AN ABEL TYPE CUBIC SYSTEM

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ABSTRACT. We consider center conditions for plane polynomial systems of Abel type consisting of a linear center perturbed by the sum of 2 homogeneous polynomials of degrees n and $2n - 1$ where $n \geq 2$. Using properties of Abel equations we obtain two general systems valid for arbitrary values on n. For the cubic $n = 2$ systems we find several sets of new center conditions, some of which show that the results in a paper by Hill, Lloyd and Pearson which were conjectured to be complete are in fact not complete. We also present a particular system which appears to be a counterexample to a conjecture by Żołądek et al. regarding rational reversibility in cubic polynomial systems.

1. INTRODUCTION

In this work we consider center conditions for the origin for differential polynomial systems in the plane having the form of a linear center perturbed by homogeneous polynomials of degrees n and $2n - 1$ where $n \geq 2$. We refer to these as *generalized cubic systems* since they contain the cubic system $(n = 2)$ as a particular case. Specifically, we assume nonlinearities such that the resulting phase plane equation is one of Abel type. The most general form of this type of system is

$$\frac{dx}{dt} = -y - p_1(x, y) - p_2(x, y), \quad \frac{dy}{dt} = x + q_1(x, y) + q_2(x, y) \qquad (1.1)$$

where

$$
\begin{aligned}
p_1(x, y) &= a_0 x^n + a_1 x^{n-1} y, \\
p_2(x, y) &= c_0 x^{2n-1} + c_1 x^{2n-2} y, \\
q_1(x, y) &= b_0 x^n + b_1 x^{n-1} y + b_2 x^{n-2} y^2 + b_3 x^{n-3} y^3, \\
q_2(x, y) &= d_0 x^{2n-1} + d_1 x^{2n-2} y + d_2 x^{2n-3} y^2 + d_3 x^{2n-4} y^3.
\end{aligned}
\qquad (1.2)
$$

where $b_3 = 0$ for the cubic system. To avoid the large number of possible subcases which can arise in (1.2), we assume throughout that $a_0 = c_0 = 0$ (except for section 6) and that all other parameters are nonzero. Systems of a more general type consisting of a linear center perturbed by polynomials of degrees n and $2n - 1$ have been studied previously. For specific examples of this see the papers by Giné and Llibre [7, 8, 9].

The ordinary differential equation

$$\frac{dy}{dx} = -\frac{x + q_1(x,y) + q_2(x,y)}{y + p_1(x,y) + p_2(x,y)} \tag{1.3}$$

corresponding to this system is an Abel equation of the second kind and the properties of these equations can be used to investigate both the integrability of the equation and the center properties of the associated system. Most of the results which we give in this paper are for the cubic system, but we will also give a general set of center conditions valid for arbitrary values of n. For the purposes of this work an integrating factor $\mu(x,y)$ of (1.3) is a function such that

$$\frac{\partial}{\partial y}\big(\mu(x,y)(x + q_1(x,y) + q_2(x,y))\big) - \frac{\partial}{\partial x}\big(\mu(x,y)(y + p_1(x,y) + p_2(x,y))\big) = 0.$$

We shall frequently make use of the fact given by Reeb [20] that if $\mu(x,y)$ is nonzero on a neighborhood of the critical point $(0,0)$ then the corresponding system is a center of $(1.1),(1.2)$. Every integrating factor that we determine in this paper will be of this type, although we might not specifically mention it.

Particular cases of the cubic systems of the type defined by (1.1), (1.2) have been studied extensively in the past. One of the earliest of these was by Kukles [15] who considered the case $a_0 = a_1 = c_0 = c_1 = 0$. He proposed a set of conditions which he thought to be both necessary and sufficient for centers of these systems. These conditions were shown to be not complete by Jin and Wang [13] who obtained a system not contained in the Kukles' condition. The complete set of conditions was given in papers by Lloyd et al. [16, 19].

The case $a_0 = c_0 = c_1 = 0$ was considered by Hill, Lloyd and Pearson in [12]. They obtained three separate conditions for nonzero parameters and conjectured that this gave a complete classification of the centers for such systems. One of the main purposes of this paper is to show that these conditions are not complete. Using properties of Abel equations, we are able to demonstrate another large class of centers not contained in those given in [12]. This approach also allows us to make further statements regarding the invariant curves and integrating factors for the systems considered therein. The results for the cases considered previously were obtained by calculating the Lyapunov coefficients of the system and then using resultants to obtain common factors. This type of development often leads to massive expressions which frequently become intractable. Fortunately, for these Abel type systems there is another approach which will reproduce the same results much more quickly and also allows for extension to the case $c_1 \neq 0$. Analyzing the problem in terms of Lyapunov coefficients allows for a discussion of both the necessity and sufficiency of the conditions so obtained, as well as a possible determination of the number of limit cycles which can bifurcate from the critical point. However since the results in this paper determine only the integrability of the systems, we consider just the sufficiency of the conditions.

In the next two sections we develop some of the fundamental ideas concerning Abel differential equations and use them to generate the systems of equations that define the solutions that we seek. In section 4 we employ the methodology previously introduced to consider certain aspects of the results given in [12]. The case $c_1 \neq 0$ is considered in the next section and we present several new sets of center conditions for these cubic systems. In the final section we look at certain results by Cherkas and Romanovski [3] which cover some of the cases defined by

$a_0 c_0 \neq 0$ which are not dealt with in this paper. We present a particular system which seems to be a counterexample to a conjecture made by Żołądek and others regarding rational (algebraic) reversibility in cubic polynomial systems.

2. Elements of the Abel differential equation

The emphasis of this paper is to use properties of the Abel differential equation to characterize center conditions for the system (1.1), (1.2). The primary reason for doing this is because all of the results having nonzero parameters in [16, 19] can be expressed in terms of easily identifiable forms of Abel equations. These are either of *Bernoulli* or *constant invariant* type, terms which we will define more fully below. In order to do so, we briefly review certain standard aspects of Abel equations as well as some less known ones.

