Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 179, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GLOBAL REGULARITY FOR GENERALIZED HALL MAGNETO-HYDRODYNAMICS SYSTEMS

RENHUI WAN

Abstract. In this article, we consider the tridimensional generalized Hall magneto-hydrodynamics (Hall-MHD) system, with $(-\Delta)^\alpha u$ and $(-\Delta)^\beta b$. For $\alpha \geq 5/4$, $\beta \geq 7/4$, we obtain the global regularity of classical solutions. For $0 < \alpha < 5/4$ and $1/2 < \beta < 7/4$, with small data, the system also possesses global classical solutions. In addition, for the standard Hall-MHD system, $\alpha = \beta = 1$, by adding a suitable condition, we give a positive answer to the open question in [3]. At last, we study the regularity criterions of generalized Hall-MHD system. In particular, we prove the regularity criterion in terms of horizontal gradient $\nabla_h u, \nabla_h b$ for $1 < \alpha < 5/4$, $5/4 \leq \beta < 7/4$.

1. Introduction

The tridimensional incompressible generalized Hall-MHD system is governed by

$$\begin{gathered}
\partial_t u + u\cdot\nabla u + (-\Delta)^\alpha u + \nabla p = b\cdot\nabla b,\\
\partial_t b + u\cdot\nabla b - b\cdot\nabla u + (-\Delta)^\beta b = -\nabla\times(J\times b),\\
\operatorname{div} u = \operatorname{div} b = 0,\\
u(0,x) = u_0(x), \quad b(0,x) = b_0(x),
\end{gathered} \tag{1.1}$$

where $t \geq 0$, $x \in \mathbb{R}^3$, p, u, b stand for scalar pressure, velocity vector and magnetic field vector, respectively, $J = \nabla\times b$ is the current density, $u_0(x)$, $b_0(x)$ are the initial velocity and magnetic field, $\alpha, \beta \geq 0$ are constants and $(-\Delta)^\alpha$ is defined by

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi),$$

and we denote $(-\Delta)^{1/2}$ by Λ. For $\alpha = \beta = 1$, the system reduces to the standard Hall-MHD system, which can be obtained from kinetic models (cf. [1]). Hall-MHD is required in many physics problem, such as magnetic reconnection [5], star formation [6]. In [2], for $\alpha = 0$, $\beta = 1$, local classical solutions were obtained and Beale-Kato-Majda type blow-up criterion was also established. In [3], for $\alpha = \beta = 1$, some blow up criterions and small data results on global existence were established.

2010 *Mathematics Subject Classification.* 35Q35, 35B65, 76W05.
Key words and phrases. Generalized Hall-mhd system; global regularity; classical solutions.

Submitted November 11, 2014. Published June 29, 2015.

Recently, Chae, Wan and Wu [4] considered the generalized Hall-MHD (1.1), and obtained the local well-posedness for $\alpha = 0$ (without velocity diffusion), $\beta > 1/2$. So it is natural to ask a question:

> Can the global classical solutions of (1.1) be obtained with some conditions on (α, β)?

One of the main goals is to give a positive answer to the question. Some related work about generalized Navier-Stokes equations and generalized MHD equations can be seen [11, 12, 13, 14, 15, 16]. Our first result is stated as follows.

Theorem 1.1. *Let $T > 0, \alpha \geq 5/4$, $\beta \geq 7/4$. Let $(u_0, b_0) \in H^s(\mathbb{R}^3)$, $s > 5/2$ satisfying $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Then (1.1) has a unique global classical solution (u, b) such that*

$$(u, b) \in L^\infty([0, T]; H^s(\mathbb{R}^3)).$$

Remark 1.2. The space $H^s(\mathbb{R}^3)$ can be equipped with the norm

$$\|f\|_{H^s} = \|f\|_{L^2} + \|f\|_{\dot{H}^s},$$

where

$$\|f\|_{\dot{H}^s}^2 = \sum_{j\in\mathbb{Z}} 2^{2js} \|\Delta_j f\|_{L^2}^2.$$

More details can be seen in section 2.

Remark 1.3. If $b = 0$, (1.1) reduces to the generlized Navier-Stokes equations. Under the scaling transformation $u_l = l^{2\alpha-1} u(l^{2\alpha} t, lx)$, $p_l = l^{4\alpha-2} p(l^{2\alpha} t, lx)$, for $\alpha \geq \frac{5}{4}$, it is well-known that the generalized Navier-Stokes equations is critical and subcritical, which can lead the global regular solutions. We refer to [11] and [12]. If $u = 0$, (1.1) reduces to the following simple Hall problem

$$\partial_t b + (-\Delta)^\beta b = \nabla \times ((\nabla \times b) \times b), \tag{1.2}$$

which is scaling invariance under $b_l = l^{2\beta-2} b(l^{2\beta} t, lx)$. The corresponding energy is

$$\begin{aligned} E(b_l) &= \operatorname{ess\,sup}_{l^{2\beta}t} \|b_l\|_{L^2}^2 + \int_0^t \|\Lambda^\beta b_l\|_{L^2}^2 d\tau \\ &= l^{4\beta-7} \big\{ \operatorname{ess\,sup}_t \|b\|_{L^2}^2 + \int_0^t \|\Lambda^\beta b\|_{L^2}^2 d\tau \big\} = l^{4\beta-7} E(b). \end{aligned}$$

This implies that the simple Hall problem (1.2) is critical system for $\beta = 7/4$ and subcritical system with $\beta > 7/4$. As generalized Navier-Stokes equations, the condition on (α, β) seems optimal.

In addition, with small initial data, we can obtain the global small classical solution for $\alpha \in (0, 5/4)$ and $\beta \in (1/2, 7/4)$.

Theorem 1.4. *Let u_0, b_0, s be as Theorem 1.1. Let $\alpha \in (0, 5/4)$, $\beta \in (1/2, 7/4)$, and if*

$$\|u_0\|_{H^s} + \|b_0\|_{H^s} < \epsilon, \tag{1.3}$$

where ϵ is sufficient small, then there exists a unique global classical solution (u, b) of (1.1).

Remark 1.5. For $\alpha = \beta = 1$, this work was proved in [2], which can be considered a special case in Theorem 1.4.

For $\alpha = \beta = 1$, the authors in [3] improved the condition (1.3) by only assuming that

$$\|u_0\|_{\dot{H}^{3/2}} + \|b_0\|_{\dot{H}^{3/2}} < \epsilon \quad \text{or} \quad \|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|b_0\|_{\dot{B}^{3/2}_{2,1}} < \epsilon, \tag{1.4}$$

where

$$\|f\|_{\dot{B}^s_{2,1}} = \sum_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^2}$$

(For more details see section 2.), ϵ is sufficient small. And they also gave an open question, whether the condition

$$\|u_0\|_{\dot{H}^{1/2}} + \|b_0\|_{\dot{H}^{3/2}} < \epsilon \tag{1.5}$$

can lead the the desired result? Motivated by these, we give a theorem as follows.

Theorem 1.6. *Let u_0, b_0, s be as Theorem 1.1. Let $\alpha = \beta = 1$. If (u_0, b_0) satisfies* (1.5)*, with an additional condition, for all $t \geq 0$,*

$$\|b(t)\|_{L^\infty} \leq C_0 < 2, \tag{1.6}$$

then there exists a unique global classical solution (u, b) of (1.1)*.*

Remark 1.7. It seems that the condition (1.6) is too strong due to global assumption. However, we find that (1.4) can be propagated to any t, which lead to the global small of $\|b(t)\|_{\dot{H}^{3/2}}$ or $\|b(t)\|_{\dot{B}^{3/2}_{2,1}}$, while (1.6) fails. In addition, $\dot{B}^{3/2}_{2,1} \hookrightarrow L^\infty$, and we do not need sufficient small of $\|b(t)\|_{L^\infty}$, which seems to make (1.5) and (1.6) weaker than the second condition of (1.4) in some sense.

For $\alpha = \beta = 1$, the authors in [3] also establish some regularity criterions involving u and ∇b. Since the presence of Hall-term $\nabla \times ((\nabla \times b) \times b)$, we find that

$$(u, b) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))$$

can not ensure the regularity criterion established in [3]. Therefore, unlike MHD system, establishing the regularity criterion in terms of horizontal gradient $\nabla_h u$ and $\nabla_h b$ seems formidable and interesting. But for system (1.1) with $1 < \alpha < 5/4$, $5/4 \leq \beta < 7/4$, we can achieve this goal. We have the following theorems, where Theorem 1.8 can be considered as a component of the proof of Theorem 1.10.

