Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 178, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR p-LAPLACIAN EQUATIONS WITH ELECTROMAGNETIC FIELDS AND CRITICAL NONLINEARITY

ZHONGYI ZHANG

Abstract. In this article, we study the perturbed p-Laplacian equation problems with critical nonlinearity in $\mathbb{R}^N$. By using the concentration compactness principle and variational method, we establish the existence and multiplicity of nontrivial solutions of the least energy.

1. Introduction

In this article we study the existence and multiplicity of solutions for the perturbed p-Laplacian equation problems with critical nonlinearity

$$\begin{aligned} &-\varepsilon^p\Big(a+b\int_{\mathbb{R}^N}|\nabla_A u|^p dx\Big)\Delta_{p,A}u+V(x)|u|^{p-2}u \\ &=|u|^{p^*-2}u+h(x,|u|^p)|u|^{p-2}u,\quad x\in\mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $\Delta_{p,A}u(x):=\operatorname{div}(|\nabla u+iA(x)u|^{p-2}(\nabla u+iA(x)u)$, here i is the imaginary unit, $p^*:=pN/(N-p)$ denotes the Sobolev critical exponent and $N\geq 3$.

We make the following assumptions on $V(x)$, $g(x)$ and $h(x)$ throughout this paper:

- (A1) $V(x)\in C(\mathbb{R}^N,\mathbb{R})$, $V(x_0)=\min V=0$ and there is $\tau_0>0$ such that the set $V^{\tau_0}=\{x\in\mathbb{R}^N:V(x)<\tau_0\}$ has finite Lebesgue measure;
- (A2) $A_j(x)\in C(\mathbb{R}^N,\mathbb{R})$ $(j=1,2,\dots,N)$ and $A(x_0)=0$;
- (A3) (1) $h\in C(\mathbb{R}^N\times[0,+\infty),\mathbb{R})$ and $h(x,t)=o(|t|)$ uniformly in x as $t\to 0$;
  - (2) there are $C_0>0$ and $q\in(p,p^*)$ such that $|h(x,t)|\leq C_0(1+t^{\frac{q-p}{p}})$;
  - (3) there $l_0>0$, $s>2p$ and $2p<\mu<p^*$ such that $H(x,t)\geq l_0|t|^{\frac{s}{p}}$ and $\mu H(x,t)\leq h(x,t)t$ for all (x,t), where $H(x,t)=\int_0^t h(x,s)ds$.

Problem (1.1) with $A(x)\equiv 0$ has an extensive literature. Different approaches have been taken to investigate this problem under various hypotheses on the potential and nonlinearity. See for example [1, 8, 12, 14, 15, 16, 18, 32, 33] and the references therein. Observe that in all these papers the nonlinearities are assumed to be subcritical together with some other technical conditions of course.

2010 *Mathematics Subject Classification.* 58E05, 58E50.
Key words and phrases. p-Laplacian equation; critical nonlinearity; magnetic fields; variational methods.

Submitted December 28, 2014. Published June 27, 2015.

The above-mentioned papers mostly concentrated on the nonlinearities with subcritical conditions. Floer and Weinstein in [20] first studied the existence of single and multiple spike solutions based on the Lyapunov-Schmidt reductions. Subsequently, Oh [32, 33, 34] extended the results in a higher dimension. Kang and Wei [25] established the existence of positive solutions with any prescribed number of spikes, clustering around a given local maximum point of the potential function. In accordance with the Sobolev critical nonlinearities, there have been many papers devoted to studying the existence of solutions to elliptic boundary-valued problems on bounded domains after the pioneering work by Brézis and Nirenberg [5]. Ding and Lin [17] first studied the existence of semi-classical solutions to the problem on the whole space with critical nonlinearities and established the existence of positive solutions, as well as of those that change sign exactly once. They also obtained multiplicity of solutions when the nonlinearity is odd.

As far as problem (1.1) in the case of $A(x) \not\equiv 0$ is concerned, we recall Bartsch [3], Cingolani [9] and Esteban and Lions [19]. This kind of paper first appeared in [19]. The authors obtained the existence results of standing wave solutions for fixed $\hbar > 0$ and special classes of magnetic fields. Cingolani [9] proved that the magnetic potential $A(x)$ only contributes to the phase factor of the solitary solutions for $\hbar > 0$ sufficiently small. For more results, we refer the reader to [2, 10, 11, 23, 27, 36, 39] and the references therein.

For general $p \geq 2$, most of the works studied the existence results to equation (1.1) with $A(x) \equiv 0$. See, for example, [13, 21, 31] and the references therein. These papers are mostly devoted to the study of the existence of solutions to the problem on bounded domains with the Sobolev subcritical nonlinearities.

In (1.1) with bounded domain, if we set $p = 2$, $A(x) \equiv 0$, $\varepsilon = 1$, $V(x) = 0$ and $g(t) = a + bt$, it reduces to the following Dirichlet problem of Kirchhoff type

$$\begin{gathered} -\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{1.2}$$

Problem (1.2) is a generalization of a model introduced by Kirchhoff [26]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. The equation (1.2) is related to the stationary analogue of problem (1.3). (1.2) received much attention only after Lions [29] proposed an abstract framework to the problem. Some important and interesting results can be found, see for example [22, 24, 28]. We note that the results dealing with the problem (1.2) with critical nonlinearity are relatively scarce.

Equation (1.1) with $p \neq 2$, $A(x) \equiv 0$, $\varepsilon = 1$, $V(x) = 0$, it reduces to the p-Kirchhoff type problem. p-Kirchhoff type problem began to attract the attention of several researchers mainly after the work of Lions [29], where a functional analysis approach was proposed to attack it. However, in this work, we use a different approach to those explored in [24], because here we are working with the p-Laplacian operator. Because p-Laplacian operator is nonlinear, some estimates for this type of operator can not be obtained using the same kind of ideas explored for the case $p = 2$. For example, We know that $W^{1,p}(\mathbb{R}^N)$ is not a Hilbert space for $1 < p < N$, except for $p = 2$. The space $W^{1,p}(\mathbb{R}^N)$ with $p \neq 2$ does not satisfy the Lieb lemma [37].