An Abel equation of the first kind is given by

$$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) \tag{2.1}$$

and an Abel equation of the second kind by

$$\frac{dy}{dx} = \frac{f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)}{g_1(x)y + g_0(x)} \tag{2.2}$$

where the coefficient functions are assumed to be suitably differentiable functions of x. The form (2.2) can always be transformed to an Abel equation of the first kind by the variable change

$$y(x) = \frac{1}{g_1(x)u(x)} - \frac{g_0(x)}{g_1(x)}.$$

If we transform (2.1) by $y(x) = u(x) - f_2(x)/(3f_3(x))$ we obtain the equation

$$\frac{du}{dx} = f_3(x)u^3 + \frac{3f_1(x)f_3(x) - f_2^2(x)}{3f_3(x)}u + \frac{s_3(x)}{f_3^2(x)} \tag{2.3}$$

where

$$s_3(x) = f_0(x)f_3^2(x) + \frac{2}{27}f_2^3(x) + \frac{1}{3}\left(f_3(x)f_2'(x) - f_2(x)f_3'(x) - f_1(x)f_2(x)f_3(x)\right). \tag{2.4}$$

Transformations of this type have been given in Kamke [14]. From $s_3(x)$ it is possible to define recursively an infinite sequence of relative invariants [2] of weight $2k + 1$ by

$$s_{2k+1}(x) = f_3(x)s'_{2k-1}(x) + (2k-1)\left(\frac{1}{3}f_2^2(x) - f_3'(x) - f_1(x)f_3(x)\right)s_{2k-1}(x) \tag{2.5}$$

for $k \geq 2$ and from these, a sequence of absolute invariants can be formed. If the first invariant $I_1(x) = s_5^3(x)/s_3^5(x)$ is constant, the Abel equation can be transformed to a separable equation. We shall refer to centers defined by this condition as *constant invariant centers*. In [7] the authors use the result

$$f_2(x)\left(\frac{f_3(x)}{f_2(x)}\right)' + f_1(x)f_3(x) = Cf_2(x)^2$$

to investigate integrability conditions for centers and foci in certain generalized cubic systems with $n = 2, 3$. In this C is a constant and if this relation is satisfied it is sufficient to guarantee that the form of (2.1) with $f_0(x) = 0$ has constant invariant. This is an easily derivable extension of the more usual form of the

condition (see Murphy [17]) $\big(f_3(x)/f_2(x)\big)' = Cf_2(x)$ for which $f_1(x) = 0$ as well. These conditions can be difficult to use with the full form (2.1) (if we wish to find conditions for which the equation has constant invariant) because it is necessary to know a particular solution in order to eliminate the $f_0(x)$ term. We can see from equation (2.6) below that we can make $f_0(x) = 0$ for cubic systems ($n = 2$, $b_3 = 0$) simply by taking $d_3 = 0$. However, we do not do this and all of our constant invariant results in this paper are based on the condition $d_3 \neq 0$.

If $s_3(x) = 0$ then (2.3) is a Bernoulli equation and the corresponding systems will also be centers. This is the case for the two systems given in [16]. The invariants of an Abel equation of the second kind are defined to be those of the corresponding Abel equation of the first kind.

When the Abel equation defined by (1.2), (1.3) is transformed to an Abel equation of the first kind, the resulting equation has coefficient functions defined by

$$
\begin{aligned}
f_0(x) &= \frac{b_3 x^{n-3} + d_3 x^{2n-4}}{1 + a_1 x^{n-1} + c_1 x^{2n-2}}, \quad f_1(x) = \frac{b_2 x^{n-2} + d_2 x^{2n-3}}{1 + a_1 x^{n-1} + c_1 x^{2n-2}}, \\
f_2(x) &= \frac{b_1 x^{n-1} + d_1 x^{2n-2}}{1 + a_1 x^{n-1} + c_1 x^{2n-2}}, \quad f_3(x) = \frac{x + b_0 x^n + d_0 x^{2n-1}}{1 + a_1 x^{n-1} + c_1 x^{2n-2}}.
\end{aligned} \tag{2.6}
$$

From these we can see that the relative and absolute invariants are rational functions. Then from the definition of $I_1(x)$ we have $s_3(x) = I_1(x)(s_3^2(x)/s_5(x))^3$ so that if $I_1 \neq 0$ is constant, $s_3(x)$ must be the cube of a rational function up to some multiplicative constant. If $I_1 = 0$ this is not necessarily the case and we will see that these two separate conditions generally define two distinct classes of centers.

We conclude the section with the following very useful result. We are certain that this must be known, but this writer has never seen it in print elsewhere.

Lemma 2.1. *A necessary and sufficient condition that the Abel differential equation* (2.1) *is of type constant invariant is that it has a particular solution of the form*

$$
y(x) = \eta(x) = \frac{1}{f_3(x)}\Big(K(s_3(x))^{(1/3)} - \frac{1}{3}f_2(x)\Big) \tag{2.7}
$$

for some constant K.

We briefly sketch a proof of this. Using a variable transformation of the form

$$
y(x) = \frac{1}{f_3^2(x)}\Big(s_3(x)u(x) - \frac{1}{3}f_2(x)f_3(x)\Big), \tag{2.8}
$$

we can transform (2.1) into

$$
\frac{du}{dx} = \tilde{f}_3(x)u^3 + \tilde{f}_1(x)u + 1 \tag{2.9}
$$

where

$$
\begin{aligned}
\tilde{f}_3(x) &= \frac{s_3^2(x)}{f_3^3(x)}, \\
\tilde{f}_1(x) &= \frac{(6f_3'(x) - f_2^2(x) + 3f_1(x)f_3(x))s_3(x) - 3f_3(x)s_3'(x)}{3f_3(x)s_3(x)}.
\end{aligned} \tag{2.10}
$$

The transformation (2.8) is of type which will retain the same equivalence class (see [2]) as that of (2.1) so the value of the invariant $I_1(x)$ is unchanged. For (2.9) we

obtain a much simpler functional form for $I_1(x)$. We find that

$$I_1(x) = -\frac{\left(\tilde{f}_3'(x) + 3\tilde{f}_1(x)\tilde{f}_3(x)\right)^3}{\tilde{f}_3(x)^4}. \tag{2.11}$$

and assuming I_1 is constant, we can solve for $\tilde{f}_1(x)$ and write (2.9) in terms of $\tilde{f}_3(x)$ as

$$\frac{du}{dx} = \tilde{f}_3(x)u^3 + \frac{1}{3}\left((-I_1)^{1/3}\tilde{f}_3(x)^{1/3} - \frac{\tilde{f}_3'(x)}{\tilde{f}_3(x)}\right)u + 1.$$

We can now show that a particular solution of this equation is $q(x) = K/(\tilde{f}_3(x))^{(1/3)}$ provided K is a solution of the equation

$$3K^3 + (-I_1)^{(1/3)}K + 3 = 0. \tag{2.12}$$

Converting back to the original form (2.1) gives (2.7).

The proof of sufficiency is quite straightforward. Assuming a solution of the form (2.9) and substituting into (2.1), we can show that the Abel equation can be reduced to

$$3K^3 - \frac{s_5(x)}{(s_3(x))^{5/3}}K + 3 = 0.$$

Since K is a constant, so is I_1.

Since the constant invariant form of the Abel equation is always solvable, it is not a complete surprise that a solution like (2.7) should exist. Depending upon the roots of (2.12) it can define up to three invariant curves of the system and these can be used to construct *Darboux* type integrating factors. For our purposes we assume that a *Darbouxian function* is a function of the form

$$\mathcal{D}(x,y) = \sum_{k=1}^{N} f_k(x,y)^{\alpha_k}$$

for constants α_k and where each $f_k(x,y)$ is a polynomial which is a particular solution of (1.3). Each solution satisfies

$$(x + q_1(x,y) + q_2(x,y))\frac{\partial f_k}{\partial y} - (y + p_1(x,y) + p_2(x,y))\frac{\partial f_k}{\partial x} = \lambda_k(x,y)f_k(x,y).$$

where $\lambda_k(x,y)$ is called the *cofactor* of $f_k(x,y)$ and is a polynomial of degree at most $2n - 2$ for generalized cubic systems. If an equation has a first integral of Darboux type or a Liouvillian first integral then it has an integrating factor of Darboux type (see [4]) with the possible inclusion of an exponential factor. For the case where I_1 is constant we can use (2.7) to generate the forms for the invariant curves.

