Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 191, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

CORDES NONLINEAR OPERATORS IN CARNOT GROUPS

GIUSEPPE DI FAZIO, MARIA STELLA FANCIULLO

Abstract. Our aim is to obtain L^p estimates for the second-order horizontal derivatives of the solutions for a nondivergence form nonlinear equation in Carnot groups.

1. Introduction

The $W^{2,p}$ estimates for elliptic differential equations and systems is a very interesting problem and many Authors have given several contributions to this problem from several different points of view (see [5, 4, 7]) using different approaches. There are essentially two main approaches to the problem: assuming on the coefficients of the equation the Cordes condition or the VMO condition. The first one consists of a geometric condition on the eigenvalues of linear operators. Cordes condition was introduced in [6] and studied by many authors (in the cases of nonlinear nonvariational equations and systems we quote [3, 4]). The second technique consists in assuming the coefficients of the operator to be in VMO-type classes (see [5, 7, 9, 10], and for more general setting see [1, 2]).

Here we obtain $W^{2,p}$ estimates for the following nonlinear nondivergence form equation

$$a(x, u, Xu, X^2u) = f,$$

where $X = (X_1, X_2, \dots, X_l)$ is a system of Hörmander's vector fields on a Carnot group, and we assume a condition that in the particular case of a linear equation gives back the Cordes condition (see [8] for the case of the Heisenberg group).

Namely, we show that there exists a critical exponent $p_0 > 2$ such that if the datum f belongs to L^p, with $2 < p < p_0$, then the second derivatives X^2u of the solutions u have the same integrability as f.

2. Preliminaries

Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. We assume $\mathcal{G} = \oplus_{i=1}^{s} V_i$, where $[V_i, V_j] \subset V_{i+j}$ for $i + j \le s$ and $[V_i, V_j] = 0$ for $i + j > s$. Let $X_1, \dots, X_l$ be a basis for V_1 and suppose that $X_1, \dots, X_l$ generate $\mathcal{G}$ as a Lie algebra. Then for $2 \le j \le s$ we choose a basis $\{X_{ij}\}$, $1 \le i \le k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i$, $i = 1, \dots, l$ and $k_1 = l$, and we call X_{i1} a commutator of length 1. If $\mathbb{G}$ is the simply connected Lie group associated to $\mathcal{G}$ then

---

2010 *Mathematics Subject Classification.* 35H20.
Key words and phrases. Cordes condition; Carnot groups; nonlinear equations.
©2015 Texas State University - San Marcos.
Submitted April 8, 2015. Published July 20, 2015.

$\mathbb{G}$ is called Carnot group. It is well known that the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{G}$ and then for any $g \in \mathbb{G}$ there exists $x = (x_{ij}) \in \mathbb{R}^n$, $1 \le i \le k_j$, $1 \le j \le s$, $n = \sum_{j=1}^{s} k_j$, such that $g = \exp(\sum x_{ij} X_{ij})$.

We now recall the definition of polynomials on the Carnot group $\mathbb{G}$ given by Folland and Stein in [15].

A function P on $\mathbb{G}$ is said to be a polynomial on $\mathbb{G}$ if $P \circ \exp$ is a polynomial on the Lie algebra $\mathcal{G}$.

Let $X_1, X_2, \dots, X_n$ be a basis of $\mathbb{G}$ and $\xi_1, \xi_2, \dots, \xi_n$ be the dual basis for $\mathcal{G}^*$ we set $\eta_i = \xi_i \circ \exp^{-1}$. Each η_i is a polynomial on $\mathbb{G}$, and $\eta_1, \eta_2, \dots, \eta_n$ form a system of global coordinates on $\mathbb{G}$. Then every polynomial on $\mathbb{G}$ can be written uniquely as

$$P(x) = \sum_I a_I \eta^I(x), \quad \eta^I = \eta_1^{i_1} \cdots \eta_n^{i_n}, \ a_I \in \mathbb{R}$$

where all but finitely many of the coefficients a_I vanish. Clearly η^I is homogeneous of degree $d(I) = \sum_{j=1}^{n} i_j d_j$, where d_j is the length of X_j as a commutator. We define the homogeneous degree of the polynomial P as $\max\{d(I) : a_I \ne 0\}$.

