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EXISTENCE OF SOLUTIONS FOR $p(x)$-KIRCHHOFF TYPE PROBLEMS WITH NON-SMOOTH POTENTIALS

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Abstract. We consider a class of $p(x)$-Kirchhoff type problem with a subdifferential term and a discontinuous perturbation. Assuming the existence of an ordered pair of appropriately defined upper and lower solutions, by the method of lower-upper solutions, penalization techniques, truncations, and results from nonlinear and multivalued analysis, we show the existence of solutions, and of extremal solutions in the interval defined by the lower and upper solution.

1. Introduction

In this article, we study the problem

$$\begin{gathered} -M(t)\,\mathrm{div}\,\big(|\nabla u|^{p(x)-2}\nabla u\big)+\partial F(x,u)+j\big(x,u(x),\nabla u(x)\big)\ni g(u(x)) \\ \text{for a.a. } x\in\Omega, \\ u\big|_{\partial\Omega}=0, \end{gathered} \tag{1.1}$$

where, $N\geq 1$, $M(t)$ is a continuous function with

$$t:=\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x,$$

$p(x)\in C(\Omega)$ with $1\leq p^-=\inf_\Omega p(x)\leq p^+=\sup_\Omega p(x)<+\infty$, $F:\Omega\times\mathbb{R}\to\mathbb{R}$ and $j:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ are not necessarily smooth potential functions. We denote $\partial F(x,u)$ the partial generalized gradient of $F(x,\cdot)$ at the point u.

The operator $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x)\equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous and in general it does not have the first eigenvalue. The study of various mathematical problems with variable exponent growth condition has caused great interest in recent years, and raised many difficult mathematical problems. Problems with variable exponent growth conditions appear in electro-rheological fluids [39, 42], stationary thermo-rheological viscous flows of non-Newtonian fluids [1, 2] and image processing [7, 24] and so on. The more details can be found in [40, 44, 43].

Problem (1.1) is a new variant of Dirichlet problem of Kirchhoff type. Indeed, if the function F is continuously differentiable with respect to the real variable u,

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$\partial F(x,u)=-f(x,u)$, $j=0$, $p(x)=2$, $g=0$ and $M(t)=a+bt$, then problem (1.1) reduces to the Dirichlet problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)\Delta u=f(x,u)\quad \text{in } \Omega,\\ u\big|_{\partial\Omega}=0, \end{gathered}\tag{1.2}$$

which is related to the stationary analogue of the equation

$$\begin{gathered} u_{tt}-\Big(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)\Delta u=f(x,u)\quad \text{in } \Omega,\\ u\big|_{\partial\Omega}=0. \end{gathered}\tag{1.3}$$

Such problems are viewed as being nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2\,\mathrm{d}x)\Delta u$, which means that problems (1.2) and (1.3) are no longer a pointwise identity and are very different from classical elliptic equations. We know that such problems are proposed by Kirchhoff in [27] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Problem (1.2) caused much attention only after Lions [31] proposed an abstract framework to the problem. Some interesting and important results can be found in [6, 16, 30, 32, 33, 36, 37] and references therein. Especially, Dai and Hao [9] studied the following $p(x)$-Kirchhoff-type problem

$$\begin{gathered} -M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\Big)\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)=f(x,u)\quad \text{in } \Omega,\\ u|_{\partial\Omega}=0, \end{gathered}\tag{1.4}$$

where f is a continuous function. By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, they established conditions ensuring the existence and multiplicity of solutions for problem (1.4).

Recently, the study of partial differential equations with nonsmooth potentials has received considerable attention. The area of nonsmooth analysis is closely related with the development of a critical point theory for nondifferentiable functions, in particular, for locally Lipschitz continuous functions based on Clarke's generalized gradient [8]. It provides an appropriate mathematical framework to extend the classic critical point theory for C^1-functionals in a natural way, and to meet specific needs in applications, such as in nonsmooth mechanics and engineering. For a comprehensive understanding, we refer to the monographs of [19, 32, 34] and References [5, 11, 17, 22, 26, 29, 41]. More precisely, if $M(t)=1$, $j=0$, and $g=0$, there exist several existence results for the problem

$$\begin{gathered} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)\in\partial F(x,u)\quad \text{in } \Omega,\\ u\big|_{\partial\Omega}=0. \end{gathered}\tag{1.5}$$

Qian and Shen [38] established conditions ensuring the existence and multiplicity of solutions for problem (1.5) via the theory of nonsmooth critical point theory and the properties of $W_0^{1,p(x)}(\Omega)$. Dai and Liu [10] obtained the existence of at least three solutions for problem (1.5) with $\partial F(x,u)$ replaced by $\lambda\partial F(x,u)$ via a version of the nonsmooth three critical points theorem. Ge et al. [21], using a variational method combined with suitable truncation techniques, proved the existence of at least five solutions under the suitable conditions for problem (1.5). Furthermore, Duan et al. [12] considered the problem

$$
\begin{gathered}
-M(t)(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)-|u|^{p(x)-2}u)\in\partial F_1(x,u)+\lambda\partial F_2(x,u)\quad\text{in }\Omega,\\
\frac{\partial u}{\partial n}=0\quad x\in\partial\Omega,
\end{gathered}
\tag{1.6}
$$

where $t=\int_\Omega\frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})\,\mathrm{d}x$. They established at least three solutions by employing the nonsmooth version three critical points for problem (1.6).

Being influenced by the reading of the above cited papers, we will study the existence and multiplicity of solutions for problem (1.1) via the method of upper and lower solutions. It is an effective tool to discuss the existence theorems for differential equations to generate monotone iterative techniques which provide constructive methods to obtain solutions (see [6, 18, 23]). Compared with the previous works (see [9, 12, 21, 38] and so on), this method can avoid complex computation. To the best of our knowledge, there exist few results to study the extremal solutions of $p(x)$-Laplacian equations with nonsmooth potentials. So our results are new even for the smooth case. The main difficulties in this paper lie in the appearances of the nonlocal term, the non-differentiable functionals and the nonhomogeneous nonlinearities. The lack of differentiability of the nonlinearity causes several technical difficulties. This implies that the variational methods for C^1 functions are not suitable in our case. Therefore our method of proof will be based on techniques from multivalued analysis and nonlinear analysis. Furthermore, our framework presents new nontrivial difficulties. In particular, the presence of set-valued reaction terms $\partial F(x,u)$ and $j(x,u,\nabla u)$ requires completely different devices than in [9, 12, 38, 21] to obtain the existence of solutions for problem (1.1). We think that our results in this direction presented here can be applied to study other different topics as well.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge. In Section 3, we prove the existence of solutions for problem (1.1) by the method of upper-lower solutions combining with two fixed points theorem. In Section 4, the extremal solutions for problem (1.1) is derived.

2. Preliminaries

We firstly give some basic notation.

- $\rightharpoonup$ means weak convergence, and $\to$ strong convergence.
- c denotes the estimated constant (the exact value may be different from line to line).
- $(X,\|\cdot\|)$ denotes a (real) Banach space and $(X^*,\|\cdot\|_*)$ its topological dual. Let (Ω,Σ,μ) be a finite measure space. $2^X\setminus\{\emptyset\}$ stands for the family of all nonempty subsets of X and $\mathfrak{B}(x)$ means the Borel σ-field of X.
- $h^-=\inf_{x\in\Omega}h(x)$, and $h^+=\inf_{x\in\Omega}h(x)$.
- $P_{f(c)}(X)=\{A\subset X: A$ is nonempty, closed (and convex)$\}$.
- $P_{wk(c)}(X)=\{A\subset X: A$ is nonempty, (weakly-)compact (and convex)$\}$.

