complex has not been studied very much, both because there are not many immediate physical applications of such equations (except the Dirac equation) and because Sturm's comparison theorems no longer apply.

In the complex coefficient case the oscillatory behavior of solutions depends not only on the roots of the characteristic equation but also on initial values. So, in applications it is possible to control the appearance of oscillatory behavior by setting appropriate initial conditions.

In this article we study which initial values lead to oscillatory solutions and which do not. The main result of the manuscript is the description of the initial values and the roots of characteristic equations that produce oscillatory solutions in the complex case.

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We think that the analysis of the complex case, even in the simplest situation where coefficients are constant, will give us a better understanding of real coefficients case as well. For example, appearance of oscillatory solutions is connected with a new algebraic condition (see (2.17)) which does not appear anywhere obvious in the study of the real case.

Proofs are based on analysis of the zeros of the linear combinations of various exponential functions. Note that asymptotic behavior of zeros of the sums of exponential functions have been studied in the classical paper of Langer and others [6, 2, 8].

Some oscillation theorems for linear differential equations with complex variable coefficients are proved in [4, 5] by using the asymptotic theory.

We are grateful to the referee for several helpful observations, including pointing out Lemma 2.2 to us.

2. Differential equations with complex constant coefficients

2.1. **Notation and a preliminary results.** For $z \in \mathbb{C}$, write $\Re[z]$ and $\Im[z]$ for (respectively) the real and imaginary parts of z.

Definition 2.1. A solution to a differential equation is said to be *oscillatory* if it possesses an infinite sequence of real zeros whose limit is ∞.

Since this is a fact we will use frequently, we emphasize that Definition 2.1 requires that an oscillatory solution must possess real zeros of arbitrarily large magnitude. Given a particular oscillatory solution u, we denote by $\{t_k : k \geq 1\}$ an unbounded increasing sequence of its zeros, so $u(t_i) = 0$ and $t_i < t_j$ for all $1 \leq i < j$.

For a given linear differential equation of order n, write $\lambda_1, \dots, \lambda_n$ for the roots of its characteristic polynomial, and write

$$x_i = \Re[\lambda_i], \quad y_i = \Im[\lambda_i], \quad \lambda_{ij} = \lambda_i - \lambda_j, \quad x_{ij} = \Re[\lambda_{ij}], \quad y_{ij} = \Im[\lambda_{ij}]$$

for all $1 \leq i, j, \leq n$.

We are going to use an important general result connected to the zeros of the solutions of linear ordinary differential equation with constant complex coefficients.

Lemma 2.2. *A linear n-th order ordinary differential equation with constant complex coefficients has a nontrivial solution with infinitely many zeros if and only if it has two distinct characteristic roots with equal real parts.*

This result follows from standard facts about the asymptotic zero distribution of exponential sums (see [6, 8]).

2.2. **Second order equations.**

Theorem 2.3. *A nontrivial solution of the initial-value problem*

$$\begin{gathered} u''(t) + au'(t) + bu(t) = 0, \\ u(0) = d_0, \quad u'(0) = d_1, \quad d_0, d_1 \in \mathbb{R}, \quad a, b \in \mathbb{C} \end{gathered} \tag{2.1}$$

is oscillatory on $(0, \infty)$ if and only if either

$$x_{12} = 0, \quad d_0 \Im[\lambda_1 + \lambda_2] = 0, \tag{2.2}$$

or the discriminant

$$D_2 = a^2 - 4b = \lambda_{12}^2 \tag{2.3}$$

is real and negative and

$$d_0\Im[a] = 0. \tag{2.4}$$

If the coefficients a and b are real, then (2.4) is automatically satisfied and the oscillatory behavior of the solutions does not depend on the initial conditions. However, the following example shows that this is not true for general complex coefficients.

Example 2.4. The solution of

$$u''(t) + (1+2i)u'(t) + iu(t) = 0, \quad u(0) = 0, \quad u'(0) = 1 \tag{2.5}$$

is oscillatory but the solution of

$$u''(t) + (1+2i)u'(t) + iu(t) = 0, \quad u(0) = 1, \quad u'(0) = 0 \tag{2.6}$$

is nonoscillatory. Specifically,

$$u_1(t) = \frac{2}{\sqrt{3}} e^{-\frac{t}{2}-it} \sin(t\frac{\sqrt{3}}{2})$$

is the oscillatory solution of (2.5) and

$$u_2(t) = \frac{1}{\sqrt{3}} e^{-\frac{t}{2}-it} \Big(\sqrt{3}\cos(t\frac{\sqrt{3}}{2}) + (2i+1)\sin(t\frac{\sqrt{3}}{2})\Big)$$

is the nonoscillatory solution of (2.6).

2.3. **Third order equations.** Now we consider the initial-value problem

$$\begin{gathered} u'''(t) + 3I_1u'(t) + 2I_2u(t) = 0, \\ u(0) = d_0, \quad u'(0) = d_1, \quad u''(0) = d_2, \end{gathered} \tag{2.7}$$

where $d_0, d_1, d_2 \in \mathbb{R}$, and $I_1, I_2 \in \mathbb{C}$.

Since the characteristic polynomial associated to (2.7) is reduced, it will always be the case that

$$\sum_{i=1}^{n} \lambda_i = 0. \tag{2.8}$$

Write

$$D_3 = -I_1^3 - I_2^2 = \lambda_{12}^2\lambda_{13}^2\lambda_{23}^2 \tag{2.9}$$

for the associated discriminant, and order the roots $\lambda_1, \lambda_2, \lambda_3$ in some way so that

$$x_1 \le x_2 \le x_3 \tag{2.10}$$

Theorem 2.5. *A nontrivial solution to* (2.7) *is oscillatory on* $(0,\infty)$ *if and only if one of the following conditions is satisfied:*

$$x_1 = x_2 < 0 < x_3, \quad d_1 = d_0x_1, \quad d_2 = (x_1^2 - y_1^2)d_0, \quad \lambda_2 = \overline{\lambda}_1, \tag{2.11}$$

or

$$x_1 < x_2 = x_3, \quad |\lambda_{13}k_2| = |\lambda_{12}k_3|, \tag{2.12}$$

or

$$x_1 = x_2 = x_3 = 0,$$

there exists a sequence of distinct natural numbers $\{m_k\}_{k=1}^{\infty}$ *such that*

