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DIRECT AND INVERSE DEGENERATE PARABOLIC DIFFERENTIAL EQUATIONS WITH MULTI-VALUED OPERATORS

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In memory of Alfredo Lorenzi

Abstract. Real interpolation spaces are used for solving some identification linear evolution problems in Banach spaces, under space regularity assumptions.

1. Introduction

This note starts with the following *direct* problem in a Banach space X,

$$\begin{gathered}\frac{d}{dt}y(t) + Ay(t) \ni f(t), \quad 0 \le t \le T, \\ y(0) = y_0.\end{gathered} \tag{1.1}$$

Here A is a possibly multivalued linear operator such that

$$\rho(A) \supset \Sigma_\alpha = \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le c(1 + |\operatorname{Im}\lambda|)^\alpha\}, \tag{1.2}$$

and the following inequality holds for $\lambda \in \Sigma_\alpha$

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \le c(1 + |\lambda|)^{-\beta}, \tag{1.3}$$

where c, α and β are positive constants satisfying $\beta \le \alpha \le 1$. It is shown in the book by Favini and Yagi [6] that $-A$ generates a C^∞-semigroup e^{-tA}, $0 < t < \infty$. We are interested in extending some of the results obtained in the paper by Favini, Lorenzi Tanabe [2], where A is assumed to be single-valued, to the case where A is multivalued. In [2] supposing that $y_0 \in D(A)$, $Ay_0 \in \tilde{X}_A^\theta$, $f \in C([0,T];X) \cap B([0,T];\widetilde{X}_A^\theta)$ or with $(X, D(A))_{\theta,\infty}$ in place of $\widetilde{X}_A^\theta$, the existence and uniqueness of a solution to (1.1) with some regularity property is established, where $B([0,T];Y)$ stands for the set of all bounded functions defined in $[0,T]$ with values in a Banach space Y. In this paper we show analogous results in case A is multivalued replacing $Ay_0 \in \tilde{X}_A^\theta$ or $Ay_0 \in (X, D(A))_{\theta,\infty}$ by $Ay_0 \cap \tilde{X}_A^\theta \ne \emptyset$ or $Ay_0 \cap (X, D(A))_{\theta,\infty} \ne \emptyset$.

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We begin by modifying the definitions of the spaces X_A^θ and $\widetilde{X}_A^\theta$ in case where A is not necessarily single valued replacing $A(t+A)^{-1}$ and Ae^{-tA} by $I - t(t+A)^{-1}$ and de^{-tA}/dt respectively and prove some preliminary results on these spaces.

The results proved for the *direct* problem are applied to the following identification problem:

$$\begin{gathered} \frac{d}{dt}y(t) + Ay(t) \ni f(t)z + h(t), \quad t \in [0,T], \\ y(0) = y_0, \\ \Phi[y(t)] = g(t), \quad t \in [0,T], \end{gathered} \tag{1.4}$$

where $y \in C([0,T];X)$, $f \in C([0,T];\mathbb{C})$ are unknown, $z \in X$, $h \in C([0,T];X)$, $g \in C([0,T];\mathbb{C})$ are given elements and $\Phi \in X^*$. Under some regularity assumptions on z, h and g it is shown that a unique solution to problem (1.4) exists. An extension of this result to equations with several unknown scalar functions

$$\begin{gathered} \frac{d}{dt}y(t) + Ay(t) \ni \sum_{j=1}^n f_j(t)z_j + h(t), \quad t \in [0,T], \\ y(0) = y_0, \\ \Phi_j[y(t)] = g_j(t), \quad j = 1, \dots, n, \quad t \in [0,T], \end{gathered} \tag{1.5}$$

is also established.