Theorem 1.8. *Let $T > 0$, $\alpha \in (1, 5/4)$, $\beta \in (1, 7/4)$. Let u_0, b_0, s be as Theorem 1.1 and (u, b) be the local classical solution to* (1.1)*. If*

$$\int_0^T \|u\|^{q_1}_{L^{p_1}} + \|\nabla b\|^{q_2}_{L^{p_2}} dt < \infty \tag{1.7}$$

for

$$\begin{gathered}
\frac{3}{p_1} + \frac{2\alpha}{q_1} \leq \min\big\{2\alpha - 1, (1 - \frac{\alpha}{\beta})\frac{3}{p_1} + (2 - \frac{1}{\beta})\alpha\big\},\\
p_1 \in (\frac{3}{2\beta - 1}, \frac{3}{\beta - 1}] \cap (\frac{3}{2\alpha - 1}, \frac{3}{\alpha - 1}],\\
\frac{3}{p_2} + \frac{2\beta}{q_2} \leq 2\beta - 1, \quad p_2 \in (\frac{3}{2\beta - 1}, \frac{3}{\beta - 1}],
\end{gathered}$$

then (u, b) remains regular in $[0, T]$.

Remark 1.9. For $1 < \alpha = \beta < 5/4$, the condition of index reduce to
$$\frac{3}{p_1} + \frac{2\alpha}{q_1} \le 2\alpha - 1 \quad \text{and} \quad \frac{3}{p_2} + \frac{2\alpha}{q_2} \le 2\beta - 1,$$
which is scaling invariance under the transformation in Remark 1.3. We can also establish the regularity criterion in terms of ∇u and ∇b, that is
$$\int_0^T \|\nabla u\|_{L^{p_1}}^{q_1} + \|\nabla b\|_{L^{p_2}}^{q_2} d\tau < \infty$$
for
$$\frac{3}{p_1} + \frac{2\alpha}{q_1} \le \min\big\{2\alpha, (1 - \frac{\alpha}{\beta})\frac{3}{p_1} + 2\alpha\big\}, \quad \max\big\{\frac{3}{2\alpha}, \frac{3}{2\beta}\big\} < p_1 \le \infty$$
$$\frac{3}{p_2} + \frac{2\beta}{q_2} \le 2\beta - 1, \quad p_2 \in \big(\frac{3}{2\beta - 1}, \frac{3}{\beta - 1}\big].$$
The proof is similar to the previous proofs.

Theorem 1.10. *Let $T > 0$, $\alpha \in (1, 5/4), \beta \in [5/4, \frac{7}{4})$. Let u_0, b_0, s be as Theorem 1.1 and (u, b) be the local classical solution to* (1.1)*. If*
$$\int_0^T \|\nabla_h u\|_{L^{p_1}}^{q_1} + \|\nabla_h b\|_{L^{p_2}}^{q_2} dt < \infty \tag{1.8}$$
for
$$\frac{3}{p_1} + \frac{2\alpha}{q_1} \le 2\alpha, \quad \frac{3}{2\alpha} < p_1 \le \infty,$$
$$\frac{3}{p_2} + \frac{2\beta}{q_2} \le 2\beta - 1, \quad \frac{3}{2\beta - 1} < p_2 \le \frac{3}{\alpha},$$
then (u, b) remains regular in $[0, T]$.

Remark 1.11. We do not know whether the regularity criterion (1.8) can be established for $\alpha \in (1, \frac{5}{4}), \beta \in (1, \frac{5}{4})$, since we only observe that
$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^{\alpha+1}(\mathbb{R}^3)),$$
$$b \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^{\beta+1}(\mathbb{R}^3))$$
can lead to (1.7) for $\alpha \in (1, 5/4), \beta \in [5/4, 7/4)$.

This article is organized as follows. In section 2, we give some notation and preliminaries. In the third section, we prove Theorem 1.1. In the fourth section, we give the proof of Theorem 1.4. In the fifth section, we give the proof of Theorem 1.6. We prove Theorem 1.8 in the following section. At the last section, we prove and Theorem 1.10.

2. Preliminaries

Let $\mathfrak{B} = \{\xi \in \mathbb{R}^d,\ |\xi| \le \frac{4}{3}\}$ and $\mathfrak{C} = \{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$. Choose two nonnegative smooth radial function χ, φ supported, respectively, in $\mathfrak{B}$ and $\mathfrak{C}$ such that
$$\chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,$$
$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote $\varphi_j = \varphi(2^{-j}\xi)$, $h = \mathfrak{F}^{-1}\varphi$ and $\tilde{h} = \mathfrak{F}^{-1}\chi$, where $\mathfrak{F}^{-1}$ stands for the inverse Fourier transform. Then the dyadic blocks Δ_j and S_j can be defined as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)f(x-y)dy,$$
$$S_j f = \sum_{k\le j-1}\Delta_k f = \chi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d}\tilde{h}(2^j y)f(x-y)dy.$$

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to annulus $\{C_1 2^j \le |\xi| \le C_2 2^j\}$, and S_j is a frequency projection to the ball $\{|\xi| \le C2^j\}$. One can easily verify that with our choice of φ,

$$\Delta_j\Delta_k f = 0 \text{ if } |j-k|\ge 2 \quad\text{and}\quad \Delta_j(S_{k-1}f\Delta_k f) = 0 \text{ if } |j-k|\ge 5.$$

With the introduction of Δ_j and S_j, let us recall the definition of the Besov space.

Let $s\in\mathbb{R}$, $(p,q)\in[1,\infty]^2$, the homogeneous space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{f\in\mathfrak{S}' : \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases}\big(\sum_{j\in\mathbb{Z}} 2^{sjq}\|\Delta_j f\|^q_{L^p}\big)^{1/q}, & \text{for } 1\le q<\infty,\\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j f\|_{L^p}, & \text{for } q=\infty,\end{cases}$$

and $\mathfrak{S}'$ denotes the dual space of

$$\mathfrak{S} = \{f\in\mathcal{S}(\mathbb{R}^d) : \partial^\alpha\hat{f}(0) = 0 : \forall\alpha\in\mathbb{N}^d \text{multi-index}\}$$

and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

We also provide the definition for the inhomogeneous Besov space. For $s>0$, and $(p,q)\in[1,\infty]^2$, the inhomogeneous Besov space $B^s_{p,q}$ can be defined as follows

$$B^s_{p,q} = \{f\in\mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}.$$

Additionally, when $p=q=2$, the Besov space and Sobolev space are equivalence; that is

$$\dot{H}^s \approx \dot{B}^s_{2,2}, \quad H^s \approx B^s_{2,2}.$$

Bernstein's inequalities are useful in this paper, so that we give it in the following proposition.

Proposition 2.1. *Let $\alpha\ge 0$. Let $1\le p\le q\le\infty$.*

(1) *If f satisfies*
$$\operatorname{supp}\widehat{f}\subset\{\xi\in\mathbb{R}^d : |\xi|\le K2^j\},$$
for some integer j and a constant $K>0$, then
$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If f satisfies*
$$\operatorname{supp}\widehat{f}\subset\{\xi\in\mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$$
for some integer j and constants $0<K_1\le K_2$, then
$$C_1 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$
where C_1 and C_2 are constants depending on α, p and q only.

For more details about Besov space such as some useful embedding relations, we refer to [7, 8]. Thanks to the Proposition 2.1, we can see that $\forall\ s > 0$,

$$\|f\|_{H^s} \approx \|f\|_{L^2} + (\sum_{j\geq 0} 2^{2js}\|\Delta_j f\|_{L^2}^2)^{1/2}, \tag{2.1}$$

which will be frequently used in our proof.

Proposition 2.2 ([10]). *Let $1 \leq p_1,\ p_2 \leq \infty$, $\sigma > 0$, $\frac{d}{p_i} - \sigma_i > 0 (i = 1, 2)$ and assume that $\sigma - \sigma_2 + \frac{d}{p_2} > 0$. Then the following inequality holds*

$$\begin{aligned} &\Big(\sum_{j\in\mathbb{Z}} 2^{2j\sigma}\|[f, \Delta_j]\cdot\nabla g\|_2^2\Big)^{1/2} \\ &\leq C\Big(\|\nabla f\|_{\dot B^{\sigma_1}_{p_1,\infty}}\|g\|_{\dot B_{2,2}^{\sigma-\sigma_1+\frac{d}{p_1}}} + \|\nabla g\|_{\dot B^{\sigma_2}_{p_2,\infty}}\|f\|_{\dot B_{2,2}^{\sigma-\sigma_2+\frac{d}{p_2}}}\Big). \end{aligned} \tag{2.2}$$

3. Proof of Theorem 1.1

From [4], one can see that the local well-posedness of (1.1) holds for $\alpha \geq 0, \beta > 1/2$. So we only need establish the global regularity, i.e. for all $0 \leq t \leq T$,

$$\|u(t)\|_{H^s} + \|b(t)\|_{H^s} \leq C(s, T, \|u_0\|_{H^s}, \|u_0\|_{H^s}).$$

At first, we give the energy estimate. Taking inner product with (u, b), integrating by parts and integrating in time $[0, t]$,

$$\|(u(t), b(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\alpha u\|_{L^2}^2 d\tau + 2\int_0^t \|\Lambda^\beta b\|_{L^2}^2 d\tau \leq \|(u_0, b_0)\|_{L^2}^2. \tag{3.1}$$