Definition 2.1. We say that the multifunction $\varphi:\Omega\to P_f(X)$ is measurable, if for all $u\in X$, the $\mathbb{R}_+$-valued function

$$\zeta\to d(u,\varphi(\zeta))=\inf\{\|u-\omega\|:\omega\in\varphi(\zeta)\}$$

is Σ-measurable. A multifunction $\varphi : \Omega \to 2^X\backslash\{\emptyset\}$ is said to be graph measurable, if its graph

$$\operatorname{Gr}\varphi = \{(\zeta,\omega) \in \Omega \times X : \omega \in \varphi(\zeta)\}$$

belongs in $\Sigma \times \mathfrak{B}(X)$.

For $P_f(X)$-valued multifunctions, measurability implies graph measurability, while the converse is true if Σ is μ-complete. Given a multifunction $\varphi : \Omega \to 2^X\backslash\{\emptyset\}$ and $1 \le r \le \infty$, we define the set

$$S^r_\varphi = \{\omega \in L^r(\Omega, X) : \omega \in \varphi(\zeta)\mu\text{-a.e.}\}.$$

This set may be empty. An easy measurable selection argument, shows that for a graph measurable multifunction $\varphi : \Omega \to 2^X \setminus \{\emptyset\}$, the set S^r_φ is nonempty if and only if the function $\zeta \to \inf\{\|\omega\| : \omega \in \partial\varphi(u)\}$ belongs to $L^r(\Omega)_+$.

Definition 2.2. Let Y and Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z\backslash\{\emptyset\}$ is said to be upper semicontinuous (usc for short), if for all $C \subseteq Z$ closed, the set

$$G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$$

is closed.

If Z is regular, then an usc multifunction $G : Y \to 2^Z \setminus \{\emptyset\}$ with closed values, has a closed graph, i.e., Gr $G = \{(y,z) \in Y \times Z : z \in G(y)\}$ is a closed subset of $Y \times Z$. The converse is true if G is locally compact, i.e., for every $y \in Y$, we can find an open neighborhood U of y such that $\overline{G(U)}$ is compact.

Definition 2.3. A map $A : D \subseteq X \to 2^{X^*}$ is said to be monotone, if for all $u, v \in D$ and all $u^* \in A(u),\ v^* \in A(v)$ we have

$$\langle u^* - v^*, u - v\rangle \ge 0.$$

We say that A is strictly monotone if $\langle u^* - v^*, u - v\rangle = 0$ implies that $u = v$.

The map $A : D \subseteq X \to 2^{X^*}$ is called maximal monotone, if it is monotone and its graph is not properly contained in the graph of another monotone map. This is equivalent to saying that $\langle u^* - v^*, u - v\rangle \ge 0$ for all $u \in D$ and all $u^* \in A(u)$, then $v \in D$ and $v^* \in A(v)$.

A maximal monotone map $A : D \subseteq X \to 2^{X^*}$ has closed and convex values and its graph

$$\operatorname{Gr} A = \{(u, u^*) \in X \times X^* : u^* \in A(u)\}$$

is sequentially closed in $X \times X^*_\omega$ and in $X_w \times X^*$. Here by X^*_w (resp. X_w) we denote the space X^* (resp. X) furnished with the corresponding weak topology. If $A : X \to X^*$ is monotone, single valued, everywhere defined (i.e., $D = X$) and demicontinuous (i.e., $u_n \to u$ in X, implies $A(u_n) \rightharpoonup A(u)$ in X^*), then A is maximal monotone.

Definition 2.4. We say that $A : D \subseteq X \to 2^{X^*}$ is weakly coercive, if D is bounded or if D is unbounded and

$$\inf\{\|u^*\| : u^* \in A(u)\} \to \infty \quad \text{as } \|u\| \to \infty,\ u \in D.$$

If $A : D \subseteq X \to 2^{X^*}$ is maximal monotone and weakly coercive, then it is surjective.

Definition 2.5. If Y, Z are Banach spaces and $K : Y \to Z$, we say that K is completely continuous, if $y_n \rightharpoonup y$ in Y implies $K(y_n) \to K(y)$ in Z.

If Y is reflexive and $K : Y \to Z$ is completely continuous, then K is compact (namely, K is continuous and for every bounded set $B \subseteq Y$, one has that $\overline{K(B)}$ is compact).

We recall some results on variable exponent Lebesgue-Sobolev spaces and list some properties of that spaces. For more details the reader is referred to [13, 15, 28] and the references therein.

Let $p \in L^\infty(\Omega)$ and $p^- > 1$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \big\{u : \mathbb{R}^N \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{p(x)}\,\mathrm{d}x < \infty\big\}$$

endowed with the norm

$$\|u\|_{p(x)} = \inf\big\{\lambda > 0 : \int_{\mathbb{R}^N} \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,\mathrm{d}x \le 1\big\}.$$

Then, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| = \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)},$$

or equivalently

$$\|u\| = \|u\|_{1,p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(|\frac{u(x)}{\lambda}|^{p(x)} + |\frac{\nabla u(x)}{\lambda}|^{p(x)}\Big)\,\mathrm{d}x \le 1\Big\}$$

for all $u \in W^{1,p(x)}(\Omega)$. From [14, 15, 28] we obtain that $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

For $p \in L^\infty(\Omega)$ with $p^- > 1$, let $p'(x) : \Omega \to \mathbb{R}$ be such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, a.e. $x \in \Omega$. We have the generalized Hölder inequality

Proposition 2.6 ([15]). *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have*

$$\Big|\int_\Omega uv\,\mathrm{d}x\Big| \le 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

Proposition 2.7. *The function $\rho : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by*

$$\rho(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x,$$

has the following properties:

(i) *If $\|u\| \ge 1$, then $\|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$;*
(ii) *If $\|u\| \le 1$, then $\|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$.*

In particular, if $\|u\| = 1$ then $\rho(u) = 1$. Moreover, $\|u_n\| \to 0$ if and only if $\rho(u_n) \to 0$.

The following fixed point theorem is due to Bader [3].

Theorem 2.8. *If Y and Z are Banach spaces, $G : Y \to P_{wkc}(Z)$ is usc from Y into Z_w, $K : Z \to Y$ is completely continuous and $\Phi = K \circ G$ maps bounded sets into relatively compact sets, then one of the following statements hold:*

(i) *the set $\Lambda = \{y \in Y : y \in \mu\Phi(y) \text{ for some } \mu \in (0,1)\}$ is unbounded, or*
(ii) *Φ has a fixed point.*

The next fixed point theorem for multifunctions in ordered Banach spaces can be found in [25].

Theorem 2.9. *If X is a separable, reflexive ordered Banach space, $C \subset X$ is a nonempty and weakly closed set and $H : C \to 2^C \setminus \{\emptyset\}$ is a multifunction with weakly closed values, $H(C)$ is bounded and*

(i) *the set $K = \{x \in C : x \le y \text{ in } X \text{ for some } y \in H(x)\}$ is nonempty;*
(ii) *if $x_1 \le y_1$ in X, $y_1 \in H(x_1)$ and $x_1 \le x_2$ in X, then there exists $y_2 \in H(x_2)$ such that $y_1 \le y_2$,*

then H has a fixed point, i.e., there exists $x \in C$ such that $x \in H(x)$.

3. Existence of solutions

In this section we discuss the existence of weak solutions for (1.1). For simplicity we set $X = W_0^{1,p(x)}(\Omega)$.