$$\frac{y_{13}}{y_{23}} \in \mathbb{Q}, \quad \frac{y_{13}}{y_{23}}(m_k + \varphi \mp \varphi_0) - \psi \pm \psi_0 \in Z, \tag{2.13}$$

and

$$|k_3y_{12}|+|k_2y_{13}|\geq|k_1y_{23}|,\quad -|k_1y_{23}|\leq|k_3y_{12}|-|k_2y_{13}|\leq|k_1y_{23}|, \tag{2.14}$$

where

$$\begin{gathered}
k_1=d_2-d_1(\lambda_2+\lambda_3)+d_0\lambda_2\lambda_3,\quad k_2=d_2-d_1(\lambda_1+\lambda_3)+d_0\lambda_1\lambda_3,\\
k_3=d_2-d_1(\lambda_1+\lambda_2)+d_0\lambda_1\lambda_2,\\
\varphi=\frac{1}{2\pi}\cos^{-1}\Big(\pm\frac{|k_1y_{23}|^2-|k_3y_{12}|^2-|k_2y_{13}|^2}{2|k_2k_3y_{12}y_{13}|}\Big),\\
\psi=\frac{1}{2\pi}\cos^{-1}\Big(\pm\frac{|k_2y_{13}|^2-|k_1y_{23}|^2-|k_3y_{12}|^2}{2|k_1k_3y_{12}y_{23}|}\Big),\\
\varphi_0=-\frac{1}{2\pi}\sin^{-1}\Big(\Im\big[\frac{k_3|k_2|}{k_2|k_3|}\big]\Big),\quad \psi_0=-\frac{1}{2\pi}\sin^{-1}\Big(\Im\big[\frac{k_3|k_1|}{k_1|k_3|}\big]\Big). \qquad (2.15)
\end{gathered}$$

Note that condition (2.14) describes the region of initial data that produce the oscillatory solutions. Condition (2.13) is similar to the condition that the roots λ_j are *commensurable* (see [6]), that is $\lambda_j=\alpha p_j$, for some $\alpha\in\mathbb{C}, p_j\in\mathbb{Z}$. For special initial values the conditions of Theorem 2.5 may be simplified.

Theorem 2.6. *A nontrivial solution to* (2.7) *with* $d_0=d_1=0$ *is oscillatory if and only if one of the following conditions is satisfied:*

$$\lambda_1=x_1<0<x_2=x_3, \tag{2.16}$$

or

$$x_1=x_2=x_3=0,\quad \frac{y_{13}}{y_{23}}\in\mathbb{Q}. \tag{2.17}$$

Furthermore, (2.16),(2.17) *are equivalent, respectively, to*

$$\Im[I_1]=0,\quad \Im[I_1^3+I_2^2]=0,\quad I_1^3+I_2^2>0, \tag{2.18}$$

$$\Re[I_2]=\Im[D_3]=0,\quad \Re[D_3]<0,\quad \frac{\sqrt{3}\Im\big[\big(-I_2+\sqrt{I_1^3+I_2^2}\big)^{1/3}\big]}{\Re\big[\big(-I_2+\sqrt{I_1^3+I_2^2}\big)^{1/3}\big]}\in\mathbb{Q}. \tag{2.19}$$

Example 2.7. The solutions to

$$u'''(t)+2I_2u(t)=0,\quad u(0)=u'(0)=0,\quad u''(0)=1,\quad I_2\neq0 \tag{2.20}$$

are nonoscillatory since $I_1=0$ and the conditions $\Re[I_2]=0$, $\Re[I_2^2]>0$ are never satisfied.

Example 2.8. The solutions to

$$u'''(t)+3(a+ib)u'(t)=0,\quad u(0)=u'(0)=0,\quad u''(0)=1 \tag{2.21}$$

are oscillatory if

$$\begin{gathered}
\Im[I_1^3+I_2^2]=3a^2b-b^3=0,\quad \Re[I_1^3+I_2^2]=a^3-3ab^2>0,\\
\frac{\sqrt{3}b}{a}\in\mathbb{Q}.
\end{gathered} \tag{2.22}$$

These conditions are satisfied if, for example, $b=0$, $I_1=a>0$ or

$$a=\sqrt{3},\quad b=3,\quad I_1=\sqrt{3}+3i.$$

Theorem 2.9. *A nontrivial solution to* (2.7) *with* $d_0 = d_2 = 0$ *is oscillatory if and only if one of the following conditions is satisfied*

$$\lambda_1 = x_1 < 0 < x_2 = x_3, \quad I_1, I_2 \in \mathbb{R}, \tag{2.23}$$

or

$$0 < x_2 = x_3, \quad 6x_2^2 + y_{12}^2 = 3y_2^2, \tag{2.24}$$

or

$$x_1 = x_2 = x_3 = 0, \quad \frac{y_{13}}{y_{23}} \in \mathbb{Q}. \tag{2.25}$$

Theorem 2.10. *A nontrivial solution to* (2.7) *with* $d_1 = d_2 = 0$ *is oscillatory if and only if*

$$\lambda_1 = x_1 < 0 < x_2 = x_3, \quad I_1, I_2 \in \mathbb{R}, \tag{2.26}$$

or equivalently

$$I_1, I_2 \in \mathbb{R}, \quad I_2 < 0, \quad D_3 < 0. \tag{2.27}$$

3. Proofs

Proof of Theorem 2.3. Note that we may assume that

$$D_2 = a^2 - 4b = \lambda_{12}^2 \neq 0, \tag{3.1}$$

since otherwise there is no distinct roots of the characteristic polynomial and in view of Lemma 2.2 there are no nontrivial oscillatory solutions of (2.1).

The solutions to (2.1) where $D_2 \neq 0$ are given by the formula

$$u(t) = \frac{(d_1 - d_0\lambda_2)e^{t\lambda_1} - (d_1 - d_0\lambda_1)e^{t\lambda_2}}{\lambda_{12}}. \tag{3.2}$$

The zeros of (3.2) satisfy

$$(d_1 - d_0\lambda_2)e^{t\lambda_1} = (d_1 - d_0\lambda_1)e^{t\lambda_2};$$

that is,

$$e^{t\lambda_{12}} = \frac{d_1 - d_0\lambda_1}{d_1 - d_0\lambda_2}. \tag{3.3}$$

Note that we may also assume that

$$(d_1 - d_0\lambda_1)(d_1 - d_0\lambda_2) \neq 0, \tag{3.4}$$

otherwise (3.2) is nonoscillatory.