Then we establish the H^s estimate. Applying the operator Δ_q to (1.1), taking the inner product with $(\Delta_q u, \Delta_q b)$, by cancelation property, integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\Delta_q u\|_{L^2}^2 + \|\Delta_q b\|_{L^2}^2) + \|\Lambda^\alpha\Delta_q u\|_{L^2}^2 + \|\Lambda^\beta\Delta_q b\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^3}[\Delta_q, u\cdot\nabla]u\cdot\Delta_q u + \int_{\mathbb{R}^3}[\Delta_q, b\cdot\nabla]b\cdot\Delta_q u - \int_{\mathbb{R}^3}[\Delta_q, u\cdot\nabla]b\cdot\Delta_q b \\ &\quad + \int_{\mathbb{R}^3}[\Delta_q, b\cdot\nabla]u\cdot\Delta_q b + \int_{\mathbb{R}^3}[\Delta_q, b\times]J\cdot\Delta_q J \\ &\leq \|[\Delta_q, u\cdot\nabla]u\|_{L^2}\|\Delta_q u\|_{L^2} + \|[\Delta_q, b\cdot\nabla]b\|_{L^2}\|\Delta_q u\|_{L^2} \\ &\quad + \|[\Delta_q, u\cdot\nabla]b\|_{L^2}\|\Delta_q b\|_{L^2} + \|[\Delta_q, b\cdot\nabla]u\|_{L^2}\|\Delta_q b\|_{L^2} \\ &\quad + \|[\Delta_q, b\times]J\|_{L^2}\|\Delta_q J\|_{L^2} \\ &= L_1(t) + L_2(t) + L_3(t) + L_4(t) + L_5(t). \end{aligned} \tag{3.2}$$

By the paraproduct decomposition, Hölder's inequality, commutator estimate [7, page. 110], and Bernstein's inequality,

$$
\begin{aligned}
|L_1(t)| &\le \sum_{|k-q|\le 4} \|\Delta_q(S_{k-1}u\cdot\nabla\Delta_k u) - S_{k-1}u\cdot\nabla\Delta_q\Delta_k u\|_{L^2}\|\Delta_q u\|_{L^2} \\
&\quad + \sum_{|k-q|\le 4} \|\Delta_q(\Delta_k u\cdot\nabla S_{k-1}u) - \Delta_k u\cdot\nabla\Delta_q S_{k-1}u\|_{L^2}\|\Delta_q u\|_{L^2} \\
&\quad + \sum_{k\ge q-3} \|\Delta_q(\tilde\Delta_k u\cdot\nabla\Delta_k u) - \tilde\Delta_k u\cdot\nabla\Delta_q\Delta_k u\|_{L^2}\|\Delta_q u\|_{L^2} \\
&\le C\|\nabla S_{q-1}u\|_{L^\infty}\|\Delta_q u\|_{L^2}^2 + C\|\Delta_q u\|_{L^2}\sum_{k\ge q-3}\|\nabla\Delta_k u\|_{L^\infty}\|\tilde\Delta_k u\|_{L^2},
\end{aligned}
\tag{3.3}
$$

where $\tilde\Delta_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Similarly,

$$
|L_2(t)| \le C\|\nabla S_{q-1}b\|_{L^\infty}\|\Delta_q b\|_{L^2}\|\Delta_q u\|_{L^2} + C\|\Delta_q u\|_{L^2}\sum_{k\ge q-3}\|\nabla\Delta_k b\|_{L^\infty}\|\tilde\Delta_k b\|_{L^2},
$$

$$
\begin{aligned}
|L_3(t)| &\le C\|\nabla S_{q-1}u\|_{L^\infty}\|\Delta_q b\|_{L^2}^2 + \|\nabla S_{q-1}b\|_{L^\infty}\|\Delta_q u\|_{L^2}\|\Delta_q b\|_{L^2} \\
&\quad + C\|\Delta_q b\|_{L^2}\sum_{k\ge q-3}\|\nabla\Delta_k b\|_{L^\infty}\|\tilde\Delta_k u\|_{L^2}, \\
|L_4(t)| &\le C\|\nabla S_{q-1}b\|_{L^\infty}\|\Delta_q u\|_{L^2}\|\Delta_q b\|_{L^2} + \|\nabla S_{q-1}u\|_{L^\infty}\|\Delta_q b\|_{L^2}^2 \\
&\quad + C\|\Delta_q b\|_{L^2}\sum_{k\ge q-3}\|\nabla\Delta_k u\|_{L^\infty}\|\tilde\Delta_k b\|_{L^2},
\end{aligned}
$$

$$
|L_5(t)| \le C2^q\|\nabla S_{q-1}b\|_{L^\infty}\|\Delta_q b\|_{L^2}^2 + C2^q\|\Delta_q b\|_{L^2}\sum_{k\ge q-3}\|\nabla\Delta_k b\|_{L^\infty}\|\tilde\Delta_k b\|_{L^2}.
$$

Multiplying 2^{2sq} and taking the summation over $q\ge 0$ in (3.2),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\sum_{q\ge 0}2^{2sq}(\|\Delta_q u\|_{L^2}^2 + \|\Delta_q b\|_{L^2}^2)\Big\} \\
&+ \sum_{q\ge 0}2^{2(s+\alpha)q}\|\Delta_q u\|_{L^2}^2 + \sum_{q\ge 0}2^{2(s+\beta)q}\|\Delta_q b\|_{L^2}^2 \\
&\le \sum_{q\ge 0}2^{2sq}(L_1(t)+\dots+L_5(t)).
\end{aligned}
\tag{3.4}
$$

By (3.3), we have

$$
\begin{aligned}
\sum_{q\ge 0}2^{2sq}|L_1(t)| &\le C\sum_{q\ge 0}2^{2sq}\|\nabla S_{q-1}u\|_{L^\infty}\|\Delta_q u\|_{L^2}^2 \\
&\quad + C\sum_{q\ge 0}2^{2sq}\|\Delta_q u\|_{L^2}\sum_{k\ge q-3}\|\nabla\Delta_k u\|_{L^\infty}\|\tilde\Delta_k u\|_{L^2} \\
&= L_{11}(t) + L_{12}(t).
\end{aligned}
$$

For the estimate of $L_{11}(t)$. Using Bernstein's inequality, Hölder's inequality and Young's inequality,

$$
|L_{11}(t)| \le C\sum_{q\ge 0}2^{2sq}\|\Delta_q u\|_{L^2}^2\sum_{m\le q-2}2^{5m/2}\|\Delta_m u\|_{L^2}
$$

$$
\begin{aligned}
&\le C\sum_{q\ge 0}2^{(s+\alpha)q}\|\Delta_q u\|_{L^2}2^{(s-\alpha)q}\|\Delta_q u\|_{L^2}\sum_{m\le q-2}2^{5m/2}\|\Delta_m u\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^\alpha u\|_{\dot H^s}^2+\underbrace{C\sum_{q\ge 0}2^{2(s-\alpha)q}\|\Delta_q u\|_{L^2}^2\Big(\sum_{m\le q-2}2^{5m/2}\|\Delta_m u\|_{L^2}\Big)^2}_{L_{111}},
\end{aligned}
$$

where

$$
\begin{aligned}
|L_{111}| &= C\sum_{q\ge 0}2^{2sq}\|\Delta_q u\|_{L^2}^2\Big(\sum_{m\le q-2}2^{\frac52 m-\alpha q}\|\Delta_m u\|_{L^2}\Big)^2\\
&= C\sum_{q\ge 0}2^{2sq}\|\Delta_q u\|_{L^2}^2\Big(\sum_{m\le -2}2^{\frac52 m-\alpha q}\|\Delta_m u\|_{L^2}\\
&\quad+\sum_{-1\le m\le q-2}2^{(\frac52-\alpha)m-\alpha q}\|\Lambda^\alpha\Delta_m u\|_{L^2}\Big)^2\\
&\le C\sum_{q\ge 0}2^{2sq}\|\Delta_q u\|_{L^2}^2\Big(\sum_{m\le -2}2^{5m/2}\|u\|_{L^2}\\
&\quad+\sum_{-1\le m\le q-2}2^{\alpha(m-q)}2^{(\frac52-2\alpha)m}\|\Lambda^\alpha u\|_{L^2}\Big)^2\\
&\le C(\|u\|_{L^2}^2+\|\Lambda^\alpha u\|_{L^2}^2)\|u\|_{H^s}^2.
\end{aligned}
$$

Thus, we obtain

$$
|L_{11}(t)|\le C(\|u\|_{L^2}^2+\|\Lambda^\alpha u\|_{L^2}^2)\|u\|_{H^s}^2+\frac18\|\Lambda^\alpha u\|_{\dot H^s}^2.
$$