Definition 3.1. We say that $u \in X$ is a weak solution of (1.1), if there exist $\omega(x) \in \partial F(x,u)$ and $\gamma(x) \in j(x,u(x),\nabla u(x))$ such that

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,\mathrm{d}x + \int_\Omega \omega v\,\mathrm{d}x + \int_\Omega \gamma v\,\mathrm{d}x = \int_\Omega g v\,\mathrm{d}x$$

for a.a. $x \in \Omega$ and all $v \in X$.

Let $A : X \to X^*$ be the nonlinear operator defined by

$$\langle A(u), v\rangle_X = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,\mathrm{d}x \quad \forall u, v \in X.$$

$$W_+ = W_0^{1,p(x)}(\Omega)_+ = \{u \in X : u(x) \ge 0 \text{ a.a. in } \Omega\}.$$

Definition 3.2. (i) A function $\bar\tau(x) \in W^{1,p(x)}$, $\bar\tau|_{\partial\Omega} > 0$ is an upper solution for problem (1.1), if there exist $\omega_+(x) \in S^{p'(\cdot)}_{\partial F(\cdot,\bar\tau(\cdot))}$ and $\gamma_+(x) \in S^{p'(\cdot)}_{j(\cdot,\bar\tau(\cdot),\nabla\bar\tau(\cdot))}$ such that

$$\langle A(\bar\tau), v\rangle_{W_0^{1,p(x)}(\Omega)} + \int_\Omega \omega_+ v\,\mathrm{d}x + \int_\Omega \gamma_+ v\,\mathrm{d}x \ge \int_\Omega g(\bar\tau) v\,\mathrm{d}x \quad \forall v \in W_+;$$

(ii) A function $\underline{\tau}(x) \in W^{1,p(x)}$, $\underline{\tau}|_{\partial\Omega} \le 0$ is an lower solution for problem (1.1), if there exist $\omega_-(x) \in S^{p'(\cdot)}_{\partial F(\cdot,\underline{\tau}(\cdot))}$ and $\gamma_-(x) \in S^{p'(\cdot)}_{j(\cdot,\underline{\tau}(\cdot),\nabla\underline{\tau}(\cdot))}$ such that

$$\langle A(\underline{\tau}), v\rangle_{W_0^{1,p(x)}(\Omega)} + \int_\Omega \omega_- v\,\mathrm{d}x + \int_\Omega \gamma_- v\,\mathrm{d}x \le \int_\Omega g(\underline{\tau}) v\,\mathrm{d}x \ \forall v \in W_+.$$

To discuss problem (1.1), we need the following hypotheses.

(H0) There exist an upper solution $\bar\tau(x)$ and a lower solution $\underline{\tau}(x)$ such that $\underline{\tau}(x) \le \bar\tau(x)$ for a.a. $x \in \Omega$.

(HM) $M(t) : [0,+\infty) \to (m_0,+\infty)$ is a continuous and increasing function with $m_0 > 0$.

(HF) $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function, such that
  (i) for all $\zeta \in \mathbb{R}$, the function $\Omega \ni x \to F(x,\zeta)$ is measurable;
  (ii) for a.a. $x \in \Omega$, $u \to F(x,u)$ is convex;
  (iii) for a.a. $x \in \Omega$, all $u \in \mathbb{R}$ and all $\omega(x) \in \partial F(x,u)$, we have

$$|\omega| \le a_0(x) + c_0|u|^{p(x)-1} \quad \text{with } a_0 \in L^{p'(x)}(\Omega)_+,\ c_0 > 0.$$

(HG1) $g:\mathbb{R}\to\mathbb{R}$ is a function which maps bounded sets to bounded sets and there exist $a>0$ and $1\le s<\infty$ such that $u\to g(u)+au^s$ is nondecreasing.

(HJ1) $j:\Omega\times\mathbb{R}\times\mathbb{R}^N\to P_{fc}(\mathbb{R})$ is a multifunction, such that
- (i) for all $\zeta,\xi\in\mathbb{R}$, $\Omega\ni x\to j(x,\zeta,\xi)$ is graph measurable;
- (ii) for a.a. $x\in\Omega$, $(\zeta,\xi)\to j(x,\zeta,\xi)$ has a closed graph;
- (iii) for a.a. $x\in\Omega$, all $\zeta\in[\underline{\tau},\bar{\tau}]$ and all $\xi\in\mathbb{R}^N$, we have
$$|j(x,\zeta,\xi)|\le a_1(x)+c_1(|\zeta|^{\nu-1}+\|\xi\|_{\mathbb{R}^N}^{\nu-1})$$
with $a_0\in L^{p'(x)}(\Omega)_+$, $c_1>0$, $1\le\nu\le p(x)$;
- (iv) for a.a. $x\in\Omega$, all $\delta>0$, all $|\zeta|,|\xi|\le\delta$, we can find $a_\delta\in L^{p'(x)}(\Omega)$ such that $|j(x,\zeta,\xi)|\le a_\delta(x)$.

Remark 3.3. (i) In [9], except hypothesis (HM), Dai and Hao also assumed that there exists $0<\mu<1$ such that $\hat{M}(t)\ge(1-\mu)M(t)t$, where $\hat{M}(t)=\int_0^t M(s)\,\mathrm{d}s$. In [12], the authors assumed that $k_0\le M(t)\le k_1$, where $k_1>k_0$ are positive constants. While, in our paper, we do not need these hypotheses at all. This means that the choice of $M(t)$ in our paper is more extensive than in [9, 12].

(ii) It is worth to point out that g need not to be continuous in hypothesis (HG1).

From [9], we have the following property.

Proposition 3.4. *If hypothesis* (HM) *holds, then A is an operator of type* $(S)_+$ *and a maximonotone operator.*

Set $\hat{A}$ be the restriction of A in $L^{p'(x)}(\Omega)$, i.e., $\hat{A}:L^{p(x)}(\Omega)\supseteq D(A)\to L^{p'(x)}(\Omega)$ is defined by
$$\hat{A}(u)=A(u)\quad\forall u\in D(A)$$
with
$$D(A)=\{u\in W_0^{1,p(x)}(\Omega):A(u)\in L^{p'(x)}(\Omega)\}$$
(recall that $L^{p'(x)}(\Omega)\subseteq W^{-1,p'(x)}(\Omega)$). It is obvious that $\hat{A}$ is a maximal monotone operator.

As is known, the method of upper and lower solutions involves truncations and penalization techniques, which aim at exploiting the fact that we control the date of problem (1.1) in the interval $[\underline{\tau}(x),\bar{\tau}(x)]$. So we define the following functions.

First, the truncation map $\psi:W^{1,p(x)}(\Omega)\to W^{1,p(x)}(\Omega)$ is defined by
$$\psi(u)(x)=\begin{cases}\bar{\tau}(x) & \text{if } \bar{\tau}(x)<u(x),\\ u(x) & \text{if } \underline{\tau}(x)\le u(x)\le\bar{\tau}(x),\\ \underline{\tau}(x) & \text{if } u(x)<\underline{\tau}(x).\end{cases}$$
It is obvious that ψ is continuous.

Second, we introduce a penalty function $\beta:\Omega\times\mathbb{R}\to\mathbb{R}$ defined by
$$\beta(x,u)=\begin{cases}\frac12\min\{m_0,1\}(|u|^{p(x)-2}u-|\bar{\tau}(x)|^{p(x)-2}\bar{\tau}(x)) & \text{if } \bar{\tau}(x)<u(x),\\ 0 & \text{if } \underline{\tau}(x)\le u(x)\le\bar{\tau}(x),\\ \frac12\min\{m_0,1\}(|u|^{p(x)-2}u-|\underline{\tau}(x)|^{p(x)-2}\underline{\tau}(x)) & \text{if } u(x)<\underline{\tau}(x).\end{cases}$$
Evidently $\beta(x,u)$ is a Carathéodory function (i.e., measurable in $x\in\Omega$, continuous in $u\in\mathbb{R}$).