To the best of our knowledge, the existence and multiplicity of solutions to problem (1.1) on $\mathbb{R}^N$ has not ever been studied by variational methods. As we shall see in the present paper, problem (1.1) can be viewed as a Schrödinger equation coupled with a non-local term. The competing effect of the non-local term with the critical nonlinearity and the lack of compactness of the embedding of $W^{1,p}(\mathbb{R}^N)$ into the space $L^p(\mathbb{R}^N)$, prevents us from using the variational methods in a standard way. Some new estimates for such a Kirchhoff equation involving Palais-Smale sequences, which are key points to apply this kinds of theory, are needed to be re-established. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all, since the appearance of non-local term, we must consider our problem for suitable space and so we need more delicate estimates.

Our main result is the following theorem.

Theorem 1.1. *Let* (A1)–(A3) *be satisfied. Then*

(i) *For any $\kappa > 0$ there is $\mathcal{E}_\kappa > 0$ such that if $\varepsilon \leq \mathcal{E}_\kappa$ problem* (1.1) *has at least one solution u_ε satisfying*

$$\frac{\theta\mu - 1}{p} \int_{\mathbb{R}^N} H(x, |u_\varepsilon|^p)dx + \Big(\frac{\theta}{p} - \frac{1}{p^*}\Big) \int_{\mathbb{R}^N} |u_\varepsilon|^{p^*} dx \leq \kappa\varepsilon^N, \tag{1.4}$$

$$\Big(\frac{\theta}{p} - \frac{1}{\mu}\Big) \int_{\mathbb{R}^N} |\nabla_A u_\varepsilon|^p dx + \Big(\frac{1}{p} - \frac{1}{\mu}\Big) \int_{\mathbb{R}^N} \lambda V(x)|u_\varepsilon|^p dx \leq \kappa\varepsilon^N. \tag{1.5}$$

Moreover, $u_\varepsilon \to 0$ in $W^{1,p}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

(ii) *Assume additionally that $h(x,t)$ is odd in t, for any $m \in \mathbb{N}$ and $\kappa > 0$ there is $\mathcal{E}_{m\kappa} > 0$ such that if $\varepsilon \leq \mathcal{E}_{m\kappa}$, problem* (1.1) *has at least m pairs of solutions $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i = 1, 2, \dots, m$ which satisfy the estimates* (1.4) *and* (1.5)*. Moreover, $u_{\varepsilon,i} \to 0$ in $W^{1,p}(\mathbb{R}^N)$ as $\varepsilon \to 0$, $i = 1, 2, \dots, m$.*

2. Main result

We set $\lambda = \varepsilon^{-p}$ and rewrite (1.1) in the form

$$\begin{aligned} &-\Big(a + b\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)\Delta_{p,A} u + \lambda V(x)|u|^{p-2}u \\ &= \lambda|u|^{p^*-2}u + \lambda h(x, |u|^p)|u|^{p-2}u,\ x \in \mathbb{R}^N. \end{aligned} \tag{2.1}$$

We are going to prove the following result.

Theorem 2.1. *Let* (A1)–(A3) *be satisfied. Then*

(1) For any $\sigma > 0$ there is $\Lambda_\sigma > 0$ such that problem (2.1) *has at least one solution u_λ for each $\lambda \geq \Lambda_\sigma$ satisfying*

$$\frac{\mu - 2}{2p} \int_{\mathbb{R}^N} H(x, |u_\lambda|^p)dx + \Big(\frac{1}{2p} - \frac{1}{p^*}\Big) \int_{\mathbb{R}^N} |u_\lambda|^{p^*} dx \leq \sigma\lambda^{-\frac{N}{p}} \tag{2.2}$$

and

$$\Big(\frac{1}{p} - \frac{1}{\mu}\Big) \int_{\mathbb{R}^N} |\nabla_A u_\lambda|^p dx + \Big(\frac{1}{p} - \frac{1}{\mu}\Big) \int_{\mathbb{R}^N} \lambda V(x)|u_\lambda|^p dx \leq \sigma\lambda^{1-\frac{N}{p}}. \tag{2.3}$$

(2) Assume additionally that $h(x,t)$ is odd in t, for any $m \in \mathbb{N}$ and $\sigma > 0$ there is $\Lambda_{m\sigma} > 0$ such that if problem (2.1) has at least m pairs of solutions u_λ which satisfy the estimates (2.2) and (2.3) whenever $\lambda \geq \Lambda_{m\sigma}$.

To prove the above theorems, we introduce the space

$$E_\lambda := \big\{u \in W^{1,p}(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx < \infty,\ \lambda > 0\big\}$$

equipped with the norm

$$\|u\|_\lambda^p = \int_{\mathbb{R}^N} (|\nabla_A u|^p + \lambda V(x)|u|^p)\, dx,$$

where $\nabla_A u := \nabla u + iAu$. It is known that E_λ is the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$. Similar to the diamagnetic inequality [19], we have the following inequality

$$|\nabla_A u(x)| \geq |\nabla|u(x)||, \quad \text{for } u \in W^{1,p}(\mathbb{R}^N, \mathbb{C}).$$

Indeed, since A is real-valued

$$|\nabla|u|(x)| = \Big|\operatorname{Re}\Big(\nabla u \frac{\overline{u}}{|u|}\Big)\Big| = \big|\operatorname{Re}\big(\nabla u + iAu\big)\frac{\overline{u}}{|u|}\big| \leq |\nabla u + iAu|,$$