3. Development of governing equations

Here we develop the ideas of the previous section into a set of conditions which will lead to the integrable Bernoulli and constant invariant forms of the Abel equation. This will generally produce a system of equations which is simpler, although not necessarily fewer in number, than the usual approach of determining center conditions in terms of Lyapunov coefficients. We do not give the specific forms for these systems because they are very straightforward to obtain. They are determined by calculating the numerators of the appropriate expressions and forming

the systems by setting the coefficients of each power of x equal to zero simultaneously. In no case that we give did the solutions of the system result in the vanishing of the denominator. There are usually a number of ways to proceed in order to accomplish this, but the systems so obtained are often not equivalent with regards to solvability. As an example of the general procedure, the Bernoulli conditions for the generalized cubic system defined by (1.2), (1.3) (with $a_0 = c_0 = 0$) are obtained by calculating $s_3(x)$ from (2.4) using the coefficient functions (2.6). This results in a system of 6 equations which is simple enough that it may be solvable without the use of a computer algebra system. We give these solutions at the end of the section. All symbolic computations in this paper were carried out using Maple 13 or later versions of the software.

It was noted in [12] ($c_1 = 0$) that $1 + a_1 x = 0$ is an invariant line for the solutions obtained therein. In the following we will find that a similar consideration holds for the more general cases $c_1 \neq 0$ that we wish to consider. For this reason it is helpful to define a new parameter by

$$c_1 = \frac{1}{4}\left(a_1^2 - C_1^2\right), \tag{3.1}$$

which allows for factorization of the denominator $1 + a_1 x + c_1 x^2$ in simple linear terms. We express the following systems in terms of C_1, which has the slight disadvantage that solving these systems often produces two solutions having similar form, only one of which is needed to define the coefficient conditions. However, it is useful because we can directly recover the solutions in [12] by setting $C_1 = \pm a_1$ in our results.

For the constant invariant form we consider only the cubic $n = 2$ case and for this we must distinguish between the two cases $I_1 \neq 0$ and $I_1 = 0$. We begin by assuming that $I_1 \neq 0$. We need to obtain the functional forms for $s_3(x)$ and $s_5(x)$ and a little consideration will show us that they must have similar functional components in order to obtain the cancellation necessary to produce a constant value for I_1. Using (2.4), (2.8) with $n = 2$ and $b_3 = 0$ gives

$$s_3(x) = \frac{k(x)}{\left((a_1 + C_1)x + 2\right)^3 \left((a_1 - C_1)x + 2\right)^3} \tag{3.2}$$

where k is a polynomial in x having maximum degree 6 and minimum degree 2. (So $s_3(x) = 0$ would produce 5 equations in this case.) We were able to use this form to produce a one parameter family of solutions for the case $c_1 \neq 0$ by taking specific values for several parameters in conjunction with some of the following ideas. What we found for these solutions was that $s_3(x)$ actually has a much simpler form in the case when I_1 is constant. With this in mind we define

$$\psi(x) = \frac{x}{(a_1 + C_1)x + 2} \tag{3.3}$$

and then take

$$s_3(x) = K_1 \psi^3(x), \quad s_5(x) = K_2 \psi^5(x) \tag{3.4}$$

where K_1, K_2 are nonzero constants. These forms now give the desired constant invariant value $I_1 = K_2^3/K_1^5$. If $C_1 = a_1$ ($c_1 = 0$), (3.3) reduces to the form of $s_3(x)$ for the constant invariant solutions in [12]. We can also express $\psi(x)$ in terms of the other linear factor of (3.2) (i.e. $C_1 \to -C_1$) but this just leads to the duality of the solutions mentioned earlier and produces nothing new. So one set of conditions

is defined by the equivalence of (3.2) with (3.3) and (3.4). This can be expressed as

$$k(x) - K_1 x^3 \left((a_1 - C_1)x + 2\right)^3 = 0 \tag{3.5}$$

where the coefficients of $k(x)$ are given in terms of the parameters of the system (1.2). This yields a system with 5 equations and in this we can see that K_1 occurs with degree at most one in any equation. So it is easily solved for and this underscores one of the advantages of this method. Using the previous results, we will introduce a second independent parameter which also occurs at low degree. Not only does this allow us to solve for these extra parameters in terms of the system parameters, but it also seems to produce the system which is the simplest to solve.

In addition to (3.5) which guarantees that $s_3(x)$ has the correct form, we need another condition to ensure that I_1 is constant. We have tried to use the form of $s_5(x)$ given in (3.4) in conjunction with the general form from (2.5), but the implementation of this is sufficiently complex that we were unable to obtain any general results using it. We can obtain a solvable system by substituting for $s_3(x)$, $s_5(x)$ from (3.4) into the definition of $s_5(x)$ from (2.5). This method gives the values of K_1, K_2 from which we could calculate I_1, but since this is not really of interest we prefer to proceed in a somewhat different direction. We will make use of some of the results arising in the proof of the Lemma in the previous section, and in this regard the simplified functional form of $I_1(x)$ given by (2.11) is quite useful. Also, the final form can be easily be modified to cover the remaining case $I_1 = 0$.

From the definition of $I_1(x)$ we see that $s_3^2(x) = \left(s_5(x)/s_3(x)\right)^3 / I_1(x)$. Then from (2.10) we have

$$\tilde{f}_3(x) = \frac{1}{I_1(x)} \left(\frac{s_5(x)}{f_3(x)s_3(x)}\right)^3.$$

Assuming I_1 is constant, (2.11) gives

$$(\tilde{f}_3(x)I_1)^{1/3} = -\frac{\tilde{f}_3'(x)}{\tilde{f}_3(x)} - 3\tilde{f}_1(x) = \frac{s_5(x)}{f_3(x)s_3(x)} = \frac{K_2}{K_1}\frac{\psi^2(x)}{f_3(x)}$$

when (3.4) is used. With (2.10), (3.4) this can be written as

$$3\frac{f_3'(x)}{f_3(x)} - 3\frac{\psi'(x)}{\psi(x)} - \frac{f_2^2(x)}{f_3(x)} + 3f_1(x) + K'\frac{\psi^2(x)}{f_3(x)} = 0, \tag{3.6}$$

where $K' = K_2/K_1$. This gives a system having 5 equations, so when combined with (3.5) it produces a system having 10 equations in total.