Third, we define a penalty multifunction $V : \Omega \times \mathbb{R} \to P_{fc}(\mathbb{R})$ defined by

$$V(x,u) = \begin{cases} [\omega_+(x), +\infty) & \text{if } \bar{\tau}(x) < u(x), \\ \mathbb{R} & \text{if } \underline{\tau}(x) \le u(x) \le \bar{\tau}(x), \\ (-\infty, \omega_-(x)] & \text{if } u(x) < \underline{\tau}(x). \end{cases}$$

Let

$$E(x,u(x)) = \partial F(x, \psi(u)(x)) \cap V(x,u),$$
$$J(x,u(x),\nabla u(x)) = j(x, \psi(u)(x), \nabla\psi(u)(x)) \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Let $\hat{\beta} : L^{p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be defined by

$$\hat{\beta}(u)(\cdot) = \beta(\cdot, u(\cdot)) \quad \text{for all } u \in L^{p(x)}(\Omega),$$

i.e., $\hat{\beta}$ is the Nemitsky operator corresponding to the function β. Note that β is a Caratheodory function and satisfies

$$|\beta(x,u)| \le \hat{a} + |u|^{p(x)-1} \quad \text{for a.a. } x \in \Omega \text{ and all } u \in \mathbb{R}$$

with $\hat{a} > 0$. So from Krasoselskii's theorem (see Gasiński and Papageorgiou [20]). We obtain that $\hat{\beta}$ is continuous and bounded.

Now set $J_1 : L^{p(x)}(\Omega) \to P_{wkc}(L^{p'(x)}(\Omega))$ be defined by $J_1(u) = S^{p'(\cdot)}_{J(\cdot,u(\cdot),\nabla u(\cdot))} + \hat{\beta}(u)$.

Proposition 3.5. *If* (HJ1) *holds, then* $J_1(u) : L^{p(x)}(\Omega) \to P_{fc}(L^{p'(x)}(\Omega))$ *is usc from* $W_0^{1,p(x)}(\Omega)$ *into* $L^{p'(x)}(\Omega)$.

Proof. Let $C \subset L^{p'(x)}(\Omega)$ be nonempty and weakly closed. We need to show that $J_1^{-1}(C) = \{u \in W_0^{1,p(x)}(\Omega) : J_1(u) \cap C \neq \emptyset\}$ is closed. So let $\{u_n\}_{n\ge1} \subset J_1^{-1}(C)$ and assume that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Set $\eta_n \in J_1(u_n) \cap C,\ n \ge 1$. Since

$$|\eta_n(x)| \le a_\delta(x) + c|u_n(x)|^{p(x)-1} + \max[\|\bar{\tau}\|_\infty^{p^--1}, \|\bar{\tau}\|_\infty^{p^+-1}, \|\underline{\tau}\|_\infty^{p^--1}, \|\underline{\tau}\|_\infty^{p^+-1}],$$

by passing to a subsequence if necessary, we may assume that $\eta_n \rightharpoonup \eta$ in $L^{p'(x)}(\Omega)$. Also, since $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ and the continuity of ψ, we have $u_n \to u$ in $C(\Omega)$, $\nabla u_n \to \nabla u$ in $L^{p(x)}(\Omega)$, $\psi(u_n) \to \psi(u)$ in $L^{p(x)}(\Omega)$ and so by passing to a subsequence if necessary we obtain that $\nabla u_n \to \nabla u$ and $\nabla\psi(u_n) \to \nabla\psi(u)$ for a.a. $x \in \Omega$. Using [19, Proposition 1.2.12], we have

$$\eta(u) \in \operatorname{conv} \limsup_{n\to+\infty} J_1(u_n(x)) \subseteq J_1(u(x))$$

for a.a. $x \in \Omega$, where the last inclusion is a consequence of the hypothesis (HJ1) (ii) and the definition of $\hat{\beta}(x,u)$. So we infer that $\eta \in J_1(u)$. Also $\eta \in C$ since the later is weakly closed. Thus $u \in J_1^-(C)$ which proves that $J_1^-(C)$ is closed and so J_1 is usc from $L^{p(x)}(\Omega)$ into $L^{p'(x)}(\Omega)_w$. $\square$

We also consider the integral function $\mathscr{F} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\mathscr{F}(u) = \int_\Omega F(x,u(x))\,\mathrm{d}x \quad \text{for all } u \in L^{p(x)}(\Omega).$$

By hypothesis (HF), $\mathscr{F}$ is continuous and convex, hence locally Lipschitz. So if $\hat{\mathscr{F}} = \mathscr{F}|_{W_0^{1,p(x)}(\Omega)}$, then, from Clarke [8, pp. 47, 36, 76] we obtain

$$\partial\tilde{\mathscr{F}}(u) = \partial\mathscr{F}(u) = S^{p'(\cdot)}_{\partial F(\cdot,u(\cdot))} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{3.1}$$

Let $H : L^{p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be defined by

$$H(u)(\cdot) = |u(\cdot)|^{p(\cdot)-2}u(\cdot) \quad \text{for all } u \in L^{p(x)}(\Omega).$$

Evidently, H is strictly monotone continuous (hence maximal monotone too) and bounded (i.e., maps bounded sets to bounded sets).

From hypothesis (HF), we obtain that $\hat{A} + H + \partial\mathscr{F} : L^{p(x)}(\Omega) \supseteq D(A) \to L^{p'(x)}(\Omega)$ is strictly monotone and surjective. So the map $L = (\hat{A} + H + \partial\mathscr{F})^{-1} : L^{p'(x)}(\Omega) \to D(A) \subseteq W_0^{1,p(x)}(\Omega)$ is well defined.

Proposition 3.6. *If hypotheses* (HM) *and* (HF) *hold, then* $L = (\hat{A}+H+\partial\mathscr{F})^{-1} : L^{p'(x)}(\Omega) \to D(A) \subseteq W_0^{1,p(x)}(\Omega)$ *is completely continuous.*

Proof. Assume that $h_n \rightharpoonup h$ in $L^{p'(x)}(\Omega)$ and let

$$u_n = L(h_n),\ n \geq 1,\ u = L(h).$$

We need to show that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$. We have $u_n \in D(A)$ for $n \geq 1$, and

$$\hat{A}(u_n) + H(u_n) + \omega_n = h_n$$

with $\omega_n \in \partial\mathscr{F}(u_n)$, $n \geq 1$. Thus

$$\langle \hat{A}(u_n), u_n\rangle_{L^{p(x)}(\Omega)} + \langle H(u_n), u_n\rangle_{L^{p(x)}(\Omega)} + \langle \omega_n, u_n\rangle_{L^{p(x)}(\Omega)} = \langle h_n, u_n\rangle_{L^{p(x)}(\Omega)}. \tag{3.2}$$