(the bar denotes complex conjugation) this fact means that if $u \in E_\lambda$, then $|u| \in W^{1,p}(\mathbb{R}^N, \mathbb{C})$, and therefore $u \in L^s(\mathbb{R}^N)$ for any $s \in [p, p^*)$. Thus, for each $s \in [p, p^*]$, there is $c_s > 0$ (independent of λ) such that if $\lambda > 1$

$$\Big(\int_{\mathbb{R}^N} |u|^s\Big)^{1/s} \leq c_s\Big(\int_{\mathbb{R}^N} |\nabla|u||^p\Big)^{1/p} \leq c_s\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p\Big)^{1/p} \leq c_s\|u\|_\lambda. \tag{2.4}$$

The energy functional $J_\lambda : E_\lambda \to \mathbb{R}$ associated with problem (2.1)

$$\begin{aligned} J_\lambda(u) := &\frac{a}{p}\int_{\mathbb{R}^N} |\nabla_A u|^p dx + \frac{b}{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)^2 + \frac{1}{p}\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx \\ &- \frac{\lambda}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} H(x, |u|^p) dx \end{aligned}$$

is well defined. Thus, it is easy to check that as arguments [35, 38] $J_\lambda \in C^1(E_\lambda, \mathbb{R})$ and its critical points are solutions of (2.1).
We call that $u \in E_\lambda$ is a weak solution of (2.1), if

$$\begin{aligned} \langle J_\lambda'(u), v\rangle = \operatorname{Re}\Big\{&a\int_{\mathbb{R}^N} \big(|\nabla_A u|^{p-2}\nabla_A u \cdot \overline{\nabla_A v}\big)\, dx + \lambda\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\overline{v}dx \\ &+ b\int_{\mathbb{R}^N} |\nabla_A u|^p dx \int_{\mathbb{R}^N} \big(|\nabla_A u|^{p-2}\nabla_A u \cdot \overline{\nabla_A v}\big)\, dx \\ &- \lambda\int_{\mathbb{R}^N} |u|^{p^*-2}u\overline{v}dx - \lambda\int_{\mathbb{R}^N} h(x, |u|^p)|u|^{p-2}u\overline{v}dx\Big\}, \end{aligned}$$

where $v \in E_\lambda$.

3. Behavior of (PS) sequences

We recall the second concentration-compactness principle by Lions [30]

Lemma 3.1 ([30]). *Let $\{u_n\}$ be a weakly convergent sequence to u in $W^{1,p}(\mathbb{R}^N)$ such that $|u_n|^{p^*} \rightharpoonup \nu$ and $|\nabla u_n|^p \rightharpoonup \mu$ in the sense of measures. Then, for some at most countable index set I,*

(i) $\nu = |u|^{p^*} + \sum_{j\in I} \delta_{x_j}\nu_j,\ \nu_j > 0,$

(ii) $\mu \geq |\nabla u|^p + \sum_{j\in I} \delta_{x_j}\mu_j,\ \mu_j > 0,$
(iii) $\mu_j \geq S\nu_j^{p/p^*}$,

where S is the best Sobolev constant, i.e. $S = \inf\big\{ \int_{\mathbb{R}^N} |\nabla u|^p dx : \int_{\mathbb{R}^N} |u|^{p^*} dx = 1\big\}$, $x_j \in \mathbb{R}^N$, δ_{x_j} *are Dirac measures at* x_j *and* μ_j, ν_j *are constants.*

Lemma 3.2 ([7]). *Let $\{u_n\}$ be a weakly convergent sequence to u in $W^{1,p}(\mathbb{R}^N)$ and define*

(i) $\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{p^*} dx$,
(ii) $\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p dx$.

The quantities ν_∞ and μ_∞ exist and satisfy

(iii) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty$,
(iv) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty$,
(v) $\mu_\infty \geq S\nu_\infty^{p/p^*}$.

We recall that a C^1 functional J_λ on Banach space E_λ is said to satisfy the Palais-Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\} \subset E_\lambda$ satisfying $\lim_{n\to\infty} J_\lambda(u_n) = c$ and $\lim_{n\to\infty} \|J_\lambda(u_n)\|_{E_\lambda^*} = 0$ has a convergent subsequence.

Lemma 3.3. *Suppose that* (A1)–(A3) *hold. Then any $(PS)_c$ sequence $\{u_n\}$ is bounded in E_λ and $c \geq 0$.*

Proof. Let $\{u_n\}$ be a sequence in E_λ such that

$$\begin{aligned} c + o(1) = J_\lambda(u_n) &= \frac{a}{p}\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx + \frac{b}{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big)^2 \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} \lambda V(x)|u_n|^p dx - \frac{\lambda}{p^*}\int_{\mathbb{R}^N} |u_n|^{p^*} dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} H(x,|u_n|^p) dx \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\langle J_\lambda'(u_n), v\rangle \\ &= \operatorname{Re}\Big\{ a\int_{\mathbb{R}^N} |\nabla_A u_n|^{p-2}\nabla_A u_n \cdot \overline{\nabla_A v} dx + \lambda\int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\overline{v} dx \\ &\quad + b\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx \int_{\mathbb{R}^N} |\nabla_A u_n|^{p-2}\nabla_A u_n \cdot \overline{\nabla_A v} dx - \lambda\int_{\mathbb{R}^N} |u_n|^{p^*-2}u_n\overline{v} dx \\ &\quad - \lambda\int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^{p-2}u_n\overline{v} dx\Big\} = o(1)\|u_n\|. \end{aligned} \tag{3.2}$$

By (3.1), (3.2) and condition (A3)(3), we have

$$\begin{aligned} &c + o(1)\|u_n\| \\ &= J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle \\ &= \Big(\frac{1}{p} - \frac{1}{\mu}\Big)a\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx + \Big(\frac{1}{p} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} \lambda V(x)|u_n|^p dx \\ &\quad + \Big(\frac{1}{2p} - \frac{1}{\mu}\Big)b\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big)^2 + \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\int_{\mathbb{R}^N} |u_n|^{p^*} dx \\ &\quad + \lambda\int_{\mathbb{R}^N} \big[\frac{1}{\mu}h(x,|u_n|^p)|u_n|^p - \frac{1}{p}H(x,|u_n|^p)\big] dx \end{aligned}$$

$$\geq \Big(\frac{1}{p}-\frac{1}{\mu}\Big)a\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx+\Big(\frac{1}{p}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}\lambda V(x)|u_n|^p dx. \tag{3.3}$$

This inequality implies that $\{u_n\}$ is bounded in E_λ. Taking the limit in (3.3) shows that $c\geq 0$. This completes the proof of Lemma 3.3. □

The main result in this section is the following compactness result.