Here we recall the definition of the Carnot-Carathéodory metric. An absolutely continuous curve $\gamma : [0, \tau] \to \mathbb{R}^n$ is called subunitary if there exists a measurable function $c = (c_1, c_2, \dots, c_l) : [0, \tau] \to \mathbb{R}^l$ such that $\gamma'(t) = \sum_{j=1}^{l} c_j(t) X_j(\gamma(t))$, for a.e. $t \in [0, \tau]$, and $\|c\|_\infty \le 1$. The Carnot-Carathéodory distance $d(x', x'')$ is defined as the infimum of those $\tau > 0$ for which there exists a subunitary curve $\gamma : [0, \tau] \to \mathbb{R}^n$ with $\gamma(0) = x'$ and $\gamma(\tau) = x''$.

We set $B_r(x_0) = \{x \in \mathbb{R}^n : d(x, x_0) < r\}$. When it is clear from the setting we will omit x_0 or r. It is well known that the Carnot-Carathéodory balls satisfy a doubling condition, that is

$$|B_{2r}(x_0)| \le 2^Q |B_r(x_0)|$$

for all $r > 0$ and $x_0 \in \mathbb{R}^n$. The constant Q is the homogeneous dimension of $\mathbb{G}$.

We define the intrinsic Sobolev spaces for a bounded domain Ω in $\mathbb{R}^n$. Let $k \in \mathbb{N}$ and $p \ge 1$ we set

$$W^{k,p}(\Omega) = \{u : \Omega \to \mathbb{R} : u, X_{i_1} \dots X_{i_j} u \in L^p(\Omega), \ 1 \le j \le k\}$$

endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{h=1}^{k} \sum_{i_j=1}^{l} \|X_{i_1} X_{i_2} \dots X_{i_h} u\|_{L^p(\Omega)} .$$

We define $W_0^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ with respect to the above norm.

For $I = (i_1, i_2, \dots, i_n)$ we denote the differential operator $X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$ by X^I and $d(I)$ the homogeneous degree of X^I. We denote by Xu the gradient of u $(X_1 u, X_2 u, \dots, X_l)$ and by $X^2 u$ the hessian matrix $\{X_{ij} u\}_{i,j=1,\dots,l}$.

In [17] the existence of approximation polynomials of Sobolev functions in Carnot groups and related Poincaré-type inequalities have been obtained. Here we state some results [17, Theorems 2.7 and 5.1], that we will use in the sequel.

Theorem 2.1. *Let k be a positive integer and u a function in $W^{k,1}(\Omega)$. Then there exists a polynomial P of degree less than k such that $\int_\Omega X^I (u - P) dx = 0$ for any $0 \le d(I) < k$.*

Choosing first $q_{10} = p = 2$ and $q_{21} = 2$, $p = \frac{2Q}{Q+2}$ in [17, Theorem 5.1], we obtain the following two inequalities that we collect in the same statement.

Theorem 2.2. *Let B_r be a ball of $\mathbb{R}^n$ and $u \in W^{2,\frac{2Q}{Q+2}}(\overline{B}_r)$. Then there exists a polynomial of degree ≤ 1 such that*

$$\int_{B_r} |u-P|^2 dx \leq cr^2 \int_{B_r} |X(u-P)|^2 dx, \tag{2.1}$$

$$\int_{B_r} |X(u-P)|^2 dx \leq c\Big(\int_{B_r} |X^2 u|^{\frac{2Q}{Q+2}} dx\Big)^{\frac{Q+2}{Q}}, \tag{2.2}$$

where the constant c is independent of B_r and u. (The polynomial P is the same as in Theorem 2.1).

The following Theorem has been proved in [14], (for different cases see [13, 11, 12]).

Theorem 2.3. *There exists a constant $C_G \geq 1$ such that for every $u \in W_0^{2,2}(\Omega)$ the following inequality holds*

$$\int_\Omega |X^2 u|^2 dx \leq C_G \int_\Omega |\Delta u|^2 dx\,, \tag{2.3}$$

where $\Delta u = \sum_{i=1}^l X_i X_i u$.