For any $\omega_0 \in \partial\mathscr{F}(0)$, we obtain

$$\langle \omega_n, u_n\rangle_{L^{p(x)}(\Omega)} = \langle \omega_n - \omega_0, u_n\rangle_{L^{p(x)}(\Omega)} + \langle \omega_0, u_n\rangle_{L^{p(x)}(\Omega)} \geq -c\|u_n\|, \tag{3.3}$$

where $\omega_n \in \partial\mathscr{F}(u_n)$, $n \geq 1$ (recall that $\partial\mathscr{F}(\cdot)$ is monotone and $\partial\mathscr{F}(0)$ is bounded). Also from Holder's inequality and the continuous compact embedding of $W_0^{1,p(x)}(\Omega)$ in $L^{p(x)}(\Omega)$, we have

$$\langle h_n, u_n\rangle_{L^{p(x)}(\Omega)} \leq c\|u_n\| \tag{3.4}$$

for all $n \geq 1$. Using (3.3), (3.4), (HM) and returning to (3.2), we have

$$\begin{aligned}
&\min\{m_0, 1\}\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,\mathrm{d}x \\
&\leq m_0\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x + \int_\Omega |u_n|^{p(x)}\,\mathrm{d}x \\
&\leq M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\Big)\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x + \int_\Omega |u_n|^{p(x)}\,\mathrm{d}x \leq c\|u_n\|
\end{aligned}$$

for all $n \geq 1$. Note that $p^+ \geq p^- > 1$, i.e.,

$$\min\{m_0, 1\}\{\|u_n\|^{p^-}, \|u_n\|^{p^+}\} \leq c\|u_n\|.$$

Noting $1 < p^- \leq p^+$, one has that the sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(x)}(\Omega)$ is bounded. By passing to a subsequence if necessary, we may assume that

$$u_n \rightharpoonup \hat{u} \quad \text{in} W_0^{1,p(x)}(\Omega) \quad \text{and} \quad u_n \to \hat{u} \quad \text{in } L^{p(x)}(\Omega)$$

as $n \to \infty$. We have $H(u_n) \to H(\hat{u})$ in $L^{p(x)}(\Omega)$ and $\langle u_n, h_n - H(u_n)\rangle \in \mathrm{Gr}(\hat{A}+\partial\mathscr{F}$ for all $n \geq 1$. Since $u_n \to \hat{u}$ in $L^{p(x)}(\Omega)$, $h_n - H(u_n) \rightharpoonup h - H(\hat{u})$ in $L^{p'(x)}(\Omega)$, and $\hat{A} + \partial\mathscr{F} : D(A) \subseteq L^{p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ is maximal monotone, we have

$$\langle \hat{u}, h - H(\hat{u}) \in \mathrm{Gr}(\hat{A} + \partial\mathscr{F}),$$

hence $\hat{u} = L(h) = u$. Also, for all $n \geq 1$, we have

$$\begin{aligned} &\langle \hat{A}(u_n), u_n - u\rangle_{L^{p(x)}(\Omega)} + \langle H(u_n), u_n - u\rangle_{L^{p(x)}(\Omega)} + \langle \omega_n, u_n - u\rangle_{L^{p(x)}(\Omega)} \\ &= \langle h_n, u_n - u\rangle_{L^{p(x)}(\Omega)}. \end{aligned} \tag{3.5}$$

So

$$\begin{aligned} &\langle A(u_n), u_n - u\rangle_{W_0^{1,p(x)}(\Omega)} + \langle H(u_n), u_n - u\rangle_{L^{p(x)}(\Omega)} + \langle \omega_n, u_n - u\rangle_{L^{p(x)}(\Omega)} \\ &= \langle h_n, u_n - u\rangle_{L^{p(x)}(\Omega)}. \end{aligned} \tag{3.6}$$

By hypothesis (HF)(iii), we derive that $\{\omega_n\}_{n\geq 1} \subseteq L^{p'(x)}(\Omega)$ is bounded. Furthermore, the sequences $\{H(u_n)\}_{n\geq 1}$, and $\{h_n\}_{n\geq 1} \subseteq L^{p'(x)}(\Omega)$ are bounded and $u_n \to u$ in $L^{p(x)}(\Omega)$, we obtain that

$$\langle \omega_n, u_n - u\rangle_{L^{p(x)}(\Omega)} \to 0, \quad \langle H(u_n), u_n - u\rangle_{L^{p(x)}(\Omega)} \to 0, \quad \langle h_n, u_n - u\rangle_{L^{p(x)}(\Omega)} \to 0,$$

and so

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle_{W_0^{1,p(x)}(\Omega)} = 0.$$

From Proposition 3.4, we already know that A is of type $(S)_+$. So it follows that

$$u_n \to u \quad \text{in } W_0^{1,p(x)}(\Omega),$$

which proves the continuity of the operator L. $\square$

Next, we introduce the order interval

$$T = [\underline{\tau}, \bar{\tau}] = \{u \in W^{1,p(x)}(\Omega) : \underline{\tau}(x) \leq u(x) \leq \bar{\tau}(x) \text{ for a.a. } x \in \Omega\}.$$

Fix $\theta(x) \in T$. We consider the auxiliary problem

$$\begin{gathered} \begin{aligned} &- M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x\Big) \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \\ &+ E(x, u(x)) + j(x, \psi(u)(x), \nabla\psi(u)(x)) + \beta(x, u(x)) \\ &\ni -a\psi(u)^s(x) + g(\theta(x)) + a\theta^s(x), \end{aligned} \\ u\big|_{\partial\Omega} = 0. \end{gathered} \tag{3.7}$$

Let $N_J(u) = S^{p'(\cdot)}_{J(\cdot,u(\cdot),\nabla u(\cdot))} + \hat{\beta}(u)$.

Proposition 3.7. *If hypotheses* (H0), (HM), (HF), (HJ1), (HG1) *hold, then problem* (3.7) *has a solution* $u_0 \in W_0^{1,p(x)}(\Omega)$.

Proof. Let $S : W_0^{1,p(x)}(\Omega) \to P_{\omega kc}(L^{p'(x)}(\Omega))$ be the multifunction defined by

$$S(x) = -N_J(u) + H(\psi(u)) - a\psi^s(u) + \hat{g}(\theta) + a\theta^s,$$

where $\hat{g}(\theta)(\cdot) = g(\theta)(\cdot) \in L^\infty(\Omega) \subseteq L^{p'(x)}(\Omega)$ (see hypothesis (HG1)). From Proposition 3.5, and noting that ψ is continuous, we can easily obtain that S is usc from $W_0^{1,p(x)}(\Omega)$ into $L^{p'(x)}(\Omega)_\omega$. So we only need to show that the set

$$\Lambda = \{u \in W_0^{1,p(x)}(\Omega) : u \in \mu(L \circ S)(u), \mu \in (0,1)\}$$

is bounded uniformly in $\mu \in (0,1)$. For convenience we assume that $u \in E$, $\|u\| \geq 1$. Then we have $\frac{1}{\mu}u \in (L \circ S)(u)$, hence

$$L^{-1}\big(\frac{1}{\mu}u\big) = S(u),$$

i.e.,

$$\big\langle A\big(\frac{1}{\mu}u\big), u\big\rangle_{W_0^{1,p(x)}(\Omega)} + \big\langle H\big(\frac{1}{\mu}u\big), u\big\rangle_{L^{p(x)}(\Omega)} + \langle \omega, u\rangle_{L^{p(x)}(\Omega)} = \langle h, u\rangle_{L^{p(x)}(\Omega)} \tag{3.8}$$

for some $\omega \in \partial\mathscr{F}(\frac{1}{\mu}u)$ and $h \in S(u)$. Note that

$$\begin{aligned} \big\langle A\big(\frac{1}{\mu}u\big), u\big\rangle_{W_0^{1,p(x)}(\Omega)} &= M\Big(\int_\Omega \frac{1}{\mu^{p(x)}p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\Big)\int_\Omega \frac{1}{\mu^{p(x)-1}}|\nabla u|^{p(x)}\,\mathrm{d}x \\ &\geq \frac{m_0}{\mu^{p^- -1}}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \\ &\geq m_0\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x \quad (\text{since } 0<\mu<1), \end{aligned} \tag{3.9}$$

and

$$\big\langle H\big(\frac{1}{\mu}u\big), u\big\rangle_{L^{p(x)}(\Omega)} = \int_\Omega \frac{1}{\mu^{p(x)-1}}|u|^{p(x)}\,\mathrm{d}x \geq \int_\Omega |u|^{p(x)}\,\mathrm{d}x. \tag{3.10}$$