Lemma 3.4. *Suppose that* (A1)–(A3) *hold. For any* $\lambda\geq 1$, J_λ *satisfies* $(PS)_c$ *condition, for all* $c\in(0,\sigma_0\lambda^{1-\frac{N}{p}})$, *where* $\sigma_0:=(\frac{1}{\mu}-\frac{1}{p^*})(aS)^{N/p}$, *that is any* $(PS)_c$*-sequence* $(u_n)\subset E_\lambda$ *has a strongly convergent subsequence in* E_λ.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence, by Lemma 3.3, $\{u_n\}$ is bounded in E_λ. Hence, by diamagnetic inequality, $\{|u_n|\}$ is bounded in $W^{1,p}(\mathbb{R}^N,\mathbb{C})$. Then, for some subsequence, there is $u\in W^{1,p}(\mathbb{R}^N,\mathbb{C})$ such that $u_n\rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N,\mathbb{C})$. We claim that

$$\int_{\mathbb{R}^N}|u_n|^{p^*}dx\to\int_{\mathbb{R}^N}|u|^{p^*}dx. \tag{3.4}$$

To prove this claim, we suppose that

$$|\nabla|u_n||^p\rightharpoonup|\nabla|u||^p+\mu\quad\text{and}\quad|u_n|^{p^*}\rightharpoonup|u|^{p^*}+\nu\quad(\text{weak}^*\text{ sense of measures}).$$

Using the concentration compactness-principle due to Lions (cf. [30, Lemma 1.2]), we obtain a countable index set I, sequences $\{x_j\}\subset\mathbb{R}^N$, $\{\mu_j\},\{\nu_j\}\subset(0,\infty)$ such that

$$\nu=\sum_{j\in I}\delta_{x_j}\nu_j,\quad \mu\geq\sum_{j\in I}\delta_{x_j}\mu_j,\quad \mu_j\geq S\nu_j^{p/p^*} \tag{3.5}$$

for all $j\in I$, where δ_{x_j} are Dirac measures at x_j and μ_j, ν_j are constants.

Now, let x_j be a singular point of the measures μ and ν. We define a function $\phi(x)\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\phi(x)=1$ in $B(x_j,\varepsilon)$, $\phi(x)=0$ in $\mathbb{R}^N\setminus B(x_j,2\varepsilon)$ and $|\nabla\phi|\leq 2/\varepsilon$ in $\mathbb{R}^N$. Since $\{u_n\phi\}$ is bounded in $W^{1,p}(\mathbb{R}^N,\mathbb{C})$ and ϕ takes values in $\mathbb{R}$, a direct calculation shows that

$$\langle J_\lambda'(u_n),u_n\phi\rangle\to 0,$$
$$\overline{\nabla_A(u_n\phi)}=i\overline{u_n}\nabla\phi+\phi\overline{\nabla_A u_n}.$$

Therefore,

$$\begin{aligned}
&a\int_{\mathbb{R}^N}|\nabla_A u_n|^p\phi dx+a\,\mathrm{Re}\Big(\int_{\mathbb{R}^N}i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla_A\phi}dx\Big)\\
&+\int_{\mathbb{R}^N}\lambda V(x)|u_n|^p\phi dx\\
&\quad=-b\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\cdot\mathrm{Re}\Big(\int_{\mathbb{R}^N}i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla_A\phi}dx\Big)\\
&\quad-b\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\int_{\mathbb{R}^N}|\nabla_A u_n|^p\phi dx+\lambda\int_{\mathbb{R}^N}h(x,|u_n|^p)|u_n|^p\phi dx\\
&\quad+\lambda\int_{\mathbb{R}^N}|u_n|^{p^*}\phi dx+o_n(1).
\end{aligned} \tag{3.6}$$

On the other hand, by Hölder's inequality we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\big|\operatorname{Re}\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla\phi}dx\big| \\
&\le \limsup_{n\to\infty}\Big(\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\Big)^{(p-1)/p}\Big(\int_{\mathbb{R}^N}|\overline{u_n\nabla_A\phi}|^p dx\Big)^{1/p} \\
&\le C_1\Big(\int_{B(x_j,2\varepsilon)}|u|^p|\nabla_A\phi|^p dx\Big)^{1/p} \\
&\le C_1\Big(\int_{B(x_j,2\varepsilon)}|\nabla_A\phi|^N dx\Big)^{1/N}\Big(\int_{B(x_j,2\varepsilon)}|u|^{p^*}dx\Big)^{1/p^*} \\
&\le C_2\Big(\int_{B(x_j,2\varepsilon)}|u|^{p^*}dx\Big)^{1/p^*}\to 0 \quad \text{as } \varepsilon\to 0\,.
\end{aligned} \tag{3.7}$$

Similarly, it follows from the definition of ϕ and condition (A3) that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^p\phi dx = 0. \tag{3.8}$$

Since ϕ has compact support, letting $n\to\infty$ in (3.6) we deduce from the lower semicontinuity of the norm, (3.7) and (3.8) that

$$a\int_{\mathbb{R}^N}\phi d\mu \le -\int_{\mathbb{R}^N}\lambda V(x)|u|^p\phi dx + \lambda\int_{\mathbb{R}^N}\phi d\nu.$$

Letting $\varepsilon\to 0$, we obtain $a\mu_j\le\lambda\nu_j$. Combing this with Lemma 3.1, we obtain $\nu_j\ge a\lambda^{-1}S\nu_j^{\frac{p}{p^*}}$. This result implies that

$$\text{(I)}\quad \nu_j = 0 \quad\text{or}\quad \text{(II)}\quad \nu_j\ge\big(a\lambda^{-1}S\big)^{N/p}\,.$$