3. Caccioppoli-type inequality and $W^{2,p}$ estimates

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $a(x,u,p,m) : \Omega \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{l^2} \to \mathbb{R}$ be a Carathéodory function satisfying the condition

(A) there exist three positive constants, α, γ and δ such that $C_G\gamma + \delta < 1$, for all $M = \{m_{ij}\}_{i,j=1,\dots,l} \in \mathbb{R}^l \times \mathbb{R}^l$, $u \in \mathbb{R}$, $p \in \mathbb{R}^l$,

$$\Big|\sum_{i=1}^l m_{ii} - \alpha[a(x,u,p,m)]\Big|^2 \leq \gamma |M|^2 + \delta \Big|\sum_{i=1}^l m_{ii}\Big|^2, \quad \text{a.e. } x \in \Omega\,.$$

We consider the nonlinear nonvariational elliptic equation

$$a(x,u,X,X^2 u) = f\,, \tag{3.1}$$

where $f \in L^2(\Omega)$.

Definition 3.1. A function $u \in W^{2,2}(\Omega)$ is called a solution of (3.1) if u satisfies (3.1) for a.e. x in Ω.

Remark 3.2. In the case of linear equation, i.e.

$$\sum_{i,j=1}^l a_{ij}(x) X_i X_j u(x) = f$$

condition (A) is stronger than the following Cordes condition (see [6, 18] for a comparison between condition (A) and Cordes condition in Euclidean setting).

Definition 3.3. The linear operator $L \equiv a_{ij}(x)X_iX_j$ satisfies the Cordes condition $K_{\epsilon,\sigma,\theta}$ if there exist $\epsilon \in (0,1]$, $\sigma > 0$ and $\theta > 0$ such that for a.e. $x \in \Omega$, $\sum_{i=1}^{l} a_{ii}(x) > 0$ and

$$0 < \frac{1}{\sigma} \le \sum_{i,j=1}^{l} a_{ij}^2(x) \le \frac{1}{l-1+\epsilon}\Big(\sum_{i=1}^{l} a_{ii}(x)\Big)^2 \le \frac{\theta^2}{l-1+\epsilon}\,.$$

Now we prove a Caccioppoli type inequality for solutions of (3.1).

Theorem 3.4. *Let condition (A) hold true and $f \in L^2(\Omega)$. Then for any $u \in W^{2,2}(\Omega)$ solution of (3.1), for any $r > 0$ such that $B_{2r} \Subset \Omega$, there exists a polynomial P of degree less than 2 such that $\int_{B_{2r}} X^I(u-P)dx = 0$ for any $0 \le d(I) < 2$, and*

$$\int_{B_r} |X^2u|^2dx \le cr^{-2}\int_{B_{2r}} |X(u-P)|^2dx + c\int_{B_{2r}} f^2dx\,. \tag{3.2}$$

Proof. Let $B_{2r} \Subset \Omega$. From Theorem 2.1 there exists a polynomial P of degree less than two such that $\int_{B_{2r}} X^I(u-P)dx = 0$, for I with $d(I) < 2$.

Let η be a $C_0^\infty(\mathbb{R}^n)$ with the properties $0 \le \eta \le 1$, $\eta = 1$ in B_r, $\eta = 0$ in $\mathbb{R}^n \setminus B_{2r}$ and $|X\eta| \le \frac{c}{r}$.

If we set $\mathcal{U} = \eta(u-P) \in W_0^{2,2}(B_{2r})$, since $X^2P = 0$ (see the proof of [17, Theorem 2.7]), we have $X^2(u-P) = X^2u$ and $X^2\mathcal{U} = X^2u$ in B_r. We have

$$\eta\Delta u = \eta(\Delta u - \alpha a(x,u,Xu,X^2u)) + \eta\alpha f\,,$$

which implies

$$\begin{aligned}|\eta\Delta u| &\le \eta|\Delta u - \alpha a(x,u,Xu,X^2u)| + |\eta\alpha f|\\ &\le \eta[\gamma|X^2u|^2 + \delta|\Delta u|^2]^{1/2} + \eta\alpha|f|.\end{aligned}$$

Note that

$$\Delta\mathcal{U} = \eta\Delta u + A(u-P), \tag{3.3}$$

$$\eta X^2u = X^2\mathcal{U} - B(u-P), \tag{3.4}$$

where

$$A(u-P) = (u-P)\Delta\eta + 2\sum X_i\eta X_i(u-P) \tag{3.5}$$

and

$$B(u-P) = \{(u-P)X_iX_j\eta + X_i\eta X_j(u-P) + X_j\eta X_i(u-P)\}_{ij}\,.$$

Then for $x \in B_{2r}$,

$$|\Delta\mathcal{U}| \le |\eta\Delta u| + |A(u-P)| \le \eta(\gamma|X^2u|^2 + \delta|\Delta u|^2)^{1/2} + \eta\alpha|f| + |A(u-P)|,$$