For any $\omega_0 \in \partial\mathscr{F}(0)$, we have

$$\begin{aligned} \langle \omega, u\rangle_{L^{p(x)}(\Omega)} &= \langle \omega - \omega_0, u\rangle_{L^{p(x)}(\Omega)} + \langle \omega_0, u\rangle_{L^{p(x)}(\Omega)} \\ &\geq \langle \omega_0, u\rangle_{L^{p(x)}(\Omega)} \geq -c\|u\|. \end{aligned} \tag{3.11}$$

Since $\omega \in \partial\mathscr{F}(\frac{1}{\mu}u)$, $\mu > 0$ and $\partial\mathscr{F}(0) \subseteq L^{p'(x)}(\Omega)$ is bounded. Furthermore, from the definition of N_J, β and ψ, we derive

$$\begin{aligned} \langle h, u\rangle_{L^{p(x)}(\Omega)} &\leq c\|u\| + \frac{\min\{m_0,1\}}{2}\int_\Omega |u|^{p(x)}\,\mathrm{d}x \\ &\leq c\|u\| + \frac{\min\{m_0,1\}}{2}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x. \end{aligned} \tag{3.12}$$

Returning to (3.8) and using (3.9)–(3.12), we have

$$\min\{m_0,1\}\int_\Omega (|\nabla u|^{p(x)}+|u|^{p(x)})\,\mathrm{d}x \leq c\|u\| + \frac{1}{2}\min\{m_0,1\}\int_\Omega (|\nabla u|^{p(x)}+|u|^{p(x)})\,\mathrm{d}x,$$

i.e.,

$$\frac{1}{2}\min\{m_0,1\}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x \leq c\|u\|.$$

Hence

$$\frac{1}{2}\min\{m_0,1\}\|u\|^{p^-} \leq c\|u\|.$$

Since $p^- > 1$ and $m_0 > 0$ we have

$$\|u\| \leq c \quad \text{for all } u \in \Lambda.$$

This implies that the set Λ is bounded. Note that S is usc from $W_0^{1,p(x)}(\Omega)$ and $L^{p'(x)}(\Omega)_w$, $L : L^{p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is completely continuous and $L \circ S$ maps bounded sets into relatively compact sets, we can employ Theorem 2.8 to obtain $u \in W_0^{1,p(x)}(\Omega)$ such that $u \in (L \circ S)(u)$. Then, we have

$$\begin{aligned} &A(u) + \omega(x) + \gamma(x) - |\psi(u)(x)|^{p(x)-2}\psi(u)(x) + |u(x)|^{p(x)-2}u(x) + \beta(x, u(x)) \\ &= -a\psi^s(u) + g(\theta(x)) + a\theta^s(x), \end{aligned} \tag{3.13}$$

where $\omega(x) \in S^{p'(x)}_{\partial F(x,u)}$ and $\gamma(x) \in S^{p'(x)}_{j(x,\psi(x),\nabla\psi(x))}$. Note that $\underline{\tau} \in W^{1,p(x)}(\Omega)$ is a lower solution of problem (1.1). Using a test function $(\underline{\tau} - u)^+ \in W_0^{1,p(x)}(\Omega)$ (recall that $\underline{\tau}\big|_{\partial\Omega} \le 0$), we have

$$\begin{aligned} &\int_\Omega \langle A(\underline{\tau}), (\underline{\tau} - u)^+(x)\rangle_{W_0^{1,p(x)}(\Omega)}\,\mathrm{d}x + \int_\Omega \omega_-(\underline{\tau} - u)^+\,\mathrm{d}x + \int_\Omega \gamma_-(\underline{\tau} - u)^+\,\mathrm{d}x \\ &\le \int_\Omega g(\underline{\tau})(\underline{\tau} - u)^+\,\mathrm{d}x. \end{aligned} \tag{3.14}$$

Analogously, acting (3.13) with $(\underline{\tau} - u)^+ \in W_0^{1,p(x)}(\Omega)$, we also have

$$\begin{aligned} &\int_\Omega \langle A(u), (\underline{\tau} - u)^+(x)\rangle_{W_0^{1,p(x)}(\Omega)}\,\mathrm{d}x + \int_\Omega \omega(\underline{\tau} - u)^+\,\mathrm{d}x + \int_\Omega \gamma(\underline{\tau} - u)^+\,\mathrm{d}x \\ &- \int_\Omega (|\psi(u)(x)|^{p(x)-2}\psi(u)(x) - |u(x)|^{p(x)-2}u(x))(\underline{\tau} - u)^+(x)\,\mathrm{d}x \\ &+ \int_\Omega \beta(x, u(x))(\underline{\tau} - u)^+(x)\,\mathrm{d}x \\ &= \int_\Omega (-a\psi^s(u) + g(\theta(x)) + a\theta^s(x))(\underline{\tau} - u)^+(x)\,\mathrm{d}x. \end{aligned} \tag{3.15}$$

Subtracting (3.15) from (3.14) and noting the definitions of $\psi(u)$ and $\beta(x, u(x))$, we derive

$$\begin{aligned} &\int_{\underline{\tau}\ge u} \langle A(\underline{\tau}) - A(u), (\underline{\tau} - u)(x)\rangle_{W_0^{1,p(x)}(\Omega)}\,\mathrm{d}x + \int_{\underline{\tau}\ge u} (\omega_- - \omega)(\underline{\tau} - u)(x)\,\mathrm{d}x \\ &+ \int_{\underline{\tau}\ge u} (\gamma_- - \gamma)(\underline{\tau} - u)(x)\,\mathrm{d}x \\ &+ \int_{\underline{\tau}\ge u} (|\underline{\tau}(x)|^{p(x)-2}\underline{\tau}(x) - |u(x)|^{p(x)-2}u(x))(\underline{\tau} - u)\,\mathrm{d}x \\ &- \frac{1}{2}\min\{m_0, 1\} \int_{\underline{\tau}\ge u} (|u(x)|^{p(x)-2}u(x) - |\underline{\tau}(x)|^{p(x)-2}\underline{\tau}(x))(\underline{\tau} - u)\,\mathrm{d}x \\ &\le \int_{\underline{\tau}\ge u} (g(\underline{\tau}) + a\underline{\tau}^s - g(\theta(x)) - a\theta^s(x))(\underline{\tau} - u)(x)\,\mathrm{d}x. \end{aligned}$$

From Proposition 3.4 and the monotonicity of $\partial F(\cdot)$, one has

$$\int_{\underline{\tau}\ge u} \langle A(\underline{\tau}) - A(u), (\underline{\tau} - u)(x)\rangle_{W_0^{1,p(x)}(\Omega)}\,\mathrm{d}x \ge 0,$$
$$\int_{\underline{\tau}\ge u} (\omega_- - \omega)(\underline{\tau} - u)(x)\,\mathrm{d}x \ge 0.$$

Recalling the definition of J, we have

$$\int_{\underline{\tau}\geq u}(\gamma_{-}-\gamma)(\underline{\tau}-u)(x)\,\mathrm{d}x\geq 0.$$

Noting that $\underline{\tau}\leq\theta$ and

$$\int_{\underline{\tau}\geq u}(|\tau(x)|^{p(x)-2}\tau(x)-|u(x)|^{p(x)-2}u(x))(\underline{\tau}-u)\,\mathrm{d}x\geq 0,$$

from hypothesis (HG1), we have

$$-\frac{1}{2}\min\{m_0,1\}\int_{\underline{\tau}\geq u}(|u(x)|^{p(x)-2}u(x)-|\underline{\tau}(x)|^{p(x)-2}\underline{\tau}(x))(\underline{\tau}-u)\,\mathrm{d}x\leq 0,$$

a contradiction unless $\underline{\tau}(x)\leq u(x)$ for a.a. $x\in\Omega$. In a similar way, we can prove that

$$u(x)\leq\bar{\tau}(x)\quad\text{for a.a. } x\in\Omega.$$

So we deduce that the auxiliary problem (3.7) has a solution $u_0\in W_0^{1,p(x)}(\Omega)$ where $\underline{\tau}(x)\leq u_0\leq\bar{\tau}(x)$. $\square$

Using Proposition 3.7 and Theorem 2.9 we obtain a solution of problem (1.1) in the order interval $T=[\underline{\tau}(x),\bar{\tau}(x)]$.