Similarly,

$$
\begin{aligned}
&|L_{12}(t)|\\
&\le C\sum_{q\ge 0}2^{2sq}\|\Delta_q u\|_{L^2}\sum_{k\ge q-3}\|\Delta_k u\|_{L^2}2^{5k/2}\|\tilde\Delta_k u\|_{L^2}\\
&\le C\Big(\sum_{p\ge 0}2^{2sq}\|\Delta_q u\|_{L^2}^2\Big)^{1/2}\Big\{\sum_{q\ge 0}2^{2sq}(\sum_{k\ge q-3}\|\tilde\Delta_k u\|_{L^2}2^{k(\frac52-\alpha)}\|\Lambda^\alpha\Delta_k u\|_{L^2})^2\Big\}^{1/2}\\
&\le C\|\Lambda^\alpha u\|_{L^2}\|u\|_{H^s}\Big\{\sum_{q\ge 0}2^{2sq}(\sum_{k\ge q-3}2^{k(\frac52-2\alpha)}\|\tilde\Delta_k\Lambda^\alpha u\|_{L^2})^2\Big\}^{1/2}\\
&\le C\|\Lambda^\alpha u\|_{L^2}\|u\|_{H^s}\Big\{\sum_{q\ge 0}2^{2sq}(\sum_{k\ge q-3}\|\tilde\Delta_k\Lambda^\alpha u\|_{L^2})^2\Big\}^{1/2}\\
&\le C\|\Lambda^\alpha u\|_{L^2}\|u\|_{H^s}\|u\|_{\dot H^{s+\alpha}}\\
&\le C\|\Lambda^\alpha u\|_{L^2}^2\|u\|_{H^s}^2+\frac18\|u\|_{\dot H^{s+\alpha}}^2,
\end{aligned}
$$

here we have used Young's inequality for series for the fifth inequality; that is,

$$
\begin{aligned}
\Big\{\sum_{q\ge 0}2^{2sq}(\sum_{k\ge q-3}\|\tilde\Delta_k\Lambda^\alpha u\|_{L^2})^2\Big\}^{1/2}&\le\Big\{\sum_{q\in\mathbb{Z}}(\sum_{k\ge q-3}2^{s(q-k)}2^{sk}\|\tilde\Delta_k\Lambda^\alpha u\|_{L^2})^2\Big\}^{1/2}\\
&\le C\|2^{-sk}\mathbf{1}_{k\ge -3}\|_{l^1(\mathbb{Z})}\|2^{sk}\|\tilde\Delta_k\Lambda^\alpha u\|_{L^2}\|_{l^2(\mathbb{Z})}\\
&\le C\|u\|_{\dot H^{s+\alpha}}.
\end{aligned}
$$

Collecting the estimates above, we have

$$\sum_{q\geq 0} 2^{2sq}|L_1(t)| \leq C(\|u\|_{L^2}^2 + \|\Lambda^\alpha u\|_{L^2}^2)\|u\|_{H^s}^2 + \frac{1}{4}\|\Lambda^\alpha u\|_{H^s}^2.$$

Similarly,

$$\sum_{q\geq 0} 2^{2sq}|L_2(t)| \leq C(\|b\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|b\|_{H^s}^2 + \frac{1}{4}\|\Lambda^\alpha u\|_{H^s}^2,$$

$$\sum_{q\geq 0} 2^{2sq}|L_3(t)| \leq C(\|(u,b)\|_{L^2}^2 + \|(\Lambda^\alpha u, \Lambda^\beta b)\|_{L^2}^2)\|b\|_{H^s}^2 + \frac{1}{4}\|(\Lambda^\alpha u, \Lambda^\beta b)\|_{H^s}^2,$$

$$\sum_{q\geq 0} 2^{2sq}|L_4(t)| \leq C(\|(u,b)\|_{L^2}^2 + \|(\Lambda^\alpha u, \Lambda^\beta b)\|_{L^2}^2)\|b\|_{H^s}^2 + \frac{1}{4}\|(\Lambda^\alpha u, \Lambda^\beta b)\|_{H^s}^2.$$

Now, we estimate the last term.

$$\begin{aligned}
\sum_{q\geq 0} 2^{2sq}|L_5(t)| &\leq C\sum_{q\geq 0} 2^{(2s+1)q}\|\Delta_q b\|_{L^2}^2 \sum_{m\leq q-2}\|\nabla\Delta_m b\|_{L^\infty} \\
&\quad + C\sum_{q\geq 0} 2^{(2s+1)q}\|\Delta_q b\|_{L^2}\sum_{k\geq q-3}\|\nabla\Delta_k b\|_{L^\infty}\|\tilde{\Delta}_k b\|_{L^2} \\
&= L_{51}(t) + L_{52}(t).
\end{aligned}$$

Similar to the estimate of $L_{11}(t)$, we have

$$\begin{aligned}
&|L_{51}(t)| \\
&\leq C\sum_{q\geq 0} 2^{(s+\beta)q}\|\Delta_q b\|_{L^2} 2^{(s+1-\beta)q}\|\Delta_q b\|_{L^2}\sum_{m\leq q-2} 2^{5m/2}\|\Delta_m b\|_{L^2} \\
&\leq \frac{1}{8}\sum_{q\geq 0} 2^{2(s+\beta)q}\|\Delta_q b\|_{L^2}^2 + \underbrace{C\sum_{q\geq 0} 2^{2(s+1-\beta)q}\|\Delta_q b\|_{L^2}^2\Big(\sum_{m\leq q-2} 2^{5m/2}\|\Delta_m b\|_{L^2}\Big)^2}_{L_{511}},
\end{aligned}$$

where

$$\begin{aligned}
|L_{511}| &\leq C\sum_{q\geq 0} 2^{2sq}\|\Delta_q b\|_{L^2}^2\Big(\sum_{m\leq -2} 2^{(1-\beta)q} 2^{5m/2}\|\Delta_m b\|_{L^2} \\
&\quad + \sum_{-1\leq m\leq q-2} 2^{(1-\beta)q} 2^{(\frac{5}{2}-\beta)m}\|\Lambda^\beta\Delta_m b\|_{L^2}\Big)^2 \\
&\leq C\sum_{q\geq 0} 2^{2sq}\|\Delta_q b\|_{L^2}^2 \sum\Big(\sum_{m\leq -2} 2^{5m/2}\|b\|_{L^2} \\
&\quad + \sum_{-1\leq m\leq q-2} 2^{(1-\beta)(q-m)} 2^{(\frac{7}{2}-2\beta)m}\|\Lambda^\beta\Delta_m b\|_{L^2}\Big)^2 \\
&\leq C(\|b\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\sum_{q\geq 0} 2^{2sq}\|\Delta_q b\|_{L^2}^2 \\
&\leq C(\|b\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|b\|_{H^s}^2.
\end{aligned}$$

So we have

$$|L_{51}(t)| \leq \frac{1}{8}\|b\|_{\dot{H}^{s+\beta}}^2 + C(\|b\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|b\|_{H^s}^2.$$

Similar to the estimate of $L_{12}(t)$, we have

$$\begin{aligned}
|L_{52}(t)| &\le C\sum_{q\ge 0} 2^{sq}\|\Delta_q b\|_{L^2} 2^{(s+1)q}\sum_{k\ge q-3}\|\nabla\Delta_q b\|_{L^\infty}\|\tilde{\Delta}_k b\|_{L^2}\\
&\le C\Big(\sum_{q\ge 0} 2^{2sq}\|\Delta_q b\|_{L^2}^2\Big)^{1/2}\Big\{\sum_{q\ge 0} 2^{2(s+1)q}(\sum_{k\ge q-3}\|\nabla\Delta_k b\|_{L^\infty}\|\tilde{\Delta}_k b\|_{L^2})^2\Big\}^{1/2}\\
&\le C\|b\|_{H^s}\Big\{\sum_{q\ge 0} 2^{2(s+1)q}\Big(\sum_{k\ge q-3} 2^{k(\frac{5}{2}-\beta)}\|\Lambda^\beta\Delta_k b\|_{L^2} 2^{-k\beta}\|\Lambda^\beta\tilde{\Delta}_k b\|_{L^2}\Big)^2\Big\}^{1/2}\\
&\le C\|b\|_{H^s}\Big\{\sum_{q\ge 0} 2^{2(s+1)q}(\sum_{k\ge q-3} 2^{-k}2^{k(\frac{7}{2}-2\beta)}\|\Lambda^\beta\Delta_k b\|_{L^2}\|\Lambda^\beta b\|_{L^2})^2\Big\}^{1/2}\\
&\le C\|\Lambda^\beta b\|_{L^2}\|b\|_{H^s}\Big\{\sum_{q\ge 0}(\sum_{k\ge q-3} 2^{(s+1)(q-k)}2^{ks}\|\Lambda^\beta\Delta_k b\|_{L^2})^2\Big\}^{1/2}\\
&\le C\|\Lambda^\beta b\|_{L^2}\|b\|_{H^s}\|b\|_{\dot{H}^{s+\beta}} \le C\|\Lambda^\beta b\|_{L^2}^2\|b\|_{H^s}^2 + \frac{1}{8}\|b\|_{\dot{H}^{s+\beta}}^2,
\end{aligned}$$

here we have used Young's inequality for the fifth inequality. Therefore,

$$\sum_{q\ge 0} 2^{2sq}|L_5(t)| \le C(\|\Lambda^\beta b\|_{L^2}^2 + \|b\|_{L^2}^2)\|b\|_{H^s}^2 + \frac{1}{4}\|\Lambda^\beta b\|_{\dot{H}^s}^2.$$