The above results are applied to the following problems

$$\begin{gathered} \frac{d}{dt}Mu(t) + Lu(t) = f(t), \quad t \in [0,T], \\ Mu(0) = Mu_0, \end{gathered} \tag{1.6}$$

and

$$\begin{gathered} \frac{d}{dt}Mu(t) + Lu(t) = \sum_{j=1}^n f_j(t)z_j + h(t), \quad t \in [0,T], \\ Mu(0) = Mu_0, \\ \Phi_j[Mu(t)] = g_j(t), \quad j = 1, \dots, n, \quad t \in [0,T], \end{gathered} \tag{1.7}$$

where L and M are linear closed operators such that

$$D(L) \subset D(M) \tag{1.8}$$

and for $\lambda \in \Sigma_\alpha$ a bounded inverse of $\lambda M + L$ exists and

$$\|M(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} \le c(1 + |\lambda|)^{-\beta}. \tag{1.9}$$

Then $A = LM^{-1}$ satisfies (1.2) and (1.3). A solution to (1.6) or (1.7) is easily obtained from that to (1.1) or (1.5) with $y_0 = Mu_0$.

We refer to the monograph [5] for diverse problems concerning regular evolution equations via Mathematical Physics. These identification problems were discussed in the papers [1] and [5] by using at all different techniques. Our present approach allows us to weaken the assumptions and to improve the regularity of solutions.

The plan of this paper is as follows. In section 2 preliminary results on intermediate spaces are collected. Section 3 is devoted to the existence and uniqueness of a solution to the direct problem (1.1). In Section 4 identification problem (1.4) is solved by transforming it to a simple Volterra integral equation for f. In Section 5 the general identification problem (1.4) is solved by applying the Banach fixed point theorem. This result is extended to equations (1.5) with several unknown scalar functions in Section 6. Finally, in Section 7 these results are applied to problems (1.6) and (1.7).

2. Preliminaries

Let A be a possibly multivalued linear operator in the complex Banach space X satisfying (1.2) and (1.3). Then it is shown in Chapter III of Favini and Yagi [6] that $-A$ generates a semigroup e^{-tA}, $0 < t < \infty$, satisfying

$$\|\frac{d^i}{dt^i}e^{-tA}\|_{\mathcal{L}(X)} \le C_0 t^{(\beta-i-1)/\alpha}, \quad i = 0,1,2, \quad C_0 > 0. \tag{2.1}$$

The set $D(A)$ makes a Banach space with norm

$$\|u\|_{D(A)} = \inf_{\phi\in Au} \|\phi\|_X \quad \text{for } u \in D(A). \tag{2.2}$$

It is known that

$$A^{-1}\frac{d}{dt}e^{-tA} = \frac{d}{dt}e^{-tA}A^{-1} = -e^{-tA}, \quad \lim_{t\to 0} e^{-tA}u = u \quad \text{for } u \in D(A). \tag{2.3}$$

If $u \in D(A)$, then in view of (2.3) and (2.1) one has for $\phi \in Au$

$$\|\frac{d}{dt}e^{-tA}u\|_X = \|\frac{d}{dt}e^{-tA}A^{-1}\phi\|_X = \|e^{-tA}\phi\|_X \le C_0 t^{(\beta-1)/\alpha}\|\phi\|_X.$$

This implies, by (2.2),

$$\|\frac{d}{dt}e^{-tA}u\|_X \le C_0 t^{(\beta-1)/\alpha}\|u\|_{D(A)}.$$

This inequality and the one obtained with the aid of an analogous argument imply

$$\|\frac{d}{dt}e^{-tA}\|_{\mathcal{L}(D(A),X)} \le C_0 t^{(\beta-1)/\alpha}, \quad \|\frac{d^2}{dt^2}e^{-tA}\|_{\mathcal{L}(D(A),X)} \le C_0 t^{(\beta-2)/\alpha}. \tag{2.4}$$

Definition. For $0 < \theta < 1$,

$$\begin{gathered} X_A^\theta = \{u \in X;\, |u|_{X_A^\theta} = \sup_{0<t<\infty} t^\theta \|u - t(t+A)^{-1}u\|_X < \infty\}, \\ \|u\|_{X_A^\theta} = |u|_{X_A^\theta} + \|u\|_X, \\ \widetilde{X}_A^\theta = \big\{u \in X; |u|_{\widetilde{X}_A^\theta} = \sup_{t>0} t^{(2-\beta-\theta)/\alpha}\|\frac{d}{dt}e^{-tA}u\|_X < \infty\big\}, \\ \|u\|_{\widetilde{X}_A^\theta} = |u|_{\widetilde{X}_A^\theta} + \|u\|_X. \end{gathered}$$