If $x_{12} > 0$ then the left-hand side of (3.3) becomes unboundedly large as $t \to \infty$, so this is impossible in view of (3.4). On the other hand, if $x_{12} < 0$ then the left-hand side of (3.3) approaches 0 as $t \to \infty$ and this is impossible as well. Consequently $x_{12} = 0$.

Solving (3.3) for t, we obtain

$$t = \frac{\ln \frac{d_1 - d_0\lambda_1}{d_1 - d_0\lambda_2}}{\lambda_{12}} = \frac{\ln |\frac{d_1 - d_0\lambda_1}{d_1 - d_0\lambda_2}| + i \arg\big(\frac{d_1 - d_0\lambda_1}{d_1 - d_0\lambda_2}\big)}{iy_{12}}$$

In order for (3.2) to be oscillatory, we need an infinite number of these t to lie in $\mathbb{R}$, which happens if and only if

$$x_{12} = 0, \quad |d_1 - d_0\lambda_1| = |d_1 - d_0\lambda_2|. \tag{3.5}$$

So we have the following infinite sequence of zeros:

$$t_k = \frac{2ki\pi}{\lambda_{12}} = \frac{2k\pi}{\Im[\lambda_{12}]}, \quad k \in \mathbb{Z}.$$

Simplifying (3.5) we have

$$x_{12} = 0, \quad d_0^2(y_1^2 - y_2^2) = 0.$$

and hence (since $y_{12} = x_{12} = 0$ would imply $D_2 = 0$, which is not the case)

$$x_{12} = 0, \quad d_0(y_1 + y_2) = 0. \tag{3.6}$$

Theorem 2.3 will now follow from the next two lemmas. □

Lemma 3.1.

$$\Re\left[\sqrt{m + in}\right] = 0, \quad m, n \in \mathbb{R}$$

if and only if

$$n = 0, \quad m \le 0.$$

Proof. From the well known formula

$$\sqrt{2(m + in)} = \sqrt{\sqrt{m^2 + n^2} + m} + i\,\mathrm{sign}(n)\sqrt{\sqrt{m^2 + n^2} - m} \tag{3.7}$$

we obtain

$$\Re\left[\sqrt{m + in}\right] = \frac{1}{\sqrt{2}}\sqrt{\sqrt{m^2 + n^2} + m}$$

which equals 0 if and only if $n = 0$, $m \le 0$. □

Lemma 3.2. $\Re[\lambda_{12}] = 0$ *if and only if*

$$\Im[D_2] = 0, \quad D_2 = a^2 - 4b \le 0. \tag{3.8}$$

Proof. As they are the roots of $\lambda^2 + a\lambda + b = 0$, λ_1 and λ_2 are given by the quadratic formula

$$\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{m + in}}{2}, \quad \lambda_{12} = \sqrt{m + in}, \quad m = \Re[D_2], \quad n = \Im[D_2].$$

Applying Lemma 3.1 we obtain

$$x_{12} = \Re\big[\sqrt{m + in}\big] = 0$$

if and only if $n = \Im[D_2] = 0, \quad m = \Re[D_2] \le 0.$ □

Proof of Theorem 2.5. First consider the case $D_3 = 0$. There are two possibilities: $I_1 = 0$ and $I_1 \ne 0$. Since in the case

$$D_3 = -I_1^3 - I_2^2 = 0, \quad I_1 = 0 \tag{3.9}$$

the equation $u'''(t) = 0$ has nonoscillatory nontrivial solutions $u = C_1 + C_2 t + C_3 t^2$, it is sufficient to consider the case

$$D_3 = \lambda_{12}^2 \lambda_{13}^2 \lambda_{23}^2 = 0, \quad I_1 \ne 0. \tag{3.10}$$

In this case there is one repeated root – for convenience, we assume $\lambda_2 = \lambda_3$. (In principle this involves a loss of generality as the λ_i are ordered, but we will not use the ordering in what follows.) So $\lambda_2 = \lambda_3 \ne 0$, $3I_1 = \lambda_2(2\lambda_1 + \lambda_2) \ne 0$, and the solutions of $u'''(t) + 3I_1 u'(t) + 2I_2 u(t) = 0$ are given by

$$u(t) = C_1 e^{t\lambda_1} + C_2 e^{t\lambda_2} + C_3 t e^{t\lambda_2}.$$

From the initial conditions

$$C_1 + C_2 = d_0, \quad C_1\lambda_1 + C_2\lambda_2 + C_3 = d_1, \quad C_1\lambda_1^2 + C_2\lambda_2^2 + 2C_3\lambda_2 = d_2,$$

we obtain

$$C_1 = \frac{d_2 + d_0\lambda_2^2 - 2d_1\lambda_2}{\lambda_{12}^2}, \quad C_2 = \frac{d_0\lambda_1^2 + 2d_1\lambda_2 - d_2 - 2d_0\lambda_1\lambda_2}{\lambda_{12}^2},$$
$$C_3 = -\frac{k_3}{\lambda_{12}}, \quad k_3 = d_2 - d_1(\lambda_1 + \lambda_2) + d_0\lambda_1\lambda_2,$$

and hence the solution is

$$u(t) = \frac{(d_2 + d_0\lambda_2^2 - 2d_1\lambda_2)e^{t\lambda_1} + (d_0\lambda_1(\lambda_1 - 2\lambda_2) + 2d_1\lambda_2 - d_2)e^{t\lambda_2}}{\lambda_{12}^2} - \frac{k_3te^{t\lambda_2}}{\lambda_{12}}. \tag{3.11}$$

The zeros of (3.11) satisfy

$$\frac{(d_2 - 2d_1\lambda_2 + d_0\lambda_2^2)}{t\lambda_{12}^2}e^{t\lambda_{12}} + \frac{d_0\lambda_1(\lambda_1 - 2\lambda_2) + 2d_1\lambda_2 - d_2}{t\lambda_{12}^2} = \frac{k_3}{\lambda_{12}};$$

that is,

$$\frac{d_2 - 2d_1\lambda_2 + d_0\lambda_2^2}{t\lambda_{12}}e^{t\lambda_{12}} = k_3 + \frac{d_2 - 2d_1\lambda_2 - d_0\lambda_1(\lambda_1 - 2\lambda_2)}{t\lambda_{12}} \tag{3.12}$$

From Lemma 2.2 it follows that if (2.7) is oscillatory then the distinct roots λ_1, λ_2 have equal real parts, that is $x_{12} = 0$.