To obtain the possible concentration of mass at infinity, similarly, we define a cut off function $\phi_R\in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x)=0$ on $|x|<R$ and $\phi_R(x)=1$ on $|x|>R+1$. Note that $\langle J'(u_n),u_n\phi_R\rangle\to 0$, this fact imply that

$$\begin{aligned}
&a\int_{\mathbb{R}^N}|\nabla_A u_n|^p\phi_R dx + a\operatorname{Re}\Big(\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla_A\phi_R}dx\Big) \\
&+\int_{\mathbb{R}^N}\lambda V(x)|u_n|^p\phi_R dx \\
&= -b\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\cdot\operatorname{Re}\Big(\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla_A\phi_R}dx\Big) \\
&\quad - b\int_{\mathbb{R}^N}|\nabla_A u_n|^p dx\int_{\mathbb{R}^N}|\nabla_A u_n|^p\phi_R dx + \lambda\int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^p\phi_R dx \\
&\quad + \lambda\int_{\mathbb{R}^N}|u_n|^{p^*}\phi_R dx + o_n(1).
\end{aligned} \tag{3.9}$$

It is easy to prove that

$$\begin{gathered}
-\lim_{R\to\infty}\lim_{n\to\infty}\operatorname{Re}\Big(\int_{\mathbb{R}^N} i|\nabla_A u_n|^{p-2}\overline{u_n}\nabla_A u_n\overline{\nabla_A\phi_R}dx\Big) = 0, \\
\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x,|u_n|^p)|u_n|^p\phi_R dx = 0.
\end{gathered}$$

Letting $R \to \infty$, we obtain $a\mu_\infty \le \lambda\nu_\infty$. By Lemma 3.2, we obtain $\nu_\infty \ge a\lambda^{-1}S\nu_\infty^{\frac{p}{p^*}}$. This result implies that

$$\text{(III)} \quad \nu_\infty = 0 \quad \text{or} \quad \text{(IV)} \quad \nu_\infty \ge \big(a\lambda^{-1}S\big)^{N/p}.$$

Next, we claim that (II) and (IV) cannot occur. If the case (IV) holds, for some $j \in I$, then by using Lemma 3.2 and condition (A3)(3), we have that

$$\begin{aligned}
c &= \lim_{n\to\infty} \Big(J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle\Big) \\
&\ge \Big(\frac{1}{p} - \frac{1}{\mu}\Big)a\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx + \Big(\frac{1}{p} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} \lambda V(x)|u_n|^p dx \\
&\quad + \lambda\int_{\mathbb{R}^N} \Big[\frac{1}{\mu}h(x,|u_n|^p)|u_n|^p - \frac{1}{p}H(x,|u_n|^p)\Big]dx + \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\int_{\mathbb{R}^N} |u_n|^{p^*} dx \\
&\ge \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\int_{\mathbb{R}^N} |u_n|^{p^*} dx \\
&\ge \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\int_{\mathbb{R}^N} |u_n|^{p^*}\phi_R dx \\
&= \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\nu_\infty \ge \sigma_0\lambda^{1-\frac{N}{p}},
\end{aligned}$$

where $\sigma_0 = (\frac{1}{\mu} - \frac{1}{p^*})(aS)^{N/p}$. This is impossible. Consequently, $\nu_j = 0$ for all $j \in I$. Similarly, if the case (II) holds, for some $j \in I$, then by condition (A3), we have

$$\begin{aligned}
c &= \lim_{n\to\infty} \Big(J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n), u_n\rangle\Big) \\
&\ge \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\int_{\mathbb{R}^N} |u_n|^{p^*} dx \\
&\ge \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\int_{\mathbb{R}^N} |u_n|^{p^*}\phi dx \\
&= \Big(\frac{1}{\mu} - \frac{1}{p^*}\Big)\lambda\nu \ge a\lambda^{1-\frac{N}{p}} \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

which leads to a contradiction. Thus, we must have (II) cannot occur for each j. Then

$$\int_{\mathbb{R}^N} |u_n|^{p^*} dx \to \int_{\mathbb{R}^N} |u|^{p^*} dx. \tag{3.10}$$

Thus, from (3.10), the lower semicontinuity of the norm and Brezis-Lieb Lemma [6], we have

$$\begin{aligned}
o(1)\|u_n\| &= \langle J_\lambda'(u_n), u_n\rangle \\
&= a\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx + \lambda\int_{\mathbb{R}^N} V(x)|u_n|^p dx + b\Big(\int_{\mathbb{R}^N} |\nabla_A u_n|^p dx\Big)^2 \\
&\quad - \lambda\int_{\mathbb{R}^N} |u_n|^{p^*} dx - \lambda\int_{\mathbb{R}^N} H(x,|u_n|^p)dx \\
&\ge \min\{a,1\}\|u_n - u\|_\lambda^p + a\int_{\mathbb{R}^N} |\nabla_A u|^p dx + \lambda\int_{\mathbb{R}^N} V(x)|u|^p dx \\
&\quad + b\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)^2 - \lambda\int_{\mathbb{R}^N} |u|^{p^*} dx - \lambda\int_{\mathbb{R}^N} H(x,|u|^p)dx
\end{aligned}$$

$$= \|u_n - u\|_\lambda^p + o(1)\|u\|_\lambda,$$

here we use $J_\lambda'(u) = 0$. Thus we prove that $\{u_n\}$ strongly converges to u in E_λ. This completes the proof of Lemma 3.4. □

4. Proof of Theorem 2.1

In the following, we always consider $\lambda \geq 1$. By the assumptions (A1)–(A3), one can see that $J_\lambda(u)$ has the mountain pass geometry.