The final integrable form that we consider occurs when $I_1 = 0$ (i.e. $s_3(x) \neq 0$, $s_5(x) = 0$), but in this case there is no reason to expect that $s_3(x)$ will appear as an exact cube. Fortunately, due to some previous work which we will describe a bit more fully in the next section, we knew that an appropriate form is $s_3(x) = K_1\omega(x)$ where K_1 is a nonzero constant and

$$\omega(x) = \frac{x^3}{\left((a_1 + C_1)x + 2\right)\left((a_1 - C_1)x + 2\right)^2}. \tag{3.7}$$

As in the previous case $I_1 \neq 0$ there is an alternative form of this in which the powers on the terms in the denominator are reversed, but this again produces nothing new. The condition corresponding to (3.5) becomes

$$k(x) - K_1 x^3 \left((a_1 - C_1)x + 2\right)\left((a_1 + C_1)x + 2\right)^2 = 0 \tag{3.8}$$

and (3.6) is replaced by

$$3\frac{f_3'(x)}{f_3(x)} - \frac{\omega'(x)}{\omega(x)} - \frac{f_2^2(x)}{f_3(x)} + 3f_1(x) = 0\,.\tag{3.9}$$

To conclude the section we give the parameter dependencies for the Bernoulli forms of the generalized cubic system.

Proposition 3.1. *Let $n \geq 2$ be an integer, $a_0 = c_0 = 0$ and (1.1), (1.3) be given by (1.2). Then if either*

$$b_0 = \frac{b_1^2 d_0 + d_1^2}{b_1 d_1}, \quad b_3 = -\frac{1}{3}(n-2)b_1, \quad c_1 = \frac{b_1 d_0(a_1 d_1 + b_1 d_0)}{d_1^2} + \frac{2}{9}\frac{d_1^2 - b_1^2 d_0}{(n-1)d_0},$$

$$d_2 = (n-1)\frac{b_1 d_0(2b_1 d_0 - a_1 d_1)}{d_1^2} + \frac{1}{9}\frac{9b_1 b_2 d_0^2 + 2d_1^3}{d_0 d_1} + \frac{2}{9}\frac{b_1^2 d_0 - d_1^2}{(n-1)d_0},$$

$$d_3 = \frac{1}{3}(n-1)\frac{b_1(2b_1 d_0 - a_1 d_1)}{d_1} + \frac{1}{3}\frac{b_1(a_1 d_1 + b_2 d_1 - b_1 d_0)}{d_1}\tag{3.10}$$

or

$$a_1 = b_0 + \frac{2}{9}\frac{b_0 d_1^2 - 4b_1 d_0 d_1 + b_0 b_1^2 d_0}{(n-1)d_0(b_0^2 - 4d_0)}, \quad b_3 = -\frac{1}{3}(n-2)b_1,$$

$$b_2 = -a_1 - (n-2)b_0 + \frac{2}{9}\frac{2b_0 d_1^2 + b_0 b_1^2 d_0 - 6b_1 d_0 d_1}{d_0(b_0^2 - 4d_0)},$$

$$c_1 = d_0 + \frac{2}{9}\frac{2b_1^2 d_0^2 + b_0^2 d_1^2 - 2b_0 b_1 d_0 d_1 - 2d_0 d_1^2}{(n-1)d_0(b_0^2 - 4d_0)},\tag{3.11}$$

$$d_2 = -c_1 - (2n-3)d_0 + \frac{2}{9}\frac{b_0^2 d_1^2 + b_0 b_1 d_0 d_1 - 4d_0 d_1^2 - 2b_1^2 d_0^2}{d_0(b_0^2 - 4d_0)},$$

$$d_3 = -\frac{1}{3}(2n-3)d_1 + \frac{1}{27}\frac{(2b_0 b_1 d_1 - 4b_1^2 d_0)d_1}{d_0(b_0^2 - 4d_0)}$$

the corresponding systems satisfy $s_3(x) = 0$ and are centers of (1.1).

Since these conditions are derived from the consideration of Bernoulli forms for (1.3) it is obvious that integrating factors can be readily obtained. We can show that an integrating factor for (3.10) is given by

$$\mu(x,y) = \frac{e^{\int \mathcal{F}(x)\,dx}}{(3b_1 d_0 x^{n-1} + b_1 d_1 x^{n-2} y + 3d_1)^3 \mathcal{D}(x)}\tag{3.12}$$

where $\mathcal{F}(x) = \mathcal{N}(x)/\mathcal{D}(x)$ with

$$\mathcal{N}(x) = x^{n-2}\big[(9(n-1)^2 b_1 d_0^2(a_1 d_1 + b_1 d_0) + 2(n-1)d_1(2d_1^3 + 9b_1 b_2 d_0^2 - 3b_1^2 d_0 d_1)$$
$$+ 4d_1^2(b_1^2 d_0 - d_1^2))x^{n-1} + 27(n-1)^2 b_1 d_0^2 d_1 + 18(n-1)b_2 d_0 d_1^2\big]$$

and

$$\mathcal{D}(x) = [9(n-1)b_1 d_0^2(a_1 d_1 - b_1 d_0) + 2d_1^2(d_1^2 - b_1^2 d_0)]x^{2n-2}$$
$$+ 9(n-1)a_1 d_0 d_1^2 x^{n-1} + 9(n-1)d_0 d_1^2.$$

We observe that (3.12) is analytic and nonzero on a neighborhood of the critical point $(0,0)$ with the same being true for the integrating factor for (3.11).

4. The article by Hill, Lloyd and Pearson

In [5] Christopher and Lloyd consider a particular case of centers for the case $a_1 = c_1 = 0$ and Lloyd and Pearson look at the general case in [16]. They find two general sets of centers which correspond to the Bernoulli centers for cubic systems from Proposition 3.1. Unfortunately, the parameter assignments for the second case in [16] are not correctly given, but correct relations for this are found in [19]. Hill, Lloyd and Pearson [12] consider the case $c_1 = 0$ and obtain three independent sets of coefficients for centers. Upon analysis, we once again find that two of these are Bernoulli centers and the third is constant invariant. Since the immediate integrability of the systems so obtained is not recognized, they undertake an exhaustive analysis to determine invariant curves and integrating factors. They also conjecture that the set of conditions which they obtain are complete. All of the results in these papers were obtained by calculating and reducing the Lyapunov coefficients of the system. In the following we will discuss various aspects of the invariant curves and integrating factors with regard to the methodology introduced in this paper and further, use it to show that the conditions in [12] are not complete. There it was conjectured [12, Conjecture 4.2] that the results given in that paper for systems satisfying $a_1 b_1 d_3 \neq 0$ are complete.

We begin by considering the question of completeness just mentioned. Some time ago this writer undertook an analysis of the cubic system $(a_1 c_1 \neq 0)$ considered in this paper. This was done by the attempted reduction of Lyapunov coefficients along the lines used by other authors. However, the general problem of insufficient memory quickly appeared and we were forced to make some assumptions in order to continue. It was decided to impose the condition $b_0 + b_2 = 0$ and this drastically reduced the size of the expressions obtained. Even so, it was necessary at one point to obtain the factorization of a resultant containing more than 144,000 terms. Fortunately, this was (indirectly) possible and through it we were able to identify a class of centers which had not been previously reported. It turned out that they were of the type $I_1 = 0$ arising from $(3.8), (3.9)$ and for future reference we denote this system by $\mathcal{S}$. These solutions were also very helpful in enabling us to set the form of $s_3(x)$ given by (3.7). So, when we became aware of the paper [12] we used these ideas to check for the possible existence of such solutions in that system. We quickly found another set of conditions which is given in the following.