On the other hand, if $x_{12} = 0$ then the left-hand side of (3.12) approaches 0 as $t \to \infty$, so for (3.11) to be oscillatory we must have $k_3 = d_2 - d_1(\lambda_1 + \lambda_2) + d_0\lambda_1\lambda_2 = 0$.

Further from (3.12)

$$e^{t\lambda_{12}} = \frac{d_2 - 2d_1\lambda_2 - d_0\lambda_1(\lambda_1 - 2\lambda_2)}{d_2 - 2d_1\lambda_2 + d_0\lambda_2^2} = \frac{d_1\lambda_{12} - d_0\lambda_1\lambda_{12}}{d_1\lambda_{12} - d_0\lambda_2\lambda_{12}} = \frac{d_1 - d_0\lambda_1}{d_1 - d_0\lambda_2}$$

which is impossible as $t \to \infty$ unless $x_{12} = 0$, that is $x_1 = x_2 = x_3 = 0$.

Since $d_0, d_1, d_2 \in \mathbb{R}$, from $k_3 = 0, \Im[k_3] = 0$ we obtain

$$d_1(y_1] + y_2) = d_0x_1(y_1 + y_2)$$

and since $\lambda_1 + \lambda_2 + \lambda_3 = 0, \lambda_2 = \lambda_3$, and $x_{12} = 0$ it follows that $y_1 + y_2 \neq 0$. Then

$$d_1 = d_0x_1, \quad |e^{t\lambda_{12}}| = |\frac{y_1}{y_2}| = 1$$

yielding $y_1 = \pm y_2$, which is a contradiction (since we know $y_1 + y_2 \neq 0$ and $\lambda_{12} \neq 0$). So: there are no oscillatory solutions in the case $D_3 = 0, I_1 \neq 0$.

In the case $D_3 = \lambda_{12}^2\lambda_{13}^2\lambda_{23}^2 \neq 0$ (λ_1, λ_2, λ_3 are distinct) the solutions of (2.7) are given by $u(t) = C_1e^{t\lambda_1} + C_2e^{t\lambda_2} + C_3e^{t\lambda_3}$, and in view of the initial conditions we have

$$u(t) = \frac{k_1\lambda_{23}e^{t\lambda_1} - k_2\lambda_{13}e^{t\lambda_2} + k_3\lambda_{12}e^{t\lambda_3}}{\lambda_{12}\lambda_{13}\lambda_{23}} \tag{3.13}$$

where, as in (2.15),

$$k_1 = d_2 - d_1(\lambda_2 + \lambda_3) + d_0\lambda_2\lambda_3, \quad k_2 = d_2 - d_1(\lambda_1 + \lambda_3) + d_0\lambda_1\lambda_3,$$
$$k_3 = d_2 - d_1(\lambda_1 + \lambda_2) + d_0\lambda_1\lambda_2$$

The zeros of (3.13) satisfy

$$k_1\lambda_{23}e^{t\lambda_1} - k_2\lambda_{13}e^{t\lambda_2} + k_3\lambda_{12}e^{t\lambda_3} = 0;$$

that is,

$$k_1\lambda_{23}e^{t\lambda_{13}} - k_2\lambda_{13}e^{t\lambda_{23}} + k_3\lambda_{12} = 0. \tag{3.14}$$

In view of assumption (2.10) it is sufficient to consider the following three cases:

$$x_1 \le x_2 < x_3, \tag{3.15}$$

$$x_1 < x_2 = x_3, \tag{3.16}$$

$$x_1 = x_2 = x_3. \tag{3.17}$$

From (2.8) we have

$$x_1 + x_2 + x_3 = 0, \quad y_1 + y_2 + y_3 = 0. \tag{3.18}$$

First we consider case (3.15). If (3.13) is oscillatory (that is, (3.14) is true for arbitrarily large values of t) then by letting $t \to \infty$ in (3.14) we obtain $k_3 = 0$ and

$$k_1 = k_1 - k_3 = \lambda_{13}(d_1 - d_0\lambda_2), \quad k_2 = k_2 - k_3 = \lambda_{23}(d_1 - d_0\lambda_1).$$

Further, from (3.14) we have

$$e^{t\lambda_{12}} = \frac{k_2\lambda_{13}}{k_1\lambda_{23}},$$

which gives the infinite sequence of zeros

$$t_n = \frac{1}{y_{12}} \sin^{-1}\Big(\Im\big[\frac{k_2\lambda_{13}}{k_1\lambda_{23}}\big]\Big) + \frac{n\pi}{y_{12}}$$

provided that either

$$\big|\frac{k_2\lambda_{13}}{k_1\lambda_{23}}\big| = 1, \quad k_3 = 0, \quad x_1 = x_2 < x_3$$

or

$$\big|\frac{d_1 - d_0\lambda_1}{d_1 - d_0\lambda_2}\big| = 1, \quad x_1 = x_2, \quad d_2 = d_1(\lambda_1 + \lambda_2) - d_0\lambda_1\lambda_2, \quad x_2 < x_3.$$

Now, since the d_j are real, from $k_3 = 0$ we obtain

$$\begin{gathered} d_2 - d_1(x_1 + x_2) + d_0(x_1x_2 - y_1y_2) = 0, \\ -d_1(y_1 + y_2) + d_0(x_1y_2 + y_1x_2) = 0, \\ \frac{d_1}{d_0} = \frac{x_1y_2 + y_1x_2}{y_1 + y_2} = x_1, \\ \frac{d_2}{d_0} = y_1y_2 - x_1x_2 + (x_1 + x_2)\frac{d_1}{d_0} = y_1y_2 + x_1x_2 = y_1y_2 + x_1^2 \end{gathered}$$

that is condition (2.11) of Theorem 2.5:

$$\big|\frac{x_1 - \lambda_1}{x_2 - \lambda_2}\big| = 1 \quad \text{or} \quad |y_1| = |y_2|, \quad y_2 = -y_1,$$

$$d_1 = x_1 d_0 = d_0 x_1, \quad d_2 = (x_1^2 + y_1y_2)d_0, \quad \lambda_2 = \overline{\lambda}_1, \quad x_1 = x_2 < 0 < x_3. \tag{3.19}$$