Lemma 4.1. *Assume* (A1)–(A3) *hold. Then There exist* $\alpha_\lambda, \rho_\lambda > 0$ *such that* $J_\lambda(u) > 0$ *if* $u \in B_{\rho_\lambda} \setminus \{0\}$ *and* $J_\lambda(u) \geq \alpha_\lambda$ *if* $u \in \partial B_{\rho_\lambda}$*, where* $B_{\rho_\lambda} = \{u \in E_\lambda : \|u\|_\lambda \leq \rho_\lambda\}$.

Proof. From condition (A3), there is $C_\delta > 0$ such that

$$\frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx + \frac{1}{p}\int_{\mathbb{R}^N} H(x,|u|^p)dx \leq \delta|u|_p^p + C_\delta|u|_{p^*}^{p^*},$$

for $\delta \leq \big(2\min\big\{\frac{a}{p}, \frac{1}{p}\big\}\lambda c_p^p\big)^{-1}$. It follows that

$$\begin{aligned}
J_\lambda(u) &:= \frac{a}{p}\int_{\mathbb{R}^N} |\nabla_A u|^p dx + \frac{b}{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)^2 + \frac{1}{p}\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx \\
&\quad - \frac{\lambda}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} H(x,|u|^p)dx \\
&\geq \min\Big\{\frac{a}{p}, \frac{1}{p}\Big\}\|u\|_\lambda^p - \lambda\delta|u|_p^p - \lambda C_\delta|u|_{p^*}^{p^*} \\
&\geq \frac{1}{2}\min\Big\{\frac{a}{p}, \frac{1}{p}\Big\}\|u\|_\lambda^p - \lambda C_\delta c_{p^*}^{p^*}\|u\|_\lambda^{p^*}.
\end{aligned}$$

Since $p^* > p$, we know that the conclusion of Lemma 4.1 holds. □

Lemma 4.2. *Under the assumption of Lemma 4.1, for any finite dimensional subspace* $F \subset E_\lambda$,

$$J_\lambda(u) \to -\infty \quad as \quad u \in F,\ \|u\|_\lambda \to \infty.$$

Proof. By using conditions (A2) and (A3), we obtain

$$J_\lambda(u) \leq \max\{\frac{a}{p}, 1\}\|u\|_\lambda^p + \frac{b}{2p}\|u\|_\lambda^{2p} - \frac{\lambda}{p^*}|u|_{p^*}^{p^*} - \lambda l_0|u|_s^s$$

for all $u \in F$. Since all norms in a finite-dimensional space are equivalent and $2p < p^*$, $p < p^*$. This completes the proof. □

Since $J_\lambda(u)$ does not satisfy the $(PS)_c$ condition for all $c > 0$, in the following we will find a special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

Recall that assumption (A2) implies that there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality we assume from now on that $x_0 = 0$. Observe that, by (A3)(3) we have

$$\frac{\lambda}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx + \lambda\int_{\mathbb{R}^N} H(x,|u|^p)dx \geq l_0\lambda\int_{\mathbb{R}^N} |u|^s dx.$$

Definite the function $I_\lambda \in C^1(E_\lambda, \mathbb{R})$ by

$$I_\lambda(u) := \frac{a}{p}\int_{\mathbb{R}^N} |\nabla_A u|^p dx + \frac{b}{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)^2 + \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx - l_0\lambda\int_{\mathbb{R}^N} |u|^s dx.$$

Then $J_\lambda(u) \le I_\lambda(u)$ for all $u \in E_\lambda$ and it suffices to construct small minimax levels for I_λ. Note that

$$\inf\Big\{\int_{\mathbb{R}^N} |\nabla\phi|^p dx : \phi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}), |\phi|_p = 1\Big\} = 0.$$

For any $1 > \delta > 0$ one can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\delta|_p = 1$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$ so that $|\nabla\phi_\delta|_p^p < \delta$. Set

$$f_\lambda = \phi_\delta(\lambda^{1/p}x), \tag{4.1}$$

then

$$\operatorname{supp} f_\lambda \subset B_{\lambda^{-1/p}r_\delta}(0).$$

Thus, for $t \ge 0$,

$$\begin{aligned} I_\lambda(tf_\lambda) &\le \frac{a}{p}t^p \int_{\mathbb{R}^N} |\nabla_A f_\lambda|^p dx + \frac{b}{2p}t^{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A f_\lambda|^p dx\Big)^2 \\ &\quad + \frac{t^p}{p}\int_{\mathbb{R}^N} \lambda V(x)|f_\lambda|^p dx - t^s l_0 \lambda \int_{\mathbb{R}^N} |f_\lambda|^s dx \\ &\le \lambda^{1-\frac{N}{p}}\Big[\frac{a}{p}t^p \int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx + \frac{b}{2p}t^{2p}\lambda^{1-\frac{N}{p}}\Big(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx\Big)^2 \\ &\quad + \frac{t^p}{p}\int_{\mathbb{R}^N} V\big(\lambda^{-1/p}x\big)|\phi_\delta|^p dx - t^s l_0 \int_{\mathbb{R}^N} |\phi_\delta|^s dx\Big] \\ &= \lambda^{1-\frac{N}{p}}\Psi_\lambda(t\phi_\delta), \end{aligned}$$

where $\Psi_\lambda \in C^1(E_\lambda,\mathbb{R})$ defined by

$$\begin{aligned} \Psi_\lambda(u) &:= \frac{a}{p}\int_{\mathbb{R}^N} |\nabla_A u|^p dx + \frac{b}{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)^2 \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} V\big(\lambda^{-1/p}x\big)|u|^p dx - l_0\int_{\mathbb{R}^N} |u|^s dx. \end{aligned}$$

Since $s > 2p$, thus there exists finite number $t_0 \in [0,+\infty)$ such that

$$\begin{aligned} \max_{t\ge 0}\Psi_\lambda(t\phi_\delta) &= \frac{a}{p}t_0^p \int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx + \frac{b}{2p}t_0^{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx\Big)^2 \\ &\quad + \frac{t_0^p}{p}\int_{\mathbb{R}^N} V\big(\lambda^{-1/p}x\big)|\phi_\delta|^p dx - t_0^s l_0 \int_{\mathbb{R}^N} |\phi_\delta|^s dx \\ &\le \frac{a}{p}t_0^p \int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx + \frac{b}{2p}t_0^{2p}\Big(\int_{\mathbb{R}^N} |\nabla_A \phi_\delta|^p dx\Big)^2 \\ &\quad + \frac{t_0^p}{p}\int_{\mathbb{R}^N} V\big(\lambda^{-1/p}x\big)|\phi_\delta|^p dx. \end{aligned}$$