Proposition 4.1. *Let $n = 2$, $a_0 = c_0 = c_1 = b_3 = 0$ and (1.1), (1.3) be given by (1.2). Set $\alpha^2 = b_0/(b_0 + 2b_2)$. Then the system defined by*

$$a_1 = 3(b_0 + b_2), \quad b_1 = 2\alpha(2b_0 + 3b_2), \quad d_0 = \frac{1}{3}\frac{b_0^2(2b_0 + 3b_2)}{b_0 + 2b_2},$$

$$d_1 = b_0\alpha(2b_0 + 3b_2), \quad d_2 = b_0(2b_0 + 3b_2), \quad d_3 = \frac{\alpha}{3}(b_0 + 2b_2)(2b_0 + 3b_2)$$

satisfies the condition $I_1 = 0$ and is a center of (1.1).

Since $s_3(x)$ for this system is nonzero, the system is not Bernoulli. It could be a member of the constant invariant family given in the paper with an added condition to make $I_1 = 0$, but it is easily shown by a number of different methods (such as the form of $s_3(x)$) that this is not the case. This would have required six parameter assignments such as the results in Proposition 4.1, but the authors only consider the possibility of up to five such assignments. They also state that there were several cases that could not be fully reduced.

An integrating factor $\mu(x, y)$ for the system satisfying $\mu(0, 0) \neq 0$ can be constructed with the help of (2.7). The phase plane equation (1.3) is an Abel equation of the second kind which can be recovered from its first kind form by letting $y \to 1/y$. Then since $K = -1$ is the only real root of (2.12) in the case $I_1 = 0$, an appropriate form for the particular solution is

$$\left(\frac{f_3(x)}{y} + \frac{1}{3} f_2(x) \right)^3 + s_3(x) = 0 \tag{4.1}$$

where $s_3(x)$ has the form (3.7) in which one of the conditions $C_1 = \pm a_1$ has been imposed. The integrating factor is of Darboux type and so its construction will involve determining the invariant curves of the system. From (4.1) we can identify two such curves. One is a polynomial $\mathcal{P}(x, y)$ of degree 4 and the other is an invariant line. The integrating factor is then given as $\mu(x, y) = (1 + 3(b_0 + b_2)x)^{-1/3}/\mathcal{P}(x, y)$.

We now briefly consider some other aspects of the results in [12, Theorem 3.1], particularly with regards to integrating factors and invariant curves. Specifically, parts (i) and (iii) are Bernoulli and part (ii) is constant invariant. In their discussion the authors state that the integrating factor for case (iii) is the most complicated that they had encountered and also that each system required an exponential term. As the Bernoulli forms of these systems are directly obtainable by setting $n = 2$ and letting $c_1 = 0$ in our results (3.10),(3.11), we can see that an integrating factor for these will include an exponential function. The one for case (i) can be found as a particular case of (3.12) and the other can be obtained as a particular case of (5.1) in the next section. With these forms known, we now consider the remaining case.

Case (ii) produces a constant invariant equation having rational coefficients and in our study of such equations ($I_1 \neq 0$) for polynomial systems we have always found that an integrating factor could be obtained in the form of a rational function. This is again of Darboux type so we must determine invariant curves for the system. In this regard it is straightforward to verify that the expression given in [12] for this case is not such a curve. It is possible to find the general forms for these and from them to construct an integrating factor. This requires an explicit description of the system and in terms of the notation in this paper this can be given by

$$a_1 = b_0 + b_2,$$
$$d_0 = \frac{(b_0 + b_2)(b_1(b_0 + b_2)\alpha + (b_0 - b_2)b_1^2 + 4b_0 b_2(b_0 + 3b_2))}{2(b_1^2 + 4b_2^2)},$$
$$d_1 = \frac{(b_0 + b_2)(3b_2(b_0 + b_2)\alpha + b_1(b_1^2 + b_2^2 - 3b_0^2 - 6b_0 b_2))}{b_1^2 + 4b_2^2},$$
$$d_2 = -\frac{(b_0 + b_2)(3b_1(b_0 + b_2)\alpha + 12b_0^2 b_2 - b_0 b_1^2 + 20b_0 b_2^2 - 3b_1^2 b_2)}{2(b_1^2 + 4b_2^2)},$$
$$d_3 = \frac{(b_0 + b_2)^2(b_1(b_0 + b_2) - b_2 \alpha)}{b_1^2 + 4b_2^2}$$

$$\tag{4.2}$$

where $\alpha^2 = b_1^2 - 4b_0^2 - 8b_0 b_2$. We can now use this representation along with (2.7) to determine information regarding invariant curves of the system. Similar to (4.1)

the appropriate form for the particular solution is

$$\frac{f_3(x)}{y} + \frac{1}{3}f_2(x) - Ks_3(x)^{1/3} = \frac{f_3(x)}{y} + \frac{1}{3}f_2(x) - \frac{K'x}{a_1 x + 1} = 0$$

for a constant K'. Of course K' is known but its form is not easy to deal with. As a component of this solution we can readily identify the hyperbola $1 + b_0 x + A(K')y + Bx^2 + C(K')xy = 0$ where B is known and A, C depend upon K'. With this it is not difficult to determine the invariant curve as

$$
\begin{aligned}
\mathcal{C}(x,y) = {}& 1 + b_0 x + \frac{1}{2}(b_1 + \alpha)y \\
& + \frac{(b_0 + b_2)(b_0 b_1^2 + 4b_0^2 b_2 + 12b_0 b_2^2 - b_1^2 b_2 + b_1(b_0 + b_2)\alpha)}{2(b_1^2 + 4b_2^2)}x^2 \\
& + \frac{(b_0 + b_2)(b_1^3 + 2b_1 b_2^2 - 2b_0^2 b_1 - 4b_0 b_1 b_2 + (2b_0 b_2 + b_1^2 + 6b_2^2)\alpha)}{2(b_1^2 + 4b_2^2)}xy
\end{aligned}
$$