Collecting the estimate above in (3.2), integrating in $[0,t]$, together with (3.1) and (2.1) yields

$$\begin{aligned}
&\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \int_0^t \|\Lambda^\alpha u(\tau)\|_{H^s}^2 d\tau + \int_0^t \|\Lambda^\beta b(\tau)\|_{H^s}^2 d\tau\\
&\le C(\|(\Lambda^\alpha u,\Lambda^\beta b)\|_{L^2}^2 + \|(u,b)\|_{L^2}^2)(\|u\|_{H^s}^2 + \|b\|_{H^s}^2) + \|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2.
\end{aligned}$$

Applying Gronwall's lemma, with (3.1), for $0\le t\le T$,

$$\begin{aligned}
&\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \int_0^t \|\Lambda^\alpha u(\tau)\|_{H^s}^2 d\tau + \int_0^t \|\Lambda^\beta b(\tau)\|_{H^s}^2 d\tau\\
&\le (\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2)\exp\Big\{C\int_0^t (\|(\Lambda^\alpha u,\Lambda^\beta b)\|_{L^2}^2 + \|(u,b)\|_{L^2}^2)d\tau\Big\}\\
&\le (\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2)\exp\big(C\|(u_0,b_0)\|_{L^2}^2(1+T)\big).
\end{aligned}$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.4

As in the proof of Theorem 1.1, we only need to show the global regularity. Since $\|u\|_{H^s} = \|u\|_{L^2} + \|u\|_{\dot{H}^s}$, we obtain

$$\begin{aligned}
&\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + 2\int_0^t \|\Lambda^\alpha u(\tau)\|_{H^s}^2 d\tau + 2\int_0^t \|\Lambda^\beta b(\tau)\|_{H^s}^2 d\tau\\
&\le \|(u_0,b_0)\|_{H^s}^2 + C\int_0^t \|(\nabla u,\nabla b)\|_{L^\infty}\|(u,b)\|_{\dot{H}^s}^2 d\tau + 2\int_0^t \|\nabla b\|_{L^\infty}\|b\|_{\dot{H}^{s+\frac{1}{2}}}^2 d\tau\\
&\le \|(u_0,b_0)\|_{H^s}^2 + C\int_0^t \|(u,b)\|_{H^s}(\|\Lambda^\alpha u\|_{H^s}^2 + \|\Lambda^\beta b\|_{H^s}^2)d\tau.
\end{aligned}$$

A similar estimate can be found in [4]. Choosing ϵ so small that $\|(u_0,b_0)\|_{H^s}^2 < \frac{1}{2C^2}$, which implies that $C\|(u_0,b_0)\|_{H^s} < 1$. Suppose there exists a first time $T^\star$, such that

$$\|u(T^\star)\|_{H^s}^2 + \|b(T^\star)\|_{H^s}^2 \geq \frac{1}{2C^2}. \tag{4.1}$$

This leads to

$$\|u(T^\star)\|_{H^s}^2 + \|u(T^\star)\|_{H^s}^2 + \int_0^{T^\star} \|\Lambda^\alpha u(t)\|_{H^s}^2 + \|\Lambda^\beta b(t)\|_{H^s}^2 dt \leq \|(u_0,b_0)\|_{H^s}^2 < \frac{1}{2C^2},$$

which contradicts (4.1). Hence, for all $t \geq 0$, we get global small solution satisfying

$$\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \int_0^t \|\Lambda^\alpha u(\tau)\|_{H^s}^2 d\tau + \int_0^t \|\Lambda^\beta b(\tau)\|_{H^s}^2 d\tau \leq \|(u_0,b_0)\|_{H^s}^2.$$

This concludes the proof of Theorem 1.4.

5. Proof of Theorem 1.6

It suffices to establish the following, for all $t \geq 0$,

$$\int_0^t \|u\|_{\dot H^{3/2}}^2 + \|\nabla b\|_{\dot H^{3/2}}^2 d\tau < \infty,$$

since [3] provided the blow up criterion in the BMO space and $\dot H^{3/2} \hookrightarrow BMO$ [7]. As the operation in section 3, we obtain

$$\begin{aligned}
&\frac12 \frac{d}{dt}(\|\Delta_q u\|_{L^2}^2 + \|\Delta_q b\|_{L^2}^2) + \|\nabla\Delta_q u\|_{L^2}^2 + \|\nabla\Delta_q b\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} [\Delta_q, u\cdot\nabla]u\cdot\Delta_q u + \int_{\mathbb{R}^3} [\Delta_q, b\cdot\nabla]b\cdot\Delta_q u - \int_{\mathbb{R}^3} [\Delta_q, u\cdot\nabla]b\cdot\Delta_q b \\
&\quad - \int_{\mathbb{R}^3} [\Delta_q, b\cdot\nabla]u\cdot\Delta_q b + \int_{\mathbb{R}^3} [\Delta_q, b\times]J\cdot\Delta_q J \\
&\leq \|[\Delta_q, u\cdot\nabla]u\|_{L^2}\|\Delta_q u\|_{L^2} + \|[\Delta_q, b\cdot\nabla]b\|_{L^2}\|\Delta_q u\|_{L^2} \\
&\quad + \|[\Delta_q, u\cdot\nabla]b\|_{L^2}\|\Delta_q b\|_{L^2} + \|[\Delta_q, b\cdot\nabla]u\|_{L^2}\|\Delta_q b\|_{L^2} \\
&\quad + \|[\Delta_q, b\times]J\|_{L^2}\|\Delta_q J\|_{L^2} \\
&= IL_1(t) + \dots + IL_5(t).
\end{aligned} \tag{5.1}$$

Multiplying (5.1) by 2^q and summing over $q \in \mathbb{Z}$,

$$\frac12 \frac{d}{dt}(\|u\|_{\dot H^{1/2}}^2 + \|b\|_{\dot H^{1/2}}^2) + \|\nabla u\|_{\dot H^{1/2}}^2 + \|\nabla b\|_{\dot H^{1/2}}^2 \leq \sum_{q\in\mathbb{Z}} 2^q |IL_1(t) + \dots + IL_5(t)|. \tag{5.2}$$

By Hölder's inequality and choosing $\sigma = \sigma_i = \frac12$, $p_i = 2$, $i = 1,2$ in (2.2),

$$\sum_{q\in\mathbb{Z}} 2^q |IL_1(t)| \leq \Big(\sum_{q\in\mathbb{Z}} 2^q \|[\Delta_q, u]\cdot\nabla u\|_{L^2}^2\Big)^{1/2} \|u\|_{\dot H^{1/2}} \leq C\|u\|_{\dot H^{1/2}}\|u\|_{\dot H^{3/2}}^2.$$

Similarly,

$$\sum_{q\in\mathbb{Z}} 2^q |IL_2(t)| \leq C\|u\|_{\dot H^{1/2}}\|b\|_{\dot H^{3/2}}^2,$$

$$\sum_{q\in\mathbb{Z}} 2^q |IL_3(t)| \leq C\|u\|_{\dot H^{1/2}}(\|u\|_{\dot H^{3/2}}^2 + \|b\|_{\dot H^{3/2}}^2),$$

$$\sum_{q\in\mathbb{Z}} 2^q|IL_4(t)| \le C\|u\|_{\dot H^{1/2}}(\|u\|^2_{\dot H^{3/2}}+\|b\|^2_{\dot H^{3/2}}),\quad \sum_{q\in\mathbb{Z}} 2^q|IL_5(t)| \le C\|b\|^3_{\dot H^{3/2}}.$$

Plugging the inequalities above in (5.1), we have

$$\begin{aligned}&\frac12\frac{d}{dt}\|(u,b)\|^2_{\dot H^{1/2}}+\|\nabla u\|^2_{\dot H^{1/2}}+\|\nabla b\|^2_{\dot H^{1/2}}\\ &\le C(\|(u,b)\|_{\dot H^{1/2}}+\|b\|_{\dot H^{3/2}})(\|u\|^2_{\dot H^{3/2}}+\|b\|^2_{\dot H^{3/2}}).\end{aligned}\tag{5.3}$$

Next, we give the $\dot H^{3/2}$ estimate of b. With a similar process, we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\|\Delta_q b\|^2_{L^2}+\|\nabla\Delta_q b\|^2_{L^2}\\ &=\int_{\mathbb{R}^3}[\Delta_q, b\cdot\nabla]u\cdot\Delta_q b+\int_{\mathbb{R}^3} b\cdot\nabla\Delta_q u\cdot\Delta_q b\\ &\quad-\int_{\mathbb{R}^3}[\Delta_q,u\cdot\nabla]b\cdot\Delta_q b+\int_{\mathbb{R}^3}[\Delta_q, b\times]J\cdot\Delta_q J\\ &\le\|[\Delta_q,b\cdot\nabla]u\|_{L^2}\|\Delta_q b\|_{L^2}+\|b\cdot\nabla\Delta_q u\|_{L^2}\|\Delta_q b\|_{L^2}\\ &\quad+\|[\Delta_q,u\cdot\nabla]b\|_{L^2}\|\Delta_q b\|_{L^2}+\|[\Delta_q,b\times]J\|_{L^2}\|\Delta_q J\|_{L^2}\\ &=K_1(t)+\dots+K_5(t).\end{aligned}$$