We move to the next case (3.16). Now the left-hand side of (3.14) approaches

$$-k_2\lambda_{13}e^{t\lambda_{23}} + k_3\lambda_{12} = 0$$

as $t \to \infty$, so if (3.13) is going to be oscillatory we must have $k_2\lambda_{13}e^{t\lambda_{23}} = k_3\lambda_{12}$ for certain arbitrarily large values of t. Since we are in case (3.16), we know that $x_{23} = 0$, so $\lambda_{23} = iy_{23}$, and also $x_{12} = x_{13}$. So in fact

$$e^{ity_{23}} = \frac{k_3\lambda_{12}}{k_2\lambda_{13}}, \quad \big|\frac{k_3\lambda_{12}}{k_2\lambda_{13}}\big| = |e^{ity_{23}}| = 1, \tag{3.20}$$

and we obtain condition (2.12) of Theorem 2.5:

$$|k_2\lambda_{13}| = |k_3\lambda_{12}|, \quad x_1 < x_2 = x_3.$$

Finally, we turn to case (3.17), in which $x_1 = x_2 = x_3$ (so by (3.18) they are all zero) and so (3.14) can be written

$$-k_1 y_{23} e^{ity_{13}} + k_2 y_{13} e^{ity_{23}} = k_3 y_{12}. \tag{3.21}$$

Denoting

$$b_1 = k_3 y_{12}, \quad b_2 = -k_1 y_{23}, \quad b_3 = k_2 y_{13}, \quad v = ty_{13}, \quad w = ty_{23}. \tag{3.22}$$

we obtain, from (3.21),

$$b_2 e^{iv} + b_3 e^{iw} = b_1, \quad b_1, b_2, b_3 \in \mathbb{C}. \tag{3.23}$$

Lemma 3.3. *The exponential equation* (3.23) *has solution* $(v, w) \in \mathbb{R} \times \mathbb{R}$ *if and only if*

$$|b_1| + |b_3| \geq |b_2|, \quad -|b_2| \leq |b_1| - |b_3| \leq |b_2|. \tag{3.24}$$

These solutions are given by the formulas

$$\begin{aligned} v &= \cos^{-1}\Big(\pm \frac{|b_1|^2 + |b_2|^2 - |b_3|^2}{2|b_1 b_2|}\Big) \mp 2\pi\tilde{\psi}_0, \\ w &= \cos^{-1}\Big(\pm \frac{|b_1|^2 + |b_3|^2 - |b_2|^2}{2|b_1 b_3|}\Big) \mp 2\pi\tilde{\varphi}_0, \end{aligned} \tag{3.25}$$

where

$$\tilde{\varphi}_0 = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im[b_1/b_3]}{|b_1/b_3|}\Big), \quad \tilde{\psi}_0 = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im[b_1/b_2]}{|b_1/b_2|}\Big). \tag{3.26}$$

Remark 3.4. Condition (3.24) is invariant with respect to the substitutions

$$b_j \to b_j e^{i\alpha}, \quad \alpha \in \mathbb{R}.$$

Remark 3.5. If equation (3.23) has a solution $(v_0, w_0) \in \mathbb{R} \times \mathbb{R}$ then it has the infinite sequence of solutions $(v_0 + 2k\pi, w_0 + 2m\pi), k, m \in \mathbb{Z}$.

Proof of Lemma 3.3. Assuming $b_3 \neq 0$ from (3.23) we have $\frac{b_1}{b_3} - e^{iw} = \frac{b_2}{b_3} e^{iv}$, and by taking the absolute value and squaring both sides of this equation we obtain

$$\big|\frac{b_1}{b_3} - e^{iw}\big|^2 = \big|\frac{b_2}{b_3} e^{iv}\big|^2.$$

Since from the definition of $\tilde{\varphi}_0$,

$$\frac{b_1}{b_3} = \pm\big|\frac{b_1}{b_3}\big|\cos(2\pi\tilde{\varphi}_0) - i\big|\frac{b_1}{b_3}\big|\sin(2\pi\tilde{\varphi}_0) = \pm\big|\frac{b_1}{b_3}\big| e^{\mp 2i\pi\tilde{\varphi}_0},$$

we have

$$\begin{gathered} \big| \pm \big|\frac{b_1}{b_3}\big| e^{\mp 2i\pi\tilde{\varphi}_0} - e^{iw}\big|^2 = \big|\frac{b_2}{b_3}\big|^2, \\ \big| \pm |b_1| - |b_3| e^{i(w \pm 2\pi\tilde{\varphi}_0)}\big|^2 = |b_2|^2, \\ \big(\pm |b_1| - |b_3|\cos(w \pm 2\pi\tilde{\varphi}_0)\big)^2 + |b_3|^2 \sin^2(w \pm 2\pi\tilde{\varphi}_0) = |b_2|^2, \end{gathered}$$

or

$$\begin{gathered} \cos(w \pm 2\pi\tilde{\varphi}_0) = \pm\frac{|b_1|^2 + |b_3|^2 - |b_2|^2}{2|b_1 b_3|}, \\ w = \cos^{-1}\Big(\pm \frac{|b_1|^2 + |b_3|^2 - |b_2|^2}{2|b_1 b_3|}\Big) \mp 2\pi\tilde{\varphi}_0, \end{gathered}$$

where
$$\tilde{\varphi}_0 = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im[b_1/b_3]}{|b_1/b_3|}\Big) = \pm\frac{1}{2\pi}\cos^{-1}\Big(\frac{\Re[b_1/b_3]}{|b_1/b_3|}\Big).$$
Note that in the case $b_3 = 0$ the real solution v of (3.23) exists if and only if $|b_2| = |b_3|$. This also follows from (3.24) in that case.

Further the real solution w exists if and only if
$$\big||b_1|^2 + |b_3|^2 - |b_2|^2\big| \le 2|b_1 b_3|.$$
To simplify this inequality one can rewrite it in the form
$$(|b_1| + |b_3|)^2 \ge |b_2|^2, \quad (|b_1| - |b_3|)^2 \le |b_2|^2,$$
and we obtain condition (3.24).