from which it follows that for all $\epsilon > 0$,

$$\begin{aligned}|\Delta\mathcal{U}|^2 &\le \eta^2(\gamma|X^2u|^2 + \delta|\Delta u|^2) + (\eta\alpha|f| + |A(u-P)|)^2\\ &\quad + 2\eta(\gamma|X^2u|^2 + \delta|\Delta u|^2)^{1/2}(\eta\alpha|f| + |A(u-P)|)\\ &\le \eta^2(\gamma|X^2u|^2 + \delta|\Delta u|^2) + (\eta\alpha|f| + |A(u-P)|)^2\\ &\quad + \epsilon\eta^2(\gamma|X^2u|^2 + \delta|\Delta u|^2) + \frac{1}{\epsilon}(\eta\alpha|f| + |A(u-P)|)^2\\ &= (1+\epsilon)\eta^2(\gamma|X^2u|^2 + \delta|\Delta u|^2) + \big(1 + \frac{1}{\epsilon}\big)(\eta\alpha|f| + |A(u-P)|)^2\end{aligned}$$

$$\le (1+\epsilon)\eta^2(\gamma|X^2u|^2+\delta|\Delta u|^2)+2\big(1+\frac{1}{\epsilon}\big)(\eta^2\alpha^2|f|^2+|A(u-P)|^2).$$

Then from (3.3) and (3.4),

$$\begin{aligned}
|\Delta\mathcal{U}|^2 &\le (1+\epsilon)\gamma|X^2(\mathcal{U})-B(u-P)|^2+(1+\epsilon)\delta|\Delta\mathcal{U}-A(u-P)|^2\\
&\quad+2\big(1+\frac{1}{\epsilon}\big)\eta^2\alpha^2|f|^2+2\big(1+\frac{1}{\epsilon}\big)|A(u-P)|^2\\
&\le (1+\epsilon)\gamma(|X^2(\mathcal{U})|^2+|B(u-P)|^2+2|X^2(\mathcal{U})||B(u-P)|)\\
&\quad+(1+\epsilon)\delta(|\Delta\mathcal{U}|^2+|A(u-P)|^2+2|\Delta\mathcal{U}||A(u-P)|)\\
&\quad+2\big(1+\frac{1}{\epsilon}\big)\eta^2\alpha^2|f|^2+2\big(1+\frac{1}{\epsilon}\big)|A(u-P)|^2\\
&\le (1+\epsilon)\gamma\big[(1+\epsilon)|X^2(\mathcal{U})|^2+\big(1+\frac{1}{\epsilon}\big)|B(u-P)|^2\big]\\
&\quad+(1+\epsilon)\delta\big[(1+\epsilon)|\Delta\mathcal{U}|^2+\big(1+\frac{1}{\epsilon}\big)|A(u-P)|^2\big]\\
&\quad+2\big(1+\frac{1}{\epsilon}\big)\eta^2\alpha^2|f|^2+2\big(1+\frac{1}{\epsilon}\big)|A(u-P)|^2\\
&\le (1+\epsilon)^2\gamma|X^2(\mathcal{U})|^2+(1+\epsilon)^2\delta|\Delta\mathcal{U}|^2\\
&\quad+c(\epsilon,\alpha,\gamma,\delta)[|A(u-P)|^2+|B(u-P)|^2+|f|^2].
\end{aligned}$$

We integrate on B_{2r} and apply (2.3) in Theorem 2.3 to obtain

$$\begin{aligned}
\int_{B_{2r}}|\Delta\mathcal{U}|^2dx &\le (1+\epsilon)^2\gamma\int_{B_{2r}}|X^2(\mathcal{U})|^2dx+(1+\epsilon)^2\delta\int_{B_{2r}}|\Delta\mathcal{U}|^2dx\\
&\quad+c\int_{B_{2r}}(|f|^2+|A(u-P)|^2+|B(u-P)|^2)dx\\
&\le (1+\epsilon)^2(\gamma C_G+\delta)\int_{B_{2r}}|\Delta\mathcal{U}|^2dx\\
&\quad+c\int_{B_{2r}}(|f|^2+|A(u-P)|^2+|B(u-P)|^2)dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&[1-(1+\epsilon)^2(\gamma C_G+\delta)]\int_{B_{2r}}|\Delta\mathcal{U}|^2dx\\
&\le c\int_{B_{2r}}(|f|^2+|A(u-P)|^2+|B(u-P)|^2)dx,
\end{aligned}$$

and then

$$\int_{B_{2r}}|\Delta\mathcal{U}|^2dx\le c\int_{B_{2r}}(|f|^2+|A(u-P)|^2+|B(u-P)|^2)dx.$$