Multiplying 2^{3q} and summing over $q\in\mathbb{Z}$,

$$\frac12\frac{d}{dt}\|b\|^2_{\dot H^{3/2}}+\|b\|^2_{\dot H^{5/2}}\le\sum_{q\in\mathbb{Z}}2^{3q}|K_1(t)+\dots+K_5(t)|.\tag{5.4}$$

By Hölder's inequality and choosing $\sigma=\sigma_i=\frac12$, $p_i=2$, $i=1,2$ in (2.2),

$$\begin{aligned}\sum_{q\in\mathbb{Z}}2^{3q}|K_1(t)|&=\sum_{q\in\mathbb{Z}}2^{3q}\|[\Delta_q,b\cdot\nabla]u\|_{L^2}\|\Delta_q b\|_{L^2}\\ &\le\Big(\sum_{q\in\mathbb{Z}}2^q\|[\Delta_q,b\cdot\nabla]u\|^2_{L^2}\Big)^{1/2}\|b\|_{\dot H^{5/2}}\\ &\le C\|b\|_{\dot H^{3/2}}(\|u\|^2_{\dot H^{3/2}}+\|b\|^2_{\dot H^{5/2}}).\end{aligned}$$

Similarly,

$$\sum_{q\in\mathbb{Z}}2^{3q}|K_3(t)|\le C\|b\|_{\dot H^{3/2}}(\|b\|^2_{\dot H^{5/2}}+\|u\|^2_{\dot H^{3/2}}).$$

By Hölder's inequality and choosing $\sigma=3/2$, $\sigma_i=\frac12$, $p_i=2$, $i=1,2$ in (2.2), then

$$\sum_{q\in\mathbb{Z}}2^{3q}|K_4(t)|\le\Big(\sum_{q\in\mathbb{Z}}2^{3q}\|[\Delta_q,b\times]J\|^2_{L^2}\Big)^{1/2}\|J\|_{\dot H^{3/2}}\le C\|b\|_{\dot H^{3/2}}\|b\|^2_{\dot H^{5/2}}.$$

Using Hölder's inequality and condition (1.6),

$$\begin{aligned}\sum_{q\in\mathbb{Z}}2^{3q}|K_2(t)|&\le\Big(\sum_{q\in\mathbb{Z}}2^q\|b\cdot\nabla\Delta_q u\|^2_{L^2}\Big)^{1/2}\|b\|_{\dot H^{5/2}}\\ &\le\|b\|_{L^\infty}\|u\|_{\dot H^{3/2}}\|b\|_{\dot H^{5/2}}\\ &\le\frac{C_0}{2}(\|u\|^2_{\dot H^{3/2}}+\|b\|^2_{\dot H^{5/2}}).\end{aligned}$$

Plugging the inequalities above in (5.4), combining with (5.3) and integrating the resulting inequality in $[0,t]$ gives

$$\begin{aligned}
&\|(u(t),b(t))\|_{\dot H^{1/2}}^2+\|b(t)\|_{\dot H^{3/2}}^2+(2-C_0)\int_0^t\|(u(\tau),b(\tau))\|_{\dot H^{3/2}}^2+\|b(\tau)\|_{\dot H^{5/2}}^2d\tau\\
&\le C\int_0^t(\|(u(\tau),b(\tau))\|_{\dot H^{1/2}}+\|b(\tau)\|_{\dot H^{3/2}})(\|(u(\tau),b(\tau))\|_{\dot H^{3/2}}^2+\|b(\tau)\|_{\dot H^{5/2}}^2)d\tau\\
&\quad+\|(u_0,b_0)\|_{\dot H^{1/2}}^2+\|b_0\|_{\dot H^{3/2}}^2.
\end{aligned}$$

Choose ϵ so small that $3(\|u_0\|_{\dot H^{1/2}}^2+\|b_0\|_{\dot H^{1/2}}^2+\|b_0\|_{\dot H^{3/2}}^2)<(\frac{2-C_0}{4C})^2$, which implies

$$C\Big(\|u_0\|_{\dot H^{1/2}}+\|b_0\|_{\dot H^{1/2}}+\|b_0\|_{\dot H^{3/2}}\Big)<\frac{2-C_0}{4}.$$

Suppose there exists a first time $T^\star$ such that

$$3\Big(\|u(T^\star)\|_{\dot H^{1/2}}^2+\|b(T^\star)\|_{\dot H^{1/2}}^2+\|b(T^\star)\|_{\dot H^{3/2}}^2\Big)\ge\big(\frac{2-C_0}{4C}\big)^2,$$

which can easily get a contradiction. Hence, for all $t\ge0$, we obtain

$$\begin{aligned}
&\|(u(t),b(t))\|_{\dot H^{1/2}}^2+\|b(t)\|_{\dot H^{3/2}}^2+(2-C_0)\int_0^t\|(u(\tau),b(\tau))\|_{\dot H^{3/2}}^2+\|b(\tau)\|_{\dot H^{5/2}}^2d\tau\\
&\le\|(u_0,b_0)\|_{\dot H^{1/2}}^2+\|b_0\|_{\dot H^{3/2}}^2.
\end{aligned}$$

This completes the proof of Theorem 1.6.

6. Proof of Theorem 1.8

Our proof contains two steps, H^1 estimates and H^s estimates.

Step 1: H^1 Estimates. Using a similar procedure as in [3], we have

$$\begin{aligned}
&\frac12\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\|\nabla\Lambda^\alpha u\|_{L^2}^2+\|\nabla\Lambda^\beta b\|_{L^2}^2\\
&=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\,dx+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla b)\cdot\nabla u\,dx+\int_{\mathbb{R}^3}\nabla(b\cdot\nabla u)\cdot\nabla b\,dx\\
&\quad-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b\,dx-\int_{\mathbb{R}^3}\nabla(J\times b)\cdot\nabla J\,dx\\
&=I_1(t)+\dots+I_5(t).
\end{aligned}\tag{6.1}$$

By integrate by parts, Hölder's inequality, interpolation inequality and Young's inequality, let $\theta_1=\frac{3-(\alpha-1)p_1}{\alpha p_1}$, we obtain

$$\begin{aligned}
|I_1(t)|&\le|\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx|\\
&\le\|u\|_{L^{p_1}}\|\nabla u\|_{L^{\frac{6p_1}{(1+2\alpha)p_1-6}}}\|\Delta u\|_{L^{\frac{6}{5-2\alpha}}}\\
&\le\|u\|_{L^{p_1}}\|\nabla u\|_{L^2}^{1-\theta_1}\|\nabla\Lambda^\alpha u\|_{L^2}^{1+\theta_1}\\
&\le C\|u\|_{L^{p_1}}^{\frac{2\alpha p_1}{(2\alpha-1)p_1-3}}\|\nabla u\|_{L^2}^2+\frac18\|\nabla\Lambda^\alpha u\|_{L^2}^2.
\end{aligned}$$

By cancelation property,

$$\int_{\mathbb{R}^3}(b\cdot\nabla)\partial_i b\cdot\partial_i u\,dx+\int_{\mathbb{R}^3}(b\cdot\nabla)\partial_i u\cdot\partial_i b\,dx=0,$$

integrate by parts, Hölder inequality and Young's inequality,

$$\begin{aligned}
|I_2(t)+I_3(t)| &= \Big|\int_{\mathbb{R}^3}(b\cdot\nabla)\partial_i b\cdot\partial_i u\,dx+\int_{\mathbb{R}^3}(\partial_i b\cdot\nabla)b\cdot\partial_i u\,dx\\
&\quad+\int_{\mathbb{R}^3}(b\cdot\nabla)\partial_i u\cdot\partial_i b\,dx+\int_{\mathbb{R}^3}(\partial_i b\cdot\nabla)u\cdot\partial_i b\,dx\Big|\\
&= \Big|\int_{\mathbb{R}^3}(\partial_i b\cdot\nabla)b\cdot\partial_i u\,dx+\int_{\mathbb{R}^3}(\partial_i b\cdot\nabla)u\cdot\partial_i b\,dx\Big|\\
&\le \int_{\mathbb{R}^3}|u||\nabla b||\nabla^2 b|\,dx\\
&\le C\|u\|_{L^{p_1}}^{\frac{2\beta p_1}{(2\beta-1)p_1-3}}\|\nabla b\|_{L^2}^2+\frac{1}{8}\|\nabla\Lambda^\beta b\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$|I_4(t)|\le C\|u\|_{L^{p_1}}^{\frac{2\beta p_1}{(2\beta-1)p_1-3}}\|\nabla b\|_{L^2}^2+\frac{1}{8}\|\nabla\Lambda^\beta b\|_{L^2}^2,$$

$$|I_5(t)|=\int_{\mathbb{R}^3}|\nabla b||\nabla b||\nabla^2 b|\le C\|\nabla b\|_{L^{p_2}}^{\frac{2\beta p_2}{(2\beta-1)p_2-3}}\|\nabla b\|_{L^2}^2+\frac{1}{8}\|\nabla\Lambda^\beta b\|_{L^2}^2,$$

where it would be used

$$\int_{\mathbb{R}^3}[(\nabla J)\times b]\cdot(\nabla J)\,dx=0.$$

Collecting the inequalities above in (6.1), together with energy estimate, integrating in time and by Gronwall's lemma yields that

$$\begin{aligned}
&\|u\|_{H^1}^2+\|b\|_{H^1}^2+\int_0^t\|\Lambda^\alpha u\|_{H^1}^2d\tau+\int_0^t\|\Lambda^\beta b\|_{H^1}^2d\tau\\
&\le\|(u_0,b_0)\|_{H^1}^2\exp\big\{C\int_0^t(\|u\|_{L^{p_1}}^{\frac{2\alpha p_1}{(2\alpha-1)p_1-3}}+\|u\|_{L^{p_1}}^{\frac{2\beta p_1}{(2\beta-1)p_1-3}}+\|\nabla b\|_{L^{p_2}}^{\frac{2\beta p_2}{(2\beta-1)p_2-3}})d\tau\big\}.
\end{aligned}\tag{6.2}$$