To solve (3.23) with respect to v we apply the substitution
$$b_2 \to b_2 e^{iw-iv}, \quad b_3 \to b_3 e^{iv-iw},$$
and obtain
$$b_2 e^{iw} + b_3 e^{iv} = b_1.$$
By transposition $b_2 \leftrightarrow b_3$ from the formula (3.25) for w we obtain the solution of this equation with respect to v
$$v = \cos^{-1}\Big(\pm\frac{|b_1|^2 + |b_2|^2 - |b_3|^2}{2|b_1 b_2|}\Big) \mp 2\pi\tilde{\psi}_0,$$
where
$$\tilde{\psi}_0 = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im[b_1/b_2]}{|b_1/b_2|}\Big) = \pm\frac{1}{2\pi}\cos^{-1}\Big(\frac{\Re[b_1/b_2]}{|b_1/b_2|}\Big).$$
The real solution v exists if and only if
$$\big||b_1|^2 + |b_2|^2 - |b_3|^2\big| \le 2|b_1 b_2|.$$
It can be shown that this inequality is equivalent to (3.24). □

Remark 3.6. To study the appearance of oscillatory solutions of the Cauchy problem for n-th order ordinary differential equations with constant complex coefficients one should study the existence of real solutions $\{v_j\}_{j=1}^n$ of the exponential equation:
$$\sum_{j=1}^{n-1} b_j e^{iv_j} = b_0, \quad b_k \in \mathbb{C}, \quad k = 0, 1, 2, \dots, n.$$

Note that the asymptotic behavior and distribution of zeros of the sums of more general exponential functions have been studied in [6, 8, 2]. and they have a complicated structure.

Continuing the proof of Theorem 2.5 we apply to equation (3.21) the condition (3.24) of Lemma 3.3, and in view of (3.22) we obtain
$$|k_3 y_{12}| + |k_2 y_{13}| \ge |k_1 y_{23}|, \quad -|k_1 y_{23}| \le |k_3 y_{12}| - |k_2 y_{13}| \le |k_1 y_{23}|,$$
or condition (2.14) of Theorem 2.5.

In view of (3.22) we have also
$$\tilde{\varphi}_0 = \varphi_0 = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im(b_1)}{|b_1|}\Big) = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im(k_3/k_2)}{|k_3/k_2|}\Big),$$
$$\tilde{\psi}_0 = \psi_0 = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im(b_1/b_2)}{|b_1/b_2|}\Big) = -\frac{1}{2\pi}\sin^{-1}\Big(\frac{\Im(k_3/k_1)}{|k_3/k_1|}\Big).$$

From (3.25) we have

$$ty_{13} = \cos^{-1}\Big(\pm\frac{|k_1y_{23}|^2+|k_3y_{12}|^2-|k_2y_{13}|^2}{2|k_1k_3y_{12}y_{23}|}\Big)\mp 2\pi\psi_0,$$
$$ty_{23} = \cos^{-1}\Big(\pm\frac{|k_3^2|y_{12}^2+|k_2^2|y_{13}^2-|k_1^2|y_{23}^2}{2|k_2k_3y_{12}y_{13}|}\Big)\mp 2\pi\varphi_0.$$

The infinite sequence of zeros $\{t_n^*\},\{t_m\}$ is given by

$$y_{13}t_n^* = 2n\pi + 2\pi\psi \mp 2\pi\psi_0, \quad n\in\mathbb{Z},$$
$$y_{23}t_m = 2m\pi + 2\pi\varphi \mp 2\pi\varphi_0, \quad m\in\mathbb{Z},$$

where $\varphi,\psi,\varphi_0,\psi_0$ are as in (2.15).

We claim that this occurs precisely when there exists a sequence of distinct integers $\{m_k\}_{k=1}^\infty$ such that

$$\frac{y_{13}}{y_{23}}\in\mathbb{Q},\quad \frac{y_{13}}{y_{23}}(m_k+\varphi\mp\varphi_0)-\psi\pm\psi_0\in Z.$$

In order for an oscillatory solution to exist, the sequences t_m, t_n^* must coincide infinitely many times. That means that there must exist sequences $\{m_k\}_{k=1}^\infty, \{n_k\}_{k=1}^\infty$ of distinct integers such that

$$t_{m_k} = \frac{2\pi(m_k+\varphi\mp\varphi_0)}{y_{23}} = t_{n_k}^* = \frac{2\pi(n_k+\psi\mp\psi_0)}{y_{13}}$$

or

$$n_k = \frac{y_{13}}{y_{23}}(m_k+\varphi\mp\varphi_0)-\psi\pm\psi_0,\quad n_k-n_j=\frac{y_{13}}{y_{23}}(m_k-m_j),\quad k,j=1,2,\dots$$

and since $\frac{n_k-n_j}{m_k-m_j}\in\mathbb{Q}$, we obtain

$$\frac{y_{13}}{y_{23}}\in\mathbb{Q},\quad \frac{y_{13}}{y_{23}}(m_k+\varphi\mp\varphi_0)-\psi\pm\psi_0\in\mathbb{Z}.$$

We have now exhausted the cases (3.15)–(3.17), which completes the proof. □

Proof of Theorem 2.6. We deduce Theorem 2.6 from Theorem 2.5. Note that case (2.11) is impossible since the condition $d_2=(x_1^2+y_1y_2)d_0$, together with $d_0=d_1=0$, implies $d_2=0$ as well. Note also that since $d_0=d_1=0$ we obtain $k_1=k_2=k_3=d_2$.

From case (2.12) we have $|\lambda_{13}|=|\lambda_{12}|$ or

$$x_{13}^2+y_{13}^2=x_{12}^2+y_{12}^2,$$

Since in this case $x_{13}=x_{12}$ we obtain $y_{13}^2=y_{12}^2$, and in view of (2.8) we obtain $y_1=0$, that is, case (2.16).

Further since $y_{23}^2=(y_{13}-y_{12})^2=y_{12}^2-2y_{12}y_{13}+y_{13}^2$ the formula (2.15) turns into

$$\varphi=\psi=\varphi_0=\psi_0=0 \tag{3.27}$$

and case (2.13) turns into case (2.17).