Finally, we get that

$$\begin{aligned}
\int_{B_r}|X^2u|^2dx &\le \int_{B_{2r}}|X^2\mathcal{U}|^2dx\le C_G\int_{B_{2r}}|\Delta\mathcal{U}|^2dx\\
&\le c\int_{B_{2r}}(|f|^2+|A(u-P)|^2+|B(u-P)|^2)dx.
\end{aligned}$$

Now we observe that from (3.5) and Poincaré inequality (2.1) we obtain

$$\begin{aligned}
&\int_{B_{2r}} |A(u-P)|^2dx \\
&\le c\int_{B_{2r}} |\Delta\eta|^2|u-P|^2dx + c\int_{B_{2r}} \sum |X_i\eta|^2|X_i(u-P)|^2dx \\
&\le cr^{-2}\Big\{r^{-2}\int_{B_{2r}} |u-P|^2dx + \int_{B_{2r}} |X(u-P)|^2dx\Big\} \\
&\le cr^{-2}\int_{B_{2r}} |X(u-P)|^2dx.
\end{aligned}$$

In the same way, we find that

$$\int_{B_{2r}} |B(u-P)|^2dx \le cr^{-2}\int_{B_{2r}} |X(u-P)|^2dx,$$

from which the Caccioppoli type inequality follows. □

Next we state [19, Theorem 3.3] which is a generalization of the Gehring lemma [16].

Lemma 3.5. *Let U and G be non-negative functions in Ω such that*

$$U \in L^t_{\rm loc}(\Omega), \quad G \in L^s_{\rm loc}(\Omega), \quad 1 < t < s.$$

If there exists $c > 1$ such that for every $B_{2r} \Subset \Omega$, $r < 1$,

$$⨍_{B_r} U^t dx \le c\Big(⨍_{B_{2r}} U dx\Big)^t + c⨍_{B_{2r}} G^t dx,$$

then there exists $\epsilon \in (0, s-t]$ such that $U \in L^p_{\rm loc}(\Omega)$, for all $p \in [t, t+\epsilon)$ and, for every $B_{2r} \Subset \Omega$, with $r < 1$, we have

$$\Big(⨍_{B_r} U^p dx\Big)^{1/p} \le K\Big[\Big(⨍_{B_{2r}} U^t dx\Big)^{1/t} + \Big(⨍_{B_{2r}} G^p dx\Big)^{1/p}\Big],$$

where the constant K depends on c, t and Q.

Our main Theorem is now an easy consequence of Caccioppoli-type inequality (3.2) and Lemma 3.5.

Theorem 3.6. *Let $u \in W^{2,2}(\Omega)$ be a solution of (3.1) then there exists $p_0 > 2$ such that, if $f \in L^p(\Omega)$, with $2 \le p < p_0$, then $u \in W^{2,p}_{\rm loc}(\Omega)$ and for all $B_{2r} \subset\subset \Omega$ we have*

$$\Big(⨍_{B_r}|X^2u|^pdx\Big)^{1/p} \le c\Big(⨍_{B_{2r}}|X^2u|^2dx\Big)^{1/2} + \Big(⨍_{B_{2r}}|f|^pdx\Big)^{1/p}.$$

Proof. Let $B_{2r} \subset\subset \Omega$, from the Caccioppoli-type inequality (3.2) and Poincaré inequality (2.2) it follows

$$\int_{B_r} |X^2u|^2dx \le cr^{-2}\Big(\int_{B_{2r}} |X^2u|^{\frac{2Q}{Q+2}}dx\Big)^{\frac{Q+2}{Q}} + \int_{B_{2r}} f^2dx\,,$$

from which

$$⨍_{B_r}|X^2u|^2dx \le c\Big(⨍_{B_{2r}}|X^2u|^{\frac{2Q}{Q+2}}dx\Big)^{\frac{Q+2}{Q}} + ⨍_{B_{2r}}f^2dx. \tag{3.6}$$

Now we can apply Lemma 3.5 with $U = |X^2u|^{\frac{2Q}{Q+2}}$, $t = \frac{Q+2}{Q}$, $G = |f|^{\frac{2Q}{Q+2}}$ and $s = \frac{p(Q+2)}{2Q}$, to obtain the thesis. $\square$

REFERENCES

[1] M. Bramanti, L. Brandolini; *L^p-estimates for nonvariational hypoelliptic operators with VMO coefficients.* Trans. Amer. Math. Soc., 352, n. 2 (2000), 781–822.