Step 2: H^s Estimates. Applying the operator Λ^s to (1.1), taking the inner product with $(\Lambda^s u,\Lambda^s b)$, with energy estimate, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|_{H^s}^2+\|b\|_{H^s}^2)+\|\Lambda^\alpha u\|_{H^s}^2+\|\Lambda^\beta b\|_{H^s}^2\\
&=-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla)u\cdot\Lambda^s u\,dx+\int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla)b\cdot\Lambda^s u\,dx+\int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla)u\cdot\Lambda^s b\,dx\\
&\quad-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla)b\cdot\Lambda^s b\,dx-\int_{\mathbb{R}^3}\Lambda^s((\nabla\times b)\times b)\cdot\Lambda^s(\nabla\times b)dx\\
&=II_1(t)+\dots+II_5(t).
\end{aligned}\tag{6.3}$$

Using Hölder's inequality and Young's inequality,

$$\begin{aligned}
|II_1(t)|&\le\|\Lambda^{s+1-\alpha}(u\otimes u)\|_{L^2}\|\Lambda^{s+\alpha}u\|_{L^2}\\
&\le C\|u\|_{L^\infty}\|u\|_{H^{s+1-\alpha}}\|\Lambda^\alpha u\|_{H^s}\\
&\le C\|u\|_{L^\infty}^2\|u\|_{H^s}^2+\frac{1}{8}\|\Lambda^\alpha u\|_{H^s}^2.
\end{aligned}$$

Similarly,

$$|II_2(t)| \le C\|b\|_{L^\infty}^2\|b\|_{H^s}^2 + \frac{1}{8}\|\Lambda^\alpha u\|_{H^s}^2,$$
$$|II_3(t), II_4(t)| \le C(\|u\|_{L^\infty}^2 + \|b\|_{L^\infty}^2)(\|u\|_{H^s}^2 + \|b\|_{H^s}^2) + \frac{1}{8}\|\Lambda^\beta b\|_{H^s}^2.$$

For the estimate of $II_5(t)$, let $\theta_2 = \frac{3-(\beta-1)p_2}{\beta p_2}$, then

$$\begin{aligned}
|II_5(t)| &\le \|\Lambda^s[(\nabla\times b)\times b] - [\Lambda^s(\nabla\times b)]\times b\|_{L^{\frac{6}{2\beta+1}}}\|\nabla\Lambda^s b\|_{L^{\frac{6}{5-2\beta}}} \\
&\le C\|\nabla b\|_{L^{p_2}}\|\Lambda^s b\|_{L^{\frac{6p_2}{(1+2\beta)p_2-6}}}\|\Lambda^\beta b\|_{H^s} \\
&\le C\|\nabla b\|_{L^{p_2}}\|\Lambda^s b\|_{L^2}^{1-\theta_2}\|\Lambda^\beta b\|_{H^s}^{1+\theta_2} \\
&\le C\|\nabla b\|_{L^{p_2}}^{\frac{2\beta p_2}{(2\beta-1)p_2-3}}\|\Lambda^s b\|_{L^2}^2 + \frac{1}{8}\|\Lambda^\beta b\|_{L^2}^2.
\end{aligned}$$

Plugging the estimate above in (6.3), applying Gronwall's lemma, we have

$$\begin{aligned}
&\|u\|_{H^s}^2 + \|b\|_{H^s}^2 + \int_0^t \|\Lambda^\alpha u\|_{H^s}^2 d\tau + \int_0^t \|\Lambda^\beta b\|_{H^s}^2 d\tau \\
&\le \|(u_0,b_0)\|_{H^s}^2 \exp\{C\exp\int_0^t (\|(u,b)\|_{L^\infty}^2 + \|\nabla b\|_{L^{p_2}}^{\frac{2\beta p_2}{(2\beta-1)p_2-3}})d\tau\}.
\end{aligned}$$

Combining the interpolation

$$\|u\|_{L^\infty} \le C\|u\|_{H^1}^{1-\frac{1}{2\alpha}}\|\nabla\Lambda^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}, \quad \|b\|_{L^\infty} \le C\|b\|_{H^1}^{1-\frac{1}{2\beta}}\|\nabla\Lambda^\beta b\|_{L^2}^{\frac{1}{2\beta}}$$

with (1.7) and (6.2) yields the desired results. This completes the proof of Theorem 1.8.

7. Proof of Theorem 1.10

Using similar operation as the proof of Theorem 1.8, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \\
&= \sum_{i,j,k=1}^3\int_{\mathbb{R}^3}\partial_k(u_i\partial_i u_j)\partial_k u_j + \sum_{i,j,k=1}^3\int_{\mathbb{R}^3}\partial_k(b_i\partial_i b_j)\partial_k u_j \\
&\quad + \sum_{i,j,k=1}^3\int_{\mathbb{R}^3}\partial_k(b_i\partial_i u_j)\partial_k b_j - \sum_{i,j,k=1}^3\int_{\mathbb{R}^3}\partial_k(u_i\partial_i b_j)\partial_k b_j \qquad (7.1) \\
&\quad - \sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i(J\times b)\cdot\partial_i J \\
&= III_1(t) + \dots + III_5(t).
\end{aligned}$$

Integrating by parts, and using

$$\int_{\mathbb{R}^3} u_i\partial_i\partial_k u_j\partial_k u_j\,dx = 0,$$

we can rewrite $III_1(t)$ as

$$III_1(t) = -\sum_{i,j,k=1}^3\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j - \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_j\partial_k u_j\\
&\quad + \sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_1 u_1+\partial_2 u_2)\partial_3 u_j\partial_3 u_j.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&III_2(t)+III_3(t)\\
&= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\\
&= \sum_{k=1}^{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 b_i\partial_i b_j\partial_3 u_j\\
&\quad + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 b_i\partial_i u_j\partial_3 b_j - \sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_1 b_1+\partial_2 b_2)(\partial_3 b_j\partial_3 u_j+\partial_3 u_j\partial_3 b_j),
\end{aligned}
$$

$$
\begin{aligned}
III_4(t) &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i b_j\partial_k b_j\\
&= \sum_{k=1}^{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i b_j\partial_k b_j + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_i\partial_i b_j\partial_3 b_j\\
&\quad + \sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_1 u_1+\partial_2 u_2)\partial_3 b_j\partial_3 b_j.
\end{aligned}
$$

Using the cancelation property as before,

$$
\begin{aligned}
III_5(t) &= -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\{\partial_i((\nabla\times b)\times b)\cdot\partial_i(\nabla\times b) - (\partial_i(\nabla\times b)\times b)\cdot\partial_i(\nabla\times b)\}\\
&= -\sum_{i=1}^{2}\int_{\mathbb{R}^3}(\nabla\times b)\times\partial_i b\cdot\partial_i(\nabla\times b)\underbrace{-\int_{\mathbb{R}^3}(\nabla\times b)\times\partial_3 b\cdot\partial_3(\nabla\times b)}_{III_{51}},
\end{aligned}
$$

where

$$
\begin{aligned}
III_{51} &= -\int_{\mathbb{R}^3}\{\partial_3 b_3(\partial_3 b_1-\partial_1 b_3)-\partial_3 b_2(\partial_1 b_2-\partial_2 b_1)\}\partial_3(\partial_2 b_3-\partial_3 b_2)\\
&\quad -\int_{\mathbb{R}^3}\{\partial_3 b_1(\partial_1 b_2-\partial_2 b_1)-\partial_3 b_3(\partial_2 b_3-\partial_3 b_2)\}\partial_3(\partial_3 b_1-\partial_1 b_3)\\
&\quad -\int_{\mathbb{R}^3}\{\partial_3 b_2(\partial_2 b_3-\partial_3 b_2)-\partial_3 b_1(\partial_3 b_1-\partial_1 b_3)\}\partial_3(\partial_1 b_2-\partial_2 b_2).
\end{aligned}
$$