It only remains to be shown that (2.16),(2.17) are equivalent correspondingly to (2.18) and (2.19). From Vieta's formulas

$$\lambda_1+\lambda_2+\lambda_3=0,\quad 3I_1=\lambda_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3,\quad 2I_2=-\lambda_1\lambda_2\lambda_3,\quad D_3=\lambda_{12}^2\lambda_{13}^2\lambda_{23}^2$$

we obtain

$$I_1,\quad iI_2,\quad D_3\in\mathbb{R},\quad D_3\le 0 \tag{3.28}$$

Conversely from (3.29) the cubic equation with real coefficients

$$\mu^3 - 3I_1\mu + 2iI_2 = 0, \quad \tilde{D}_3 = -(-I_1)^3 - (iI_2)^2 = -D_3 \geq 0 \tag{3.29}$$

has non-negative discriminant, and so it has three real roots; that is, $\Im[\mu_j] = 0$, and by substitution $\mu = -i\lambda$ equation (3.29) turns to the characteristic equation $\lambda^3 + 3I_1\lambda + 2I_2 = 0$. It follows that the characteristic equation has imaginary roots; that is, (2.18) is true.

To rewrite the last part of condition (2.17) we will use the cubic formula. Denote

$$z_1 = -I_2 + \sqrt{-D_3}, \quad z_2 = -I_2 - \sqrt{-D_3}, \quad D_3 = -I_1^3 - I_2^2. \tag{3.30}$$

The roots of $\lambda^3 + 3I_1\lambda + 2I_2 = 0$ are given by the cubic formula:

$$\begin{gathered} \lambda_1 = z_1^{1/3} + z_2^{1/3} = \xi + \eta, \quad \xi = (-I_2 + \sqrt{D})^{1/3}, \\ \eta = (-I_2 - \sqrt{D})^{1/3}, \quad D = I_1^3 + I_2^2, \\ \lambda_2 = -\frac{z_1^{1/3} + z_2^{1/3}}{2} + \frac{i\sqrt{3}(z_1^{1/3} - z_2^{1/3})}{2} = -\frac{\xi + \eta}{2} + \frac{i\sqrt{3}(\xi - \eta)}{2}, \\ \lambda_3 = -\frac{z_1^{1/3} + z_2^{1/3}}{2} - \frac{i\sqrt{3}(z_1^{1/3} - z_2^{1/3})}{2} = -\frac{\xi + \eta}{2} - \frac{i\sqrt{3}(\xi - \eta)}{2}, \end{gathered} \tag{3.31}$$

from which we obtain

$$\begin{gathered} \lambda_{23} = i\sqrt{3}(z_1^{1/3} - z_2^{1/3}) = i\sqrt{3}(\xi - \eta), \\ \lambda_{13} = \frac{3}{2}\left(z_1^{1/3} + z_2^{1/3}\right) + \frac{i\sqrt{3}}{2}\left(z_1^{1/3} - z_2^{1/3}\right) = \frac{3}{2}(\xi + \eta) + \frac{i\sqrt{3}}{2}(\xi - \eta). \end{gathered}$$

Finally, we can use the cubic formula to rewrite the last part of condition (2.18):

$$\begin{gathered} \frac{\Im[\lambda_{13}]}{\Im[\lambda_{23}]} = \frac{\frac{3}{2}\Im[\xi + \eta] + \frac{\sqrt{3}}{2}\Re[\xi - \eta]}{\sqrt{3}\Re[\xi - \eta]} = \frac{\sqrt{3}\Im[\xi + \eta]}{2\Re[\xi - \eta]} + \frac{1}{2} \\ \frac{\Im[\lambda_{13}]}{\Im[\lambda_{23}]} = \frac{\sqrt{3}\Im[z_1^{1/3} + z_2^{1/3}]}{2\Re[z_1^{1/3} - z_2^{1/3}]} + \frac{1}{2} \end{gathered}$$

and we have $z_2 = -\overline{z_1}$, and so

$$\frac{\Im[\lambda_{13}]}{\Im[\lambda_{23}]} = \frac{\sqrt{3}\Im[z_1^{1/3} - \overline{z_1}^{1/3}]}{2\Re[z_1^{1/3} + \overline{z_1}^{1/3}]} + \frac{1}{2} = \frac{\sqrt{3}\Im[z_1^{1/3}]}{2\Re[z_1^{1/3}]} + \frac{1}{2}.$$

It follows that $\frac{\Im[\lambda_{13}]}{\Im[\lambda_{23}]} \in \mathbb{Q}$ if and only if

$$\frac{\sqrt{3}\Im[(-I_2 + \sqrt{-D_3})^{1/3}]}{\Re[(-I_2 + \sqrt{-D_3})^{1/3}]} \in \mathbb{Q}.$$

So we have established that (2.17) is equivalent to (2.19). □

Proof of Theorem 2.9. Again, we deduce Theorem 2.9 from Theorem 2.5. Note that case (2.11) leads to the trivial solution since it follows from $d_0 = d_2 = 0$ that $d_1 = 0$.

The case (2.12), in view of

$$k_1 = -d_1(\lambda_2 + \lambda_3), \quad k_2 = -d_1(\lambda_1 + \lambda_3), \quad k_3 = -d_1(\lambda_1 + \lambda_2),$$

turns into

$$1 = \Big|\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - \lambda_3^2}\Big| = \frac{(x_{12} + iy_{12})(x_1 + x_2 + iy_1 + iy_2)}{(x_{13} + iy_{13})(x_1 + x_3 + iy_1 + iy_3)}.$$

Further,

$$0 = \Big|\frac{(x_{12}+iy_{12})(x_1+x_2+iy_1+iy_2)}{(x_{13}+iy_{13})(x_1+x_3+iy_1+iy_3)}\Big|^2 - 1. \tag{3.32}$$

Noting that we have $x_{12} = x_{13}$, $x_{23} = 0$, and that (from (2.8)) $x_1 = -2x_2$ and $y_3 = -y_1 - y_2$, the right-hand side of (3.32) can now be shown (by some effort) to be equal to

$$\frac{y_1 y_{23}(3y_2^2 - 6x_2^2 - y_{12}^2)}{(9x_2^2 + y_{12}^2)(x_2^2 + y_3^2)}. \tag{3.33}$$

We may assume that $x_2 \neq 0$, because then $x_2 = x_3$ and (2.8) would imply $x_1 = 0$ and we would not be in case (2.12). By assumption, $y_{23} \neq 0$ in this case (otherwise $\lambda_2 = \lambda_3$ and so $D_3 = 0$), so this says that in case (2.12) the solution is oscillatory if and only if either $y_1 = 0$ or $3y_2^2 - 6x_2^2 - y_{12}^2 = 0$. If $y_1 = 0$, then $\lambda_1 \in \mathbb{R}$ and $y_2 = -y_3$, so λ_2 and λ_3 are conjugate to one another, so I_1 and I_2 are real as well. This is the first case (2.23) in the statement of Theorem 2.9, since $x_1 < x_2$ and $x_1 = -2x_2$ imply that $x_1 < 0 < x_2$. On the other hand, if $3y_2^2 - 6x_2^2 - y_{12}^2 = 0$ then we are in the second case (2.24) in the statement of Theorem 2.9.