In addition to this the system also has three invariant lines, one of which is $a_1 x + 1 = 0$. The other two have the form $Ax + By + C = 0$ and can be obtained by assuming that they are solutions of (1.3). Letting $\ell_k(x,y) = A_k x + B_k y + C_k$ for $k = 1, 2$, we find

$$
\begin{aligned}
A_1 &= (b_0 + b_2)((b_1 + \alpha)\beta + b_0(4b_0^2 + 8b_0 b_2 - b_1^2 - b_1\alpha)), \\
B_1 &= 2b_0(b_0 + b_2)^2(b_1 + \alpha), \\
C_1 &= (b_1 + \alpha)\beta + (b_0 + 2b_2)(b_1^2 - 4b_0 b_2 + b_1\alpha), \\
A_2 &= A_1 - 2(b_0 + b_2)(b_1 + \alpha)\beta, \\
B_2 &= B_1, \quad C_2 = C_1 - 2(b_1 + \alpha)\beta
\end{aligned}
$$

where

$$\beta = \sqrt{b_0(12b_0 b_2^2 + 8b_0^2 b_2 - b_0 b_1^2 - 2b_1^2 b_2 - 2b_0 b_1(b_0 + b_2)\alpha)}.$$

The lines $\ell_1(x,y)$ and $\ell_2(x,y)$ will be real if $I_1 > 729/4$ and complex conjugates if $I_1 < 729/4$. In the latter case their product will have the form of an empty conic. It can then be confirmed with the help of a computer algebra system that the rational function

$$\mu(x,y) = \frac{1 + (b_0 + b_2)x}{\ell_1(x,y)\ell_2(x,y)\mathcal{C}(x,y)} \tag{4.3}$$

is an integrating factor of the system defined by (4.2).

5. The case $c_1 \neq 0$

Here we extend the ideas presented in sections 2 and 3 to the system in which $c_1 \neq 0$. We know that there are two sets of Bernoulli centers given as follows.

Corollary 5.1. *Let $n = 2$, $a_0 = c_0 = 0$ and (1.1), (1.3) be given by (1.2). Then if either*

$$
b_0 = \frac{b_1^2 d_0 + d_1^2}{b_1 d_1}, \quad b_3 = 0, \quad c_1 = \frac{9a_1 b_1 d_0^2 d_1 - 2b_1^2 d_0 d_1^2 - 9b_1^2 d_0^3 + 2d_1^4}{9d_0 d_1^2},
$$

$$
d_2 = \frac{b_1(9b_2 d_0 d_1 - 9a_1 d_0 d_1 + 2b_1 d_1^2 + 18b_1 d_0^2)}{9d_1^2}, \quad d_3 = \frac{b_1(b_1 d_0 + b_2 d_1)}{3d_1}
$$

or

$$a_1 = b_0 + \frac{2}{9}\frac{b_0 d_1^2 - 4b_1 d_0 d_1 + b_0 b_1^2 d_0}{d_0(b_0^2 - 4d_0)}, \quad b_3 = 0,$$

$$b_2 = -a_1 + \frac{2}{9}\frac{2b_0 d_1^2 + b_0 b_1^2 d_0 - 6b_1 d_0 d_1}{d_0(b_0^2 - 4d_0)},$$

$$c_1 = d_0 + \frac{2}{9}\frac{2b_1^2 d_0^2 + b_0^2 d_1^2 - 2b_0 b_1 d_0 d_1 - 2d_0 d_1^2}{d_0(b_0^2 - 4d_0)},$$

$$d_2 = -c_1 - d_0 + \frac{2}{9}\frac{b_0^2 d_1^2 + b_0 b_1 d_0 d_1 - 4d_0 d_1^2 - 2b_1^2 d_0^2}{d_0(b_0^2 - 4d_0)},$$

$$d_3 = -\frac{1}{3}d_1 + \frac{1}{27}\frac{(2b_0 b_1 d_1 - 4b_1^2 d_0)d_1}{d_0(b_0^2 - 4d_0)}$$

the corresponding systems satisfy $s_3(x) = 0$ and are centers of (1.1).

An integrating factor for the first case can be obtained from (3.12) and one for the second case is given by

$$\mu(x,y) = \frac{e^{\int \mathcal{F}(x)\,dx}}{(3d_0 x^2 + d_1 xy + b_1 y + 3)^3 \mathcal{D}(x)} \tag{5.1}$$

where $\mathcal{F}(x) = \mathcal{N}(x)/\mathcal{D}(x)$ with

$$\mathcal{N}(x) = 2(3b_0^2 d_1^2 + 9b_0^2 d_0^2 + 4b_1^2 d_0^2 - 4d_0 d_1^2 - 6b_0 b_1 d_0 d_1 - 36d_0^3)x$$
$$+ 9b_0^3 d_0 - 36b_0 d_0^2 - 8b_1 d_0 d_1 + 4b_0 d_1^2$$

and

$$\mathcal{D}(x) = (9b_0^2 d_0^2 + 2b_0^2 d_1^2 - 4b_0 b_1 d_0 d_1 + 4b_1^2 d_0^2 - 4d_0 d_1^2 - 36d_0^3)x^2$$
$$+ (9b_0^3 d_0 + 2b_0 b_1^2 d_0 + 2b_0 d_1^2 - 36b_0 d_0^2 - 8b_1 d_0 d_1)x + 9d_0(b_0^2 - 4d_0).$$

The system also has general constant invariant solutions and using the results from section 3, we obtain the following result.

Proposition 5.2. *Let $n = 2$, $a_0 = c_0 = b_3 = 0$ and* (1.1), (1.3) *be given by* (1.2). *Then the system defined by*

$$b_0 = \frac{1}{4}\frac{\alpha}{C}, \quad b_2 = -b_0 - \frac{C}{2},$$

$$b_1 = -\frac{1}{8}\frac{C^4 + 2a_1 C^3 - 4d_0 C^2 - 8a_1 d_0 C + 32d_0^2 + 32d_3^2}{d_3 C} + \frac{1}{16}\frac{C^2 - 8d_0}{d_3 C}\alpha,$$

$$d_1 = -\frac{1}{2}b_1 C - 3d_3, \quad d_2 = \frac{1}{2}C^2 + a_1 C - 3d_0 - \frac{\alpha}{4}$$

where $\alpha = 2a_1 C - 8d_0 \pm 2\sqrt{C^2 C_1^2 - 16d_3^2}$, $C = C_1 - a_1$ and C_1 is given by (3.1) *satisfies the condition I_1 constant and is a center of* (1.1).

The system has a rational integrating factor having the same basic form as (4.3) except for the addition of a second invariant line which appears as a factor in the denominator. This arises from the factorization of the denominators of the coefficient functions $f_0, \ldots, f_3$. For later reference we also specifically note that no member of this family satisfies the condition $b_0 + b_2 = 0$ since C cannot be zero.

In addition to the general constant invariant solutions, the system also has analogs of the $I_1 = 0$ solution given in Proposition 4.1.