On the one hand, since $V(0) = 0$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$, there is $\Lambda_\delta > 0$ such that

$$V\big(\lambda^{-1/p}x\big) \le \frac{\delta}{|\phi_\delta|_p^p} \quad \text{for all } |x| \le r_\delta \text{ and } \lambda \ge \Lambda_\delta.$$

This implies

$$\max_{t\ge 0}\Psi_\lambda(t\phi_\delta) \le \frac{a}{p}t_0^p\delta + \frac{b}{2p}t_0^{2p}\delta^2 + \frac{t_0^p}{p}\delta \le T^*\delta. \tag{4.2}$$

where $T^* := (\frac{a}{p}t_0^p + \frac{b}{2p}t_0^{2p} + \frac{t_0^p}{p})$. Therefore, for all $\lambda \ge \Lambda_\delta$,

$$\max_{t\ge 0} J_\lambda(t\phi_\delta) \le T^*\delta\lambda^{1-\frac{N}{p}}. \tag{4.3}$$

Thus we have the following lemma.

Lemma 4.3. *Under the assumption of Lemma 4.1, for any $\kappa > 0$ there exists $\Lambda_\kappa > 0$ such that for each $\lambda \geq \Lambda_\kappa$, there is $\widehat{f}_\lambda \in E_\lambda$ with $\|\widehat{f}_\lambda\| > \rho_\lambda$, $J_\lambda(\widehat{f}_\lambda) \leq 0$ and*

$$\max_{t\in[0,1]} J_\lambda(t\widehat{f}_\lambda) \leq \kappa\lambda^{1-\frac{N}{p}}. \tag{4.4}$$

Proof. Choose $\delta > 0$ so small that $T^*\delta \leq \kappa$. Let $f_\lambda \in E_\lambda$ be the function defined by (4.1). Taking $\Lambda_\kappa = \Lambda_\delta$. Let $\widehat{t}_\lambda > 0$ be such that $\widehat{t}_\lambda\|f_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(tf_\lambda) \leq 0$ for all $t \geq \widehat{t}_\lambda$. By (4.3), let $\widehat{f}_\lambda = \widehat{t}_\lambda f_\lambda$ we know that the conclusion of Lemma 4.3 holds. $\square$

For any $m^* \in \mathbb{N}$, one can choose m^* functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\delta^i \cap \operatorname{supp}\phi_\delta^k = \emptyset$, $i \neq k$, $|\phi_\delta^i|_s = 1$ and $|\nabla\phi_\delta^i|_p^p < \delta$. Let $r_\delta^{m^*} > 0$ be such that $\operatorname{supp}\phi_\delta^i \subset B_{r_\delta}^i(0)$ for $i = 1, 2, \dots, m^*$. Set

$$f_\lambda^i(x) = \phi_\delta^i(\lambda^{1/p}x), \quad \text{for } i = 1, 2, \dots, m^* \tag{4.5}$$

and

$$H_{\lambda\delta}^{m^*} = \operatorname{span}\{f_\lambda^1, f_\lambda^2, \dots, f_\lambda^{m^*}\}.$$

Observe that for each $u = \sum_{i=1}^{m^*} c_i f_\lambda^i \in H_{\lambda\delta}^{m^*}$,

$$\int_{\mathbb{R}^N} |\nabla_A u|^p dx = \sum_{i=1}^{m^*} |c_i|^p \int_{\mathbb{R}^N} |\nabla_A f_\lambda^i|^p dx,$$

$$\int_{\mathbb{R}^N} V(x)|u|^p dx = \sum_{i=1}^{m^*} |c_i|^p \int_{\mathbb{R}^N} V(x)|f_\lambda^i|^p dx,$$

$$\frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx = \frac{1}{p^*}\sum_{i=1}^{m^*} |c_i|^{p^*} \int_{\mathbb{R}^N} |f_\lambda^i|^{p^*} dx,$$

$$\int_{\mathbb{R}^N} H(x, |u|^p) dx = \sum_{i=1}^{m^*} \int_{\mathbb{R}^N} H(x, c_i f_\lambda^i) dx.$$

On the other hand, by mathematical induction we have the inequality

$$\Big(\sum_{i=1}^{m} a_i\Big)^2 \leq m\sum_{i=1}^{m} a_i^2 \quad \text{for all } a_i \geq 0. \tag{4.6}$$

Thus by (4.6), one has

$$\Big(\int_{\mathbb{R}^N} |\nabla u|^p dx\Big)^2 = \Big(\sum_{i=1}^{m^*} |c_i|^p \int_{\mathbb{R}^N} |\nabla f_\lambda^i|^p dx\Big)^2 \leq m^* \sum_{i=1}^{m^*} |c_i|^{2p}\Big(\int_{\mathbb{R}^N} |\nabla f_\lambda^i|^p dx\Big)^2.$$

Therefore

$$J_\lambda(u) \leq m^* \sum_{i=1}^{m^*} J_\lambda(c_i f_\lambda^i)$$

and as before

$$J_\lambda(c_i f_\lambda^i) \leq \lambda^{1-\frac{N}{p}} \Psi(|c_i| f_\lambda^i).$$

Set

$$\beta_\delta := \max\{|\phi_\delta^i|_p^p : j = 1, 2, \dots, m^*\}$$

and choose $\Lambda_{m^*\delta} > 0$ so that

$$V(\lambda^{-1/p}x) \le \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \le r_\delta^{m^*} \text{ and } \lambda \ge \Lambda_{m^*\delta}.$$

As before, we can obtain

$$\max_{u\in H_{\lambda\delta}^{m^*}} J_\lambda(u) \le (m^*)^p T^* \delta \lambda^{1-\frac{N}{p}} \tag{4.7}$$

for all $\lambda \ge \Lambda_{m^*\delta}$. Using this estimate we have the following result.