Finally, we deal with case (2.13), in which $x_1 = x_2 = x_3$ (and all are zero because of (2.8)). Now (2.15) becomes

$$\varphi = \psi = \varphi_0 = \psi_0 = 0$$

so (2.13) turns to (2.25), and this completes the proof. □

Proof of Theorem 2.10. As in the previous two theorems, case (2.11) is impossible. Indeed, in case (2.11) from $d_1 = d_2 = 0$ we obtain $d_0 x_1 = 0$, which yields $d_0 = 0$ since here $x_1 < 0$.

Consider case (2.12). In this case from $d_1 = d_2 = 0$ we obtain

$$k_2 = d_0\lambda_1\lambda_3, \quad k_3 = d_0\lambda_1\lambda_2.$$

so (2.12) turns into

$$|\lambda_{13}\lambda_1\lambda_3| = |\lambda_{12}\lambda_1\lambda_2|, \quad x_1 < x_2 = x_3.$$

If $\lambda_1 = 0$, then (2.8) requires that $x_2 + x_3 = 0$, hence $x_2 = x_3 = 0$ which contradicts the condition $x_1 < x_2$. If $\lambda_1 \neq 0$, then

$$|\lambda_{13}\lambda_3| = |\lambda_{12}\lambda_2|;$$

that is,

$$\Big|\frac{\lambda_2\lambda_{12}}{\lambda_3\lambda_{13}}\Big|^2 = 1.$$

This means that

$$\frac{(x_2^2 + y_2^2)(x_{12}^2 + y_{12}^2)}{(x_3^2 + y_3^2)(x_{13}^2 + y_{13}^2)} - 1 = 0, \quad x_1 < x_2 = x_3. \tag{3.34}$$

In this case we know that $x_2 = x_3$ and so that $x_1 = -2x_2$ and $y_1 + y_2 + y_3 = 0$. Using these facts and performing some calculation, (3.35) becomes

$$-\frac{4y_1(y_1 + 2y_2)(3x_2^2 + y_1^2 + y_1y_2 + y_2^2)}{(x_2^2 + y_1^2 + 2y_1y_2 + y_2^2)(9x_2^2 + 4y_1^2 + 4y_1y_2 + y_2^2)} = 0.$$

If $y_1 = 0$ then λ_1 is real (and negative, by (2.8) and (3.34)), and $y_2 = -y_3$, so I_1 and I_2 are real and we are in the situation given in the statement of the theorem. We claim that this is the only possibility. If $y_1 + 2y_2 = 0$ then (2.8) gives that

$y_2 = y_3$, which in case (3.35) means that $\lambda_2 = \lambda_3$, contradicting the assumption that $D_3 \neq 0$. On the other hand, if

$$3x_2^2 + y_1^2 + y_1 y_2 + y_2^2 = 0$$

then by completing the square we obtain

$$3x_2^2 + \Big(y_1 + \frac{y_2}{2}\Big)^2 + \frac{3}{4} y_2^2 = 0.$$

This requires that $x_2 = y_2 = 0$. But $x_1 < x_2 = x_3 = 0$ violates (2.8), so this is also impossible.

The final case is (2.13), in which $x_1 = x_2 = x_3 = 0, y_3 = -y_1 - y_2$, from which (2.15) becomes $\varphi_0 = \psi_0 = 0$ and

$$\cos(2\pi\varphi) = -\frac{2y_1^4 + 7y_1^3 y_2 + 11 y_1^2 y_2^2 + 8 y_1 y_2^3 + 4 y_2^4}{2y_1^2 y_2 (y_1 + y_2)} \tag{3.35}$$

which we can put in terms of the variable $\gamma = y_1/y_2$:

$$\cos(t y_{23}) = -\frac{2\gamma^4 + 7\gamma^3 + 11\gamma^2 + 8\gamma + 4}{2\gamma^2(\gamma+1)} = f(\gamma). \tag{3.36}$$

Analysis of $f(\gamma)$ reveals that $-1 \leq f(\gamma) \leq 1$ only when $\gamma = -2$, and $f(-2) = -1$ – that is, (3.35) is only possible when $y_1 + 2y_2 = 0$, and that again gives a contradiction as it would imply $y_2 = y_3$ and hence $\lambda_2 = \lambda_3$ and $D_3 = 0$. Consequently, the case (2.13) gives rise to no oscillatory solutions and the proof is complete. □

As noted in Remark 3.6, similar results for higher order equations will depend on analysis of larger systems of exponential equations, which is difficult. Our results have some limited applicability: for instance, in degree 4 if

$$\sum_{j=1}^{4} x_j^2 \neq 0$$

then one can reduce to various instances of Theorem 2.5. However, as with the third order results presented here, the most troublesome case is when all the x_is are zero.

References

[1] J. H. Barrett; *Oscillation theory of ordinary differential equations,* Advances in Math., (1969), 3, pp. 415-509

[2] P. Bleher, R. Mallison Jr.; *Zeros of sections of exponential sums,* IMRN International Mathematics Research Notices Volume 2006 (2006) Article ID 38937, pp. 1 - 49.

[3] L. Erbe; *Existence of oscillatory solutions and asymptotic behavior of a class of third order linear differential equations,* Pacific J. Math. 64 (1976), pp. 369-385.

[4] G. Hovhannisyan; *On oscillations of solutions of third-order dynamic equations.* Abstract and Applied Analysis, (2012), Article ID 715981, pp.1-15.

[5] G. Hovhannisyan; *Asymptotic solutions of n-th order dynamic equation and oscillations.* ISRN Mathematical Analysis, Volume 2013 (2013), Article ID 946453, pp.1-11.

[6] R. E. Langer; *On the zeros of exponential sums and integrals.* Bull. Am. Math. Soc., (1931), 37, pp. 213-239

[7] C. A. Swanson; *Comparison and oscillation theory of linear differential equations.* Academic Press, New York and London, 1968.

[8] B. Shapiro; *Problems around polynomials: the good, the bad and ugly.* Arnold Math. J. (2015), Volume 1, Issue 1, 91-99

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