[2] M. Bramanti, M. S. Fanciullo; *BMO estimates for nonvariational operators with discontinuous coefficients structured on Hormander's vector fields on Carnot groups*, Advances in Diff. Eq., **18**, 9-10, (2013), 955-1004.

[3] S. Campanato; *Sistemi ellitticiin forma di divergenza. Regolaritá all'interno*, Quaderni SNS di Pisa, (1980).

[4] S. Campanato; *$\mathcal{L}^{2,\lambda}$ theory for nonlinear nonvariational differential systems*, Rend. Mat. Appl., **10** (1990), 531-549.

[5] F. Chiarenza, M. Frasca, P. Longo; *Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients*, Ric. di Mat., **60**, (1991), 149-168.

[6] H. O. Cordes; *Zero order a priori estimates for solutions of elliptic differential equations*, Proceedings of Symposia in Pure Mathematics IV (1961).

[7] G. Di Fazio; *L^p estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. Un. Mat. Ital. A (7) **10**, n.2, (1996) 409–420.

[8] G. Di Fazio, M. S. Fanciullo; *$W^{2,p}_{\mathrm{loc}}$ estimates for Cordes nonlinear operators in the Heisenberg group*, J. of Math. Analysis and Appl., **411**, 947-052, (2014).

[9] G. Di Fazio, M. S. Fanciullo, P. Zamboni; *Interior L^p estimates for degenerate elliptic equations in divergence form with VMO coefficients*, Differential and Integral Equations, **25**, 7-8 (2012), 619-628.

[10] G. Di Fazio, M. S. Fanciullo, P. Zamboni; *L^p estimates for degenerate elliptic systems with VMO coefficients* , St. Petersburg Math. J., **25**, 6 (2014).

[11] G. Di Fazio, A. Domokos, M. S. Fanciullo, J. J. Manfredi; *Subelliptic Cordes estimates in the Grushin plane*, Manuscripta Mathematica, **120**, 419-433 (2006).

[12] A. Domokos, M. S. Fanciullo; *On the best constant for the Friedrichs-Knapp-Stein inequality in free nilpotent Lie groups of step two and applications to subelliptic PDE*, J. of Geometric Analysis, **17**,2, (2007), 245-252.

[13] A. Domokos, J. J. Manfredi; *Subelliptic Cordes estimates*, Proc. AMS, **133**, no. 4, 1047-1056 (2005).

[14] G. B. Folland; *Applications of analysis on nilpotent groups to partial differential equations*, Bull. Amer. Math. Soc. **83** (1977) 912-930.

[15] G. B. Folland, E. M. Stein; *Hardy spaces on homogeneous groups*, Mathematical Notes **28**, Princeton Univ. Press and Univ. of Tokyo Press 1982.

[16] M. Giaquinta; *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Math. Studies No. 105, Princeton Univ. Press, Princeton 1983.

[17] G. Lu; *Polynomials, Higher Order Sobolev Extension Theorems and Interpolation Inequalities on Weighted Folland-Stein Spaces on Stratified group*, Acta Mathematica Sinica **16**, n. 3, (2000), 405-444.

[18] A. Tarsia; *On Cordes and Campanato conditions*, Arch. of Inequalities and Applications, **2**, (2004), 25-40.

[19] A. Zatorska-Goldstein; *Very weak solutions of nonlinear subelliptic equations*, Annales Acc. Scientiarum Fennicae, **30**, (2005), 407-436.

GIUSEPPE DI FAZIO
DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CATANIA, VIALE A. DORIA 6, 95125, CATANIA, ITALY
E-mail address: `difazio@dmi.unict.it`

MARIA STELLA FANCIULLO
DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CATANIA, VIALE A. DORIA 6, 95125, CATANIA, ITALY
E-mail address: `fanciullo@dmi.unict.it`