Theorem 3.8. *If hypotheses* (H0), (HM), (HJ1), (HF), (HG1) *hold, then problem* (1.1) *has at least one nontrivial solution* $u\in W_0^{1,p(x)}(\Omega)\cap T$.

Proof. It is obvious that T is weakly closed in $W_0^{1,p(x)}(\Omega)$. We consider the multifunction $I:T\to 2^{W_0^{1,p(x)}(\Omega)}\setminus\{\emptyset\}$ which to all $\theta\in T$ assigns the set of solution to the auxiliary problem (3.7). From Proposition 3.7 we have

$$I(\theta)\neq\emptyset\quad\forall\theta\in T.$$

Moreover, it is clear from Proposition 3.7 that for all $\theta\in T$, the set $I(\theta)\subset W_0^{1,p(x)}(\Omega)$ is weakly closed and $I(T)\subset W_0^{1,p(x)}(\Omega)$ is bounded. So it remains to verify statements (i) and (ii) in Theorem 2.9.

If $\theta=\underline{\tau}$, then from Proposition 3.7, $I(\underline{\tau})\neq\emptyset$ and $I(\underline{\tau})\subseteq T$. So if $u\in I(\underline{\tau})$, we have $\underline{\tau}\leq u$ and we have verified statement (i) of Theorem 2.9.

Next, we verify statement (ii) of Theorem 2.9. If $\theta_1\in T$, $\theta_1\leq u_1$, $u_1\in I(\theta_1)$ and $\theta_1\leq\theta_2$, then we can find $u_2\in I(\theta_2)$ such that $u_1\leq u_2$ (In $W_0^{1,p(x)}(\Omega)$, we consider the partial order induced by the positive cone $L^{p(x)}(\Omega)_+$, i.e., the pointwise partial order).

Since $u_1\in I(\theta_1)\subseteq T$, we have

$$\beta(x,u_1(x))=0,\quad\psi(u_1)=u_1,\quad\nabla\psi(u_1)=\nabla u_1,\quad V(x,u_1(x))=\mathbb{R}.$$

So we can write that

$$A(u_1)+\omega_1+\gamma_1=-au_1^s+g(\theta_1)+a\theta_1^s,$$

where $\omega_1\in\partial F(x,u_1)$, $\gamma_1\in j(x,u_1,\nabla u_1)$. Noting that $\theta_1\leq\theta_2$, by hypothesis (HG1), we obtain

$$g(\theta_1(x))+a\theta_1^s\leq g(\theta_2(x))+a\theta_2^s$$

for a.a. $x\in\Omega$. So for all $v\in W_+$ we derive

$$\langle A(u_1),v\rangle_{W_0^{1,p(x)}(\Omega)}+\int_\Omega\omega_1 v\,\mathrm{d}x+\int_\Omega\gamma_1 v\,\mathrm{d}x\leq -a\int_\Omega u_1^s v\,\mathrm{d}x+\int_\Omega(g(\theta_2(x))+a\theta_2^s)v\,\mathrm{d}x,$$

from which we infer that $u_1 \in W_0^{1,p(x)}(\Omega)$ is a lower solution for problem

$$\begin{gathered} - M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x\Big) \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \partial F(x,u(x)) + j(x,u,\nabla u) \\ \ni -au^s(x) + g(\theta_2(x)) + a\theta_2^s(x), \quad \text{for a.a. } x\in\Omega, \\ u\big|_{\partial\Omega} = 0. \end{gathered} \tag{3.16}$$

Recall that $\bar\tau \in W^{1,p(x)}(\Omega)$ is an upper solution of problem (3.16) too. Arguing as for for the auxiliary problem (3.7) via truncation and penalization techniques, we have a solution $u_2 \in W_0^{1,p(x)}(\Omega)$ of problem (3.16) such that

$$u_1(x) \le u_2(x) \le \bar\tau(x) \quad \text{for a.a. } x\in\Omega.$$

Therefore $u_2 \in I(\theta_2)$ and $u_1 \le u_2$. This verifies statement (ii) of Theorem 2.9. So we can obtain $u \in W_0^{1,p(x)}(\Omega)\cap T$ such that $u\in I(u)$. Evidently this is a solution of problem (1.1). □

4. Extremal solutions

In this section we produce a greatest and a smallest solution of (1.1) in the order interval $T = [\underline{\tau}, \bar\tau]$. These solutions are called extremal solutions of (1.1) in T.

To produce the extremal solutions of (1.1) in the order interval $T = [\underline{\tau}, \bar\tau]$, we need to strengthen hypotheses (HG1) and (HJ1).

- (HG2) $g:\mathbb{R}\to\mathbb{R}$ is a function which maps bounded sets to bounded sets, it is upper semicontinuous and there exist $a>0$ and $1\le s<\infty$ such that $u\to g(u)+au^s$ is nondecreasing;
- (HJ2) $j:\Omega\times\mathbb{R}\to\mathbb{R}$ is a function, such that
  - (i) for all $\zeta\in\mathbb{R}$, $\Omega\ni x\to j(x,\zeta)$ is measurable;
  - (ii) for a.a. $x\in\Omega$, $\zeta\to j(x,\zeta)$ is continuous and nonincreasing;
  - (iii) $j(\cdot,\underline{\tau}(\cdot)), j(\cdot,\bar\tau(\cdot)) \in L^{p'(x)}(\Omega)$.

Note that a subset Λ of a partially ordered space is a chain (or a totally order set), if for every $u,v\in\Lambda$, either $u\le v$ or $v\le u$. Now set

$$T_1 = \{u\in W_0^{1,p(x)}(\Omega) : u \text{ is a solution of (1.1) and } u\in T\}.$$

Proposition 4.1. *If hypotheses* (H0), (HM), (HF), (HG2), (HJ2) *are satisfied, and* $K\subseteq T_1$ *is a chain, then* K *has an upper bound.*

Proof. Note that $\Lambda\subset L^{p(x)}(\Omega)$ is bounded and $L^{p(x)}(\Omega)$ is a complete lattice. So we can define $u=\sup T$ in $L^{p(x)}(\Omega)$. In fact we can find an increasing sequence $\{u_n\}_{n\ge1}$ such that $u=\sup_{n\ge1}u_n$, hence $u_n\to u$ in $L^{p(x)}(\Omega)$ (monotone convergence theorem). By definition we have

$$-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\Big)\operatorname{div}(|\nabla u_n|^{p(x)-2}\nabla u_n) + \omega_n(x) + j(x,u_n) = g(u_n(x)),$$

where $\omega_n(x)\in\partial F(x,u_n(x))$ for a.a. $x\in\Omega$ and all $n\ge1$. By hypotheses (HJ2) (ii) and (iii) we derive

$$|j(x,u_n(x))| \le \max\{j(x,\underline{\tau}(x)), -j(x,\bar\tau(x))\}$$

for a.a. $x\in\Omega$ and all $n\ge1$. From hypotheses (HF)(iii), (HM), (HG2), the above inequality, and noting the fact $|u_n(x)|\le\max\{-\underline{\tau}(x),\bar\tau(x)\}$, we deduce that the sequence $\{\nabla u_n\}_{n\geq 1} \subset L^{p(x)}(\Omega;\mathbb{R}^N)$ is bounded. Therefore