Proposition 5.3. *Let* $n = 2$, $a_0 = c_0 = b_3 = 0$ *and* (1.1), (1.3) *be given by* (1.2). *If any one of the following conditions holds, then the corresponding system satisfies* $I_1 = 0$ *and is a center of* (1.1).
(i)

$$C_1 = \alpha, \quad b_1 = \beta, \quad b_2 = \frac{1}{6}\alpha + \frac{1}{2}a_1 - b_0,$$

$$d_0 = \frac{1}{12}\alpha^2 + \frac{1}{6}(a_1 - b_0)\alpha - \frac{1}{4}a_1(a_1 - 2b_0),$$

$$d_1 = \frac{1}{4}\frac{4a_1 - 5b_0}{\beta}\alpha^2 + \frac{1}{2}\frac{(2a_1 - b_0)(2a_1 - 5b_0)}{\beta}\alpha$$
$$- \frac{3}{4}\frac{4a_1^3 - 15a_1^2 b_0 + 26a_1 b_0^2 - 16b_0^3}{\beta},$$

$$d_2 = \frac{3(2a_1 - 3b_0)}{b_0}d_0 - a_1\alpha + a_1^2 - 8a_1 b_0 + 8b_0^2,$$

$$d_3 = -\frac{1}{12}\frac{a_1 - 2b_0}{\beta}\alpha^2 - \frac{1}{6}\frac{(a_1 + b_0)(a_1 - 2b_0)}{\beta}\alpha$$
$$+ \frac{1}{12}\frac{3a_1^3 - 12a_1^2 b_0 + 24a_1 b_0^2 - 16b_0^3}{\beta}$$

where α *is a root of the cubic in* X,

$$X^3 + (5a_1 - 8b_0)X^2 + 3a_1(a_1 - 4b_0)X - 9a_1^3 + 36a_1^2 b_0 - 72a_1 b_0^2 + 48b_0^3 = 0,$$

and

$$\beta^2 = \frac{1}{2b_0}((3a_1 - 2b_0)\alpha^2 + (6a_1^2 - 19a_1 b_0 + 7b_0^2)\alpha - 9a_1^3 + 33a_1^2 b_0 - 63a_1 b_0^2 + 42b_0^3).$$

(ii)

$$b_1 = \alpha, \quad b_2 = \frac{C}{2} - b_0, \quad d_0 = -\frac{1}{4}C(C - 2b_0),$$

$$d_1 = -\frac{3}{4}\frac{(a_1 - b_0)C(C - 2b_0)}{\alpha}, \quad d_2 = -d_0, \quad d_3 = \frac{1}{6}\frac{C_1(a_1 - b_0)(C - 2b_0)}{\alpha}$$

where $\alpha^2 = (3b_0 C_1 - 3a_1 C + 3b_0(3a_1 - 2b_0))/2$ *and* $C = C_1 + a_1$.
(iii)

$$a_1 = \alpha, \quad b_1 = \beta, \quad b_2 = \frac{1}{2}\alpha + \frac{1}{6}C_1 - b_0,$$

$$d_1 = \frac{3(\alpha - b_0)d_0}{\beta} + \frac{3C_1 d_0^2}{(b_0^2 - 3d_0)\beta}, \quad d_2 = -d_0 + \frac{2C_1 d_0^2}{b_0(b_0^2 - 3d_0)},$$

$$d_3 = -\frac{1}{6}\frac{b_0 C_1(b_0^2 - 6d_0)\alpha}{(b_0^2 - 3d_0)\beta} + \frac{1}{6}\frac{b_0 C_1^2(b_0^2 - 2d_0)}{(b_0^2 - 3d_0)\beta} + \frac{1}{3}\frac{b_0^2 C_1(b_0^2 - 5d_0)}{(b_0^2 - 3d_0)\beta}$$

where α *is a root of the quadratic in* X,

$$b_0(b_0^2 - 3d_0)X^2 - 2b_0^2(b_0^2 - 3d_0)X - b_0(b_0^2 - 3d_0)C_1^2 - 2(b_0^2 - 4d_0)(b_0^2 - d_0)C_1$$
$$+ 4b_0 d_0(b_0^2 - 3d_0) = 0,$$

and

$$\beta^2 = \frac{3}{2}b_0\alpha - \frac{3}{2}\frac{b_0 C_1(b_0^2 - 5d_0)}{b_0^2 - 3d_0} - 3b_0^2 + 6d_0.$$

(iv)

$$a_1 = \alpha, \quad b_1 = \beta, \quad b_2 = \frac{1}{2}\alpha + \frac{1}{6}C_1 - b_0,$$

$$d_0 = \frac{1}{12}(\alpha + C_1)(3\alpha - C_1),$$

$$d_1 = -\frac{1}{2}\frac{C_1 + 3b_0}{\beta}\alpha^2 - \frac{1}{6}\frac{2C_1^3 + 9b_0C_1^2 - 9b_0^2C_1 - 45b_0^3}{(C_1 + 3b_0)\beta}\alpha$$
$$+ \frac{1}{6}\frac{C_1^4 + 9b_0C_1^3 + 24b_0^2C_1^2 + 15b_0^3C_1 - 18b_0^4}{(C_1 + 3b_0)\beta},$$

$$d_2 = \frac{1}{12}\frac{(\alpha + C_1)(2C_1^2 + 9b_0C_1 - 6C_1\alpha - 15b_0\alpha + 12b_0^2)}{b_0},$$

$$d_3 = -\frac{1}{6}\frac{(C_1 + 3b_0)(C_1 - 2b_0)}{b_0\beta}\alpha^2 - \frac{1}{18}\frac{2C_1^3 - b_0C_1^2 - 3b_0^2C_1 + 36b_0^3}{b_0\beta}\alpha$$
$$+ \frac{1}{18}\frac{C_1^4 + 4b_0C_1^3 + 3b_0^2C_1^2 + 18b_0^4}{b_0\beta}$$

where α is a root of the cubic in X,

$$9(C_1 + 3b_0)X^3 + 3(C_1^2 - 9b_0^2)X^2 - (5C_1^3 + 21\,b_0C_1^2 + 36b_0^2C_1 + 36b_0^3)X$$
$$+ C_1^4 + 6b_0C_1^3 + 15b_0^2C_1^2 + 24b_0^3C_1 + 36b_0^4 = 0,$$

and

$$\beta^2 = -\frac{1}{2b_0}(3(C_1 + 2b_0)\alpha^2 + (2C_1^2 + b_0C_1 - 12b_0^2)\alpha - C_1^3 - 5b_0C_1^2 - 6b_0^2C_1 + 6b_0^3).$$

These are the systems which remain after removing other simpler forms which appear as special cases of these. At this time we believe the systems to be independent, although it is certainly possible that it may be shown that one or more of them belong to a larger, more general system. The fact that there are multiple solutions is somewhat surprising, although it is similar to what happens for the $I_1 = 0$ case [18] for the homogeneous $n = 4$ system. It can be shown by considering resultants or numerically that each of the 4 systems does produce real–valued parameters. Also, each of them has a form for which $b_0 + b_2 = 0$, although the one for (iv) is not real-valued. This is in contrast to the result from Proposition 5.2 for which the general system did not have this property and shows that the two classes of centers are distinct. We do not give integrating factors for these systems because they are similar to those already described and can be obtained in the same manner.