Therefore, we can easily obtain the estimates

$$\begin{aligned}|III_1(t)| &\le C\int_{\mathbb{R}^3}|\nabla_h u||\nabla u|^2dx,\\ |III_2(t)+III_3(t)| &\le C\int_{\mathbb{R}^3}|\nabla_h u||\nabla b|^2dx+\underbrace{\int_{\mathbb{R}^3}|\nabla_h b||\nabla b||\nabla u|dx}_{\Xi},\\ |III_4(t)| &\le C\int_{\mathbb{R}^3}|\nabla_h u||\nabla b|^2dx+\int_{\mathbb{R}^3}|\nabla_h b||\nabla b|^2dx,\\ |III_5(t)| &\le C\int_{\mathbb{R}^3}|\nabla_h b||\nabla b||\Delta b|dx.\end{aligned} \tag{7.2}$$

Using Hölder's inequality, interpolation inequality and Young's inequality, let $\theta_3=\frac{3}{2p_1\alpha}$,

$$\begin{aligned}|III_1(t)| &\le C\|\nabla_h u\|_{L^{p_1}}\|\nabla u\|_{L^2}^{2(1-\theta_3)}\|\nabla\Lambda^\alpha u\|_{L^2}^{2\theta_3}\\ &\le C\|\nabla_h u\|_{L^{p_1}}^{\frac{1}{1-\theta_3}}\|\nabla u\|_{L^2}^2+\frac18\|\nabla\Lambda^\alpha u\|_{L^2}^2\\ &= C\|\nabla_h u\|_{L^{p_1}}^{\frac{2p_1\alpha}{2p_1\alpha-3}}\|\nabla u\|_{L^2}^2+\frac18\|\nabla\Lambda^\alpha u\|_{L^2}^2.\end{aligned}$$

For the estimate of the second term, let $\theta_4=\frac{6-2p_2\alpha}{2\beta p_2}$, we have

$$\begin{aligned}\Xi &\le \|\nabla_h b\|_{L^{p_2}}\|\nabla b\|_{L^{\frac{6p_2}{(3+2\alpha)p_2-6}}}\|\nabla u\|_{L^{\frac{6}{3-2\alpha}}}\\ &\le \|\nabla_h b\|_{L^{p_2}}\|\nabla b\|_{L^2}^{1-\theta_4}\|\nabla\Lambda^\beta b\|_{L^2}^{\theta_4}\|\nabla\Lambda^\alpha u\|_{L^2}\\ &\le C\|\nabla_h b\|_{L^{p_2}}^{\frac{2}{1-\theta_4}}\|\nabla b\|_{L^2}^2+\frac18\|(\nabla\Lambda^\alpha u,\nabla\Lambda^\alpha b\|_{L^2}^2\\ &\le C\|\nabla_h b\|_{L^{p_2}}^{\frac{2p_2\beta}{(\alpha+\beta)p_2-3}}\|\nabla b\|_{L^2}^2+\frac18\|(\nabla\Lambda^\alpha u,\nabla\Lambda^\alpha b\|_{L^2}^2.\end{aligned}$$

Thus,

$$\begin{aligned}|III_2(t)+III_3(t)| &\le C(\|\nabla_h u\|_{L^{p_1}}^{\frac{2p_1\beta}{2p_1\beta-3}}+\|\nabla_h b\|_{L^{p_2}}^{\frac{2p_2\beta}{(\alpha+\beta)p_2-3}})\|\nabla b\|_{L^2}^2\\ &\quad+\frac18(\|\nabla\Lambda^\alpha u\|_{L^2}^2+\|\nabla\Lambda^\beta b\|_{L^2}^2).\end{aligned}$$

Similarly,

$$\begin{aligned}|III_4(t)| &\le C(\|\nabla_h u\|_{L^{p_1}}^{\frac{2p_1\beta}{2p_1\beta-3}}+\|\nabla_h b\|_{L^{p_2}}^{\frac{2p_2\beta}{2p_2\beta-3}})\|\nabla b\|_{L^2}^2+\frac14\|\nabla\Lambda^\beta b\|_{L^2}^2,\\ |III_5(t)| &\le C\|\nabla_h b\|_{L^{p_2}}^{\frac{2\beta p_2}{(2\beta-1)p_2-3}}\|\nabla b\|_{L^2}^2+\frac18\|\nabla\Lambda^\beta b\|_{L^2}^2.\end{aligned}$$

Plugging the inequalities above in (7.1), integrating in $[0,t]$, with energy estimate,

$$\begin{aligned}&\|(u,b)\|_{H^1}^2+\int_0^t\|\Lambda^\alpha u\|_{H^1}^2d\tau+\int_0^t\|\Lambda^\beta b\|_{H^1}^2d\tau\\ &\le C\int_0^t(\|\nabla_h b\|_{L^{p_2}}^{\frac{2\beta p_2}{(2\beta-1)p_2-3}}+\|\nabla_h u\|_{L^{p_1}}^{\frac{2p_1\alpha}{2p_1\alpha-3}}+\|\nabla_h u\|_{L^{p_1}}^{\frac{2p_1\beta}{2p_1\beta-3}}\\ &\quad+\|\nabla_h b\|_{L^{p_2}}^{\frac{2p_2\beta}{(\alpha+\beta)p_2-3}}+\|\nabla_h b\|_{L^{p_2}}^{\frac{2p_2\beta}{2p_2\beta-3}})\|(u,b)\|_{H^1}^2+\|(u_0,b_0)\|_{H^1}^2.\end{aligned}$$

Thanks to Gronwall's lemma, with assumption (1.8), we obtain

$$\|(u,b)\|_{H^1}^2 + \int_0^T \|\Lambda^\alpha u\|_{H^1}^2 dt + \int_0^T \|\Lambda^\beta b\|_{H^1}^2 dt < \infty,$$

which implies that

$$\sup_{0\le t\le T} \|u\|_{L^3} < \infty, \quad \int_0^T \|\nabla b\|_{L^3}^{\frac{\beta}{\beta-1}} dt < \infty.$$

Combining with regularity criterion (1.8) can lead the desired result. So we complete the proof of Theorem 1.10.

Acknowledgments. The author would like to thank Prof. Dongho Chae and Prof. Zhifei Zhang for their interest and comments.

References

[1] M. Acheritogaray, P. Degond, A. Frouvelle, J. Liu; Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system, *Kinet. Relat. Models* **4** (2011), 901-918.

[2] D. Chae, P. Degond, J. Liu; Well-posedness for Hall-magnetohydrodynamic; *Ann. Inst. H. Poincar Anal. Non linaire* **31** (2014), 555-565.

[3] D. Chae, J. Lee; On the blow-up criterion and small data global existence for the Hall-Magneto-hydrodynamics, *J. Differ. Equ.* **256** (2014), 3835-3858.

[4] D. Chae, R. Wan, J. Wu; Local well-posedness for the Hall-MHD equations with fractional magnetic diffusion, arXiv:1404.0486v2 [math. AP] 30 Jan 2015, to appear in J. Math. Fluid Mech.

[5] T. Forbes; Magentic reconnection in solar flares, *Geophys. Astrophys. Fluid Dyn.*, **62** (1991), 15-36.

[6] M. Wardle; Star formation and the Hall effect, *Astrophys. Space Sci.*, **292** (2004), 317-323.

[7] H. Bahouri, J.-Y. Chemin, R. Danchin; *Fourier Analysis and Nonliear Partial Differential Equations*, Springer, 2011.

[8] L. Grafakos; *Modern fourier analysis*, secnod ed., Grad. Text in Math., vol. 250, Springer-Verlag, 2008.

[9] G. Galdi; *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, 2nd Edition, Springer, 2011.

[10] Q. Chen, C. Miao; Existence theorem and blow-up criterion of strong solutions to the two-fluid MHD equation in $\mathbb{R}^3$, *J. Differ. Equ.*, **239** (2007), 251-271.

[11] N. Katz, N. Pavlović; A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation, *Geom. Funct. Anal.* **12** (2002), 355-379.

[12] T. Tao; Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation. *Anal. PDE*, **3** (2009), 361-366.

[13] J. Wu; Generalized MHD equations, *J. Differ. Equ.*, **195** (2003), 284-312.

[14] J. Wu; Global regularity for a class of generalized magnetohydrodynamic equations, *J. Math. Fluid Mech.*, **13** (2011), 295-305.

[15] Y. Zhou; Reguality criteria for the generalized viscous MHD equations, *Ann. Inst. H. Poincare, Anal. Non Lineaire*, **24** (2007), 491-505.

[16] C. Tran, X. Yu, Z. Zhai; Note on solution regularity of the generalized nagnetohydrodynamic equations with partial dissipation, *Nonlinear Anal.*, **85** (2013), 43-51.

Renhui Wan

Department of mahtematics, Zhejiang University, Hangzhou, Zhejiang 310027, China

E-mail address: rhwanmath@zju.edu.cn, rhwanmath@163.com