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega).$$

Recall that $A : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is the nonlinear operator defined by

$$\langle A(u), v\rangle_{W_0^{1,p(x)}(\Omega)} = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,\mathrm{d}x$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. We know that A is a maximal monotone and bounded operator. We derive

$$A(u_n) + \omega_n + N_j(u_n) = g(u_n)\ \forall n \geq 1$$

(note that $N_j(y)(\cdot) = j(\cdot, y(\cdot))$), where $\omega_n \in \partial F(x, u_n)$. By passing to a subsequence if necessary, we may assume that

$$\omega_n \rightharpoonup \omega \quad \text{in} \quad L^{p'(x)}(\Omega)$$

and since $\operatorname{Gr}\partial F(x,\cdot)$ is closed for a.a. $x \in \Omega$, we obtain that $\omega(x) \in S^{p'(\cdot)}_{\partial F(\cdot,u(\cdot))}$. Also

$$N_j(u_n) \to N_j(u) \quad \text{in } L^{p'(x)}(\Omega)$$

(see hypotheses (HJ2)(ii) and (iii)) and as in previous proofs, we have

$$A(u_n) \rightharpoonup A(u) \quad \text{in } W^{-1,p'(x)}(\Omega).$$

Because of hypothesis (HG2) and $\{u_n(x)\}_{n\geq 1}$ is increasing for a.a. $x \in \Omega$, we have

$$g(u(x)) + au^s(x) \geq g(u_n(x)) + au_n^s(x)$$

for a.a. $x \in \Omega$ and all $n \geq 1$, hence

$$g(u(x)) + au^s(x) \geq \limsup_{n\geq 1} g(u_n(x)) + au^s(x) \geq g(u(x)) + au^s(x)$$

for a.a. $x \in \Omega$, therefore

$$g(u_n(x)) \to g(u(x)) \quad \text{as } n \to \infty \text{ for a.a. } x \in \Omega.$$

Thus as $n \to +\infty$, we have

$$A(u) + \omega + N_j(u) = g(u),$$

where $\omega \in \partial F(x, u)$. Then

$$\begin{gathered}
- M\big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \partial F(x, u(x)) + j(x, u) \\
\ni g(u), \quad \text{for a.a. } x \in \Omega, \\
u\big|_{\partial\Omega} = 0.
\end{gathered}$$

So we conclude that $u \in W^{1,p(x)}(\Omega)$ is a solution of problem (1.1) and $u \in T$. Clearly $u \in T_1$ is an upper bound of K. $\square$

Recall that if $(Y, \leq)$ is a partially ordered set, we say that Y is directed, if for each $y_1, y_2 \in Y$, there exists $y_3 \in Y$ such that $y_1 \leq y_3$ and $y_2 \leq y_3$.

Proposition 4.2. *If hypotheses* (H0), (HM), (HF), (HJ2), (HG2) *are satisfied, then the partially ordered set T_1 is directed.*

Proof. Set $u_1,\ u_2 \in T_1$ and $u_3 = \max\{u_1, u_2\}$. We have $u_3 = (u_1 - u_2)^+ + u_2$ and so it follows that $u_3 \in W_0^{1,p(x)}(\Omega)$. We introduce the following truncation and penalty function and multifunctions:

$$\hat{\psi}(u)(x) = \begin{cases} \bar{\tau}(x), & \text{if } \bar{\tau}(x) < u(x), \\ u(x), & \text{if } \underline{u}_3(x) \le u(x) \le \bar{\tau}(x), \\ u_3(x), & \text{if } u(x) < u_3(x), \end{cases}$$

$$\hat{V}(x,u) = \begin{cases} [\omega_+(x), +\infty), & \text{if } \bar{\tau}(x) < u(x), \\ \mathbb{R}, & \text{if } u_3(x) \le u(x) \le \bar{\tau}(x), \\ (-\infty, \hat{\omega}_-(x)], & \text{if } u(x) < u_3(x), \end{cases}$$

where $\hat{\omega}_- = \min\{\omega_1^-, \omega_2^-\}$, $\omega_+ \in S^{p'(\cdot)}_{\partial F(\cdot,\bar{\tau}(\cdot))}$, ω_i^- $(i = 1,2)$ are the $L^{p'(x)}(\Omega)$-selector of $\partial F(\cdot, u_i)$ corresponding to the solution u_i (see Definition 3.1) and

$$\hat{\beta}(x,u) = \begin{cases} \frac{1}{2}\min\{m_0, 1\}(|u|^{p(x)-2}u - |\bar{\tau}(x)|^{p(x)-2}\bar{\tau}(x)), & \text{if } \bar{\tau}(x) < u(x), \\ 0, & \text{if } u_3(x) \le u(x) \le \bar{\tau}(x), \\ \frac{1}{2}\min\{m_0, 1\}(|u|^{p(x)-2}u - |u_3|^{p(x)-2}u_3), & \text{if } u(x) < u_3(x). \end{cases}$$

Employing theses items, we introduce the following modification of the multivalued nonlinearity, namely

$$\hat{E}(x,u) = \partial F(x, \hat{\psi}(u)(x)) \cap \hat{V}(x,u).$$

In a similar way as in the Section 3, we can find a solution $u \in W_0^{1,p(x)}(\Omega)$ of (1.1) such that $u_3(x) \le u(x) \le \bar{\tau}(x)$ for a.a. $x \in \Omega$. Hence $u \in T_1$ and $u_1 \le u$, $u_2 \le u$, therefore T_1 is directed. □

To produce both smallest and greatest solutions in $[\underline{\tau}, \bar{\tau}]$, we need to strengthen further (HG2) as follows:

(HG3) $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist $a > 0$ and $1 \le s < \infty$ such that $u \mapsto g(u) + au^s$ is nondecreasing.

Theorem 4.3. *If hypotheses* (H0), (HM), (HJ2), (HF), (HG3) *hold, then* (1.1) *has extremal solutions in the order interval* $[\underline{\tau}, \bar{\tau}]$.

Proof. By Proposition 4.1 and Zorn's lemma, we can find $u^* \in T_1$ a maximal element for the pointwise ordering of $W_0^{1,p(x)}(\Omega)$. If $u \in T_1$, then from Proposition 4.2, we can find $y \in T_1$ such that $u \le y$, and $u^* \le y$. The maximality of u^* means that $u^* = y$. Noting that $u \in T_1$ is arbitrary, we have $u \le u^*$ for all $u \in T_1$ and so u^* is the greatest solution of problem (1.1) in the order interval T. If on $W_0^{1,p(x)}(\Omega)$ we use the partial order $\le_\circ$ defined by $u \le_\circ y$ if and only if $y(x) \le u(x)$ for a.a. $x \in \Omega$, then, from the same argument we can produce $u_* \in T_1$, which is the smallest element of T_1. Hence $\{u_*, u^*\}$ are the extremal solutions of problem (1.1) in the interval $[\underline{\tau}, \bar{\tau}]$. □

Remark 4.4. If (HG2) holds, we can only generate the great solution in $[\underline{\tau}, \bar{\tau}]$. Similarly, if g is only lower semicontinuous, we can only produce the smallest solution in $[\underline{\tau}, \bar{\tau}]$.

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