Any of the systems for which $I_1 = 0, s_3(x) \neq 0$ should be obtainable simply by solving the system of equations defined by $s_5(x) = 0$ (denote this system by $\mathcal{T}$) with the condition that $s_3(x)$ is nonzero. In fact, if this had been possible these results would have been obtained much sooner than they actually were since these were some of the first solutions that we checked for following the Bernoulli cases. However, $\mathcal{T}$ turned out to be totally intractable for the software (Maple) and it was not until we began adding more specific forms for the various expressions that any results were forthcoming. Obviously, the system $\mathcal{S}$ described in the previous section is a solution of $\mathcal{T}$. It was obtained through the reduction of Lyapunov constants, but even then in the factored forms of the resultants Maple was unable to solve the much simplified system. Fortunately, we were able to find a parametric form for

the solution and use it to confirm the $I_1 = 0$ nature of the system. The system $\mathcal{S}$ is equivalent to (iii) with the added condition $b_0 + b_2 = 0$.

We have indicated that we had previously attempted an analysis of this system ($c_1 \neq 0$) in terms of Lyapunov coefficients. At that time we were unable to calculate a sufficient number of these coefficients to carry out an analysis without imposing additional conditions on the system parameters. This author is of the opinion that the results presented in this section are not obtainable at present using the conventional development just described because of the inability of computers and computer algebra systems to deal with the massive expressions required for this type of analysis.

6. THE CASE $a_0 c_0 \neq 0$

This case is not the general focus of this paper, but we recently learned of the paper [3] by Cherkas and Romanovski which addressed certain aspects of these types of systems. They considered certain cubic systems which could be transformed to a system of Liénard type and used the known center conditions for these systems (see [6]) to obtain center conditions for the original systems. Specifically, their results apply to systems of the form

$$\frac{dx}{dt} = -y - a_0 x^2 - a_1 xy - c_0 x^3, \quad \frac{dy}{dt} = x + b_0 x^2 + b_1 xy + b_2 y^2 + d_0 x^3 + d_1 x^2 y.$$

We have only analyzed a few of their results in detail, restricting for now our attention to case (β) of [3, Theorem 5] where 41 separate systems are listed. Each of these depends upon at least one arbitrary parameter which is fortunate because it is necessary to use it in order to transform the results (β) to cubic systems. Otherwise, the system of equations which defines this transformation is overdetermined and has no solution. Even so, some cases do not have solutions and others degenerate to quadratic systems. Of those which define cubic systems some are Bernoulli and others are constant invariant. However, there is one system defined by result 9 which is particularly interesting. This is given by

$$\frac{dx}{dt} = -y - Ax^2 - xy - Ax^3,$$
$$\frac{dy}{dt} = x + x^2 + (2A - 1)xy - \frac{2}{3}y^2 + 2A(1 - 5A)x^3 + \frac{1}{3}(2A - 1)x^2 y. \tag{6.1}$$

where A is an arbitrary parameter. We have found 6 integrable cases for this system. Converting it to a first kind form, we see that only $f_3(x)$ is dependent upon A with

$$f_3(x) = \frac{A(1 - 4A)x^4 + 9A(1 - 4A)x^3 + 3x^2 + 3x}{3(x + 1)}$$

and since the terms in A are symmetric about $A = 1/8$, the integrable forms occur in pairs. Calculation of $s_5(x)$ easily shows that if $A = 1/12, 1/6$ the equation is solvable with $I_1 = 0$. For $A = 1/20, 1/5$ the systems have a Darboux first integral which arises from a non–constant invariant Abel equation. For $A = 1/5$ this is given by $\mathcal{U}(x, y) = \mathcal{P}^2(x, y)/\mathcal{Q}^3(x, y)$ where

$$\mathcal{P}(x, y) = 648x^3 - 900x^2 y + 450xy^2 - 125y^3$$
$$+ 4500x^2 - 3060xy + 675y^2 + 7650x - 2295y + 3825, \tag{6.2}$$
$$\mathcal{Q}(x, y) = 84x^2 - 60xy + 25y^2 + 300x - 90y + 225.$$

The other forms are obtained by noting that $f_3(x)$ will reduce to just x if $A = 0, 1/4$. The resulting Abel equation

$$\frac{dy}{dx} = xy^3 - \frac{1}{3}\frac{x(x+3)}{x+1}y^2 - \frac{1}{3}\frac{2}{x+1}y$$

(where we have retained y as the dependent variable) is transformable to a Riccati equation which is solvable in terms of special functions. The exact form of the transformation which achieves this is difficult to give because of its implicit nature. However, if we set

$$\frac{dt}{dx} = \frac{1}{\sqrt[3]{81}}\frac{x}{(x+1)^{4/3}}, \quad t = \int t'(x)\,dx = \frac{\sqrt[3]{9}}{6}\frac{x+3}{(x+1)^{1/3}},$$
$$u(t) = \frac{1}{\sqrt[3]{9}}\frac{1}{(x+1)^{2/3}}y(x)$$

the Abel equation transforms directly to

$$\frac{du}{dt} = u^3 - 2tu^2$$

which is the representative equation for Class 2 given in [2, Appendix A] and which is shown there to be solvable in terms of Airy functions. These are the only cubic systems that we know of with this property, although Żołądek [22, Remark 2.4] has discussed this phenomenon as a common occurrence with regards to complex cubic systems. No real, homogeneous ($p_2 = q_2 = 0$) system of degree 3 is solvable in such a manner although there are homogeneous systems which are so integrable for each $n \geq 4$. The separate cases also show that we needn't bother trying to find an integrating factor of Darboux type for the general system even though such expressions exist for the $I_1 = 0$ and Darboux integrable forms.

It has been conjectured by Żołądek [21] that all real centers for polynomial systems are either rationally reversible or have a Liouvillian first integral. Other similar conjectures have been made. See for example Christopher and Llibre [4]. It has been shown by Berthier, Cerveau and Lins-Neto [1] that this is not true in general, however the conjecture still remains for cubic systems. In the major work [21] and the follow-up paper [23] the author gives a complete list of rationally reversible cubic systems. Minor corrections to this list are given by von Bothmer [10] and von Bothmer and Kröker [11]. There are 17 such systems given denoted by CR_j for $j = 1, \ldots, 17$. It has been shown by von Bothmer that CR_5, CR_7, CR_{12} and CR_{16} are fully integrable and of these CR_5 and CR_7 are constant invariant. Our brief recent study of these systems has added CR_2 and CR_3 as being of Bernoulli type and CR_{17} is of type $I_1 = 0$ when expressed as $dx/dy = \dot{x}/\dot{y}$. In addition to these several other systems are almost integrable in the sense that one additional parameter assignment will lead to an integrable Liouvillian form. In [23] he also gives a list of 35 Darboux integrable cubic systems which does not appear to include the system (6.2) given above.

The system (6.1) has a non–Liouvillian first integral if $A = 0$, $1/4$, so unless these two cases are rationally reversible the conjectures just mentioned are not true. Our study of the rationally reversible cubic systems CR_j has shown that (6.1) is not a member of these systems. In particular, it cannot be a member of any of the fully integrable systems described above.

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