Lemma 4.4. *Under the assumptions of Lemma 4.1, for any $m^* \in \mathbb{N}$ and $\kappa > 0$ there exists $\Lambda_{m^*\kappa} > 0$ such that for each $\lambda \ge \Lambda_{m^*\kappa}$, there exists an m^*-dimensional subspace $F_{\lambda m^*}$ satisfying*

$$\max_{u\in F_{\lambda m^*}} J_\lambda(u) \le \kappa \lambda^{1-\frac{N}{p}}.$$

Proof. Choose $\delta > 0$ so small that $(m^*)^p T^* \delta \le \kappa$. Taking $F_{\lambda m^*} = H_{\lambda\delta}^{m^*} = \operatorname{span}\{f_\lambda^1, f_\lambda^2, \dots, f_\lambda^{m^*}\}$, where $f_\lambda^i(x) = \phi_\delta^i(\lambda^{1/p}x)$, for $i = 1, 2, \dots, m^*$ are given by (4.5). From (4.7), the statement of the lemma follows. □

We now establish the existence and multiplicity results.

Proof of Theorem 2.1. Using Lemma 4.3, we choose $\Lambda_\sigma > 0$ and define for $\lambda \ge \Lambda_\sigma$, the minimax value

$$c_\lambda := \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} J_\lambda(t\widehat{f}_\lambda)$$

where

$$\Gamma_\lambda := \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0 \text{ and } \gamma(1) = \widehat{f}_\lambda\}.$$

By Lemma 4.1, we have $\alpha_\lambda \le c_\lambda \le \sigma_0 \lambda^{1-\frac{N}{p}}$. By Lemma 3.4, we know that J_λ satisfies the $(PS)_{c_\lambda}$ condition, there is $u_\lambda \in E_\lambda$ such that $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$. Then u_λ is a solution of (2.1). Moreover, it is well known that such a Mountain-Pass solution is a least energy solution of (2.1). Such u_λ is a critical point of J_λ, for $\tau \in [2p, p^*]$,

$$\begin{aligned}
\sigma\lambda^{1-\frac{N}{p}} &\ge J_\lambda(u_\lambda) = J_\lambda(u_\lambda) - \frac{1}{\tau} J'_\lambda(u_\lambda)u_\lambda \\
&= \Big(\frac{1}{p} - \frac{1}{\tau}\Big) a \int_{\mathbb{R}^N} |\nabla_A u_\lambda|^p dx + \Big(\frac{1}{2p} - \frac{1}{\tau}\Big) b \Big(\int_{\mathbb{R}^N} |\nabla_A u|^p dx\Big)^2 \\
&\quad + \Big(\frac{1}{p} - \frac{1}{\tau}\Big) \int_{\mathbb{R}^N} \lambda V(x)|u_\lambda|^p dx + \Big(\frac{1}{\tau} - \frac{1}{p^*}\Big)\lambda \int_{\mathbb{R}^N} |u_\lambda|^{p^*} dx \\
&\quad + \lambda \int_{\mathbb{R}^N} \big[\frac{1}{\tau} h(x, |u_\lambda|^p)|u_\lambda|^p - \frac{1}{p} H(x, |u_\lambda|^p)\big] dx \\
&\ge \Big(\frac{1}{p} - \frac{1}{\tau}\Big) a \int_{\mathbb{R}^N} |\nabla_A u_\lambda|^p dx + \Big(\frac{1}{p} - \frac{1}{\tau}\Big) \int_{\mathbb{R}^N} \lambda V(x)|u_\lambda|^p dx \\
&\quad + \Big(\frac{1}{\tau} - \frac{1}{p^*}\Big)\lambda \int_{\mathbb{R}^N} |u_\lambda|^{p^*} dx + \Big(\frac{\mu}{\tau} - \frac{1}{p}\Big)\lambda \int_{\mathbb{R}^N} H(x, |u_\lambda|^p) dx,
\end{aligned} \tag{4.8}$$

where μ is the constant in (A3). Taking $\tau = 2p$ yields the estimate (2.2), and taking $\tau = \mu$ gives the estimate (2.3) hence the existence is proved.
Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ, for each $Z \in \Sigma$. Let $\operatorname{gen}(Z)$ be the Krasnoselski genus and

$$i(Z) := \min_{h\in\Gamma_{m^*}} \operatorname{gen}(h(Z) \cap \partial B_{\rho_\lambda}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $h \in C(E_\lambda, E_\lambda)$ and ρ_λ is the number from Lemma 4.1. Then i is a version of Benci's pseudoindex [4]. Let

$$c_{\lambda i} := \inf_{i(Z)\geq i} \sup_{u\in Z} J_\lambda(u), \quad 1 \leq i \leq m^*.$$

Since $J_\lambda(u) \geq \alpha_\lambda$ for all $u \in \partial B^+_{\rho\lambda}$ and since $i(F_{\lambda m^*}) = \dim F_{\lambda m^*} = m^*$, we have

$$\alpha_\lambda \leq c_{\lambda 1} \leq \dots \leq c_{\lambda m^*} \leq \sup_{u\in H_{\lambda m^*}} J_\lambda(u) \leq \sigma\lambda^{1-\frac{N}{p}}.$$

It follows from Lemma 3.4 that J_λ satisfies the $(PS)_{c_\lambda}$ condition at all levels c_i. By the usual critical point theory, all c_i are critical levels and J_λ has at least m^* pairs of nontrivial critical points. □

Acknowledgments. The author wants to thank the anonymous referees for their carefully reading this paper and their useful comments.

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Zhongyi Zhang
College of Mathematics, Jilin University, Changchun 130011, Jilin, China
E-mail address: zhyzhang66@163.com