One of the definition of $(X, D(A))_{\theta,\infty}$ is

$$\begin{aligned} (X, D(A))_{\theta,\infty} = \Big\{ & u = u_0(t) + u_1(t) \ \forall t \in (0,\infty);\ \sup_{0<t<\infty} \|t^\theta u_0(t)\|_X < \infty, \\ & \sup_{0<t<\infty} \|t^{\theta-1}u_1(t)\|_{D(A)} < \infty\Big\}, \end{aligned}$$

$$\begin{aligned} &\|u\|_{(X,D(A))_{\theta,\infty}} \\ &= \inf_{u=u_0(t)+u_1(t)\ \forall t\in(0,\infty)} \big\{\sup_{0<t<\infty} \|t^\theta u_0(t)\|_X + \sup_{0<t<\infty} \|t^{\theta-1}u_1(t)\|_{D(A)}\big\}. \end{aligned}$$

Lemma 2.1. *The following inclusion relation holds for $0 < \theta < 1$,*

$$X_A^\theta \subset (X, D(A))_{\theta,\infty}.$$

Proof. Suppose $u \in X_A^\theta$. Set

$$u_0(t) = u - t(t+A)^{-1}u, \quad u_1(t) = t(t+A)^{-1}u.$$

Then $u = u_0(t) + u_1(t)$ and

$$\sup_{0<t<\infty} \|t^\theta u_0(t)\|_X = \sup_{0<t<\infty} t^\theta \|u - t(t+A)^{-1}u\|_X < \infty.$$

Since $(t+A)u_1(t) \ni tu$, one has $Au_1(t) \ni t(u - u_1(t))$, and hence

$$\|u_1(t)\|_{D(A)} \le t\|u - u_1(t)\|_X = t\|u - t(t+A)^{-1}u\|_X.$$

Therefore,

$$\sup_{0<t<\infty} \|t^{\theta-1}u_1(t)\|_{D(A)} \le \sup_{0<t<\infty} t^\theta \|u - (t+A)^{-1}u\|_X < \infty.$$

□

Lemma 2.2. *The following inclusion relation holds for $0 < \theta < 1$,*

$$(X, D(A))_{\theta,\infty} \subset \widetilde{X}_A^\theta.$$

Proof. Let $u \in (X, D(A))_{\theta,\infty}$. From (2.1), with $i = 1$, and the first inequality of (2.4) it follows that

$$\|\frac{d}{dt}e^{-tA}u\|_X \le C_0 t^{(\beta-2+\theta)/\alpha}\|u\|_{(X,D(A))_{\theta,\infty}}. \tag{2.5}$$

This readily implies $u \in \widetilde{X}_A^\theta$. □

Lemma 2.3. *Suppose $\alpha + \beta + \theta > 2$. Then for $u \in (X, D(A))_{\theta,\infty}$, $e^{-tA}u \to u$ as $t \to 0$. If $v \in D(A)$, the set $Av \cap (X, D(A))_{\theta,\infty}$ consists of at most a single element $-\lim_{t\to 0} \frac{d}{dt}e^{-tA}v$.*

Proof. Let $u \in D(A)$ and $\phi \in Au$. Then in view of (2.3) one gets

$$e^{-tA}u - u = \int_0^t \frac{d}{d\tau}e^{-\tau A}u d\tau = \int_0^t \frac{d}{d\tau}e^{-\tau A}A^{-1}\phi d\tau = -\int_0^t e^{-\tau A}\phi d\tau.$$

Hence, noting that the assumption implies $\alpha + \beta > 1$, one deduces

$$\begin{aligned}
\|e^{-tA}u - u\|_X &= \|\int_0^t e^{-\tau A}\phi d\tau\|_X \\
&\le C_0 \int_0^t \tau^{(\beta-1)/\alpha}\|\phi\|_X d\tau \\
&= C_0 \frac{t^{(\beta-1)/\alpha+1}}{(\beta-1)/\alpha+1}\|\phi\|_X.
\end{aligned}$$

This implies

$$\|e^{-tA}u - u\|_X \le C_0 \frac{t^{(\beta-1)/\alpha+1}}{(\beta-1)/\alpha+1}\|u\|_{D(A)}. \tag{2.6}$$

For $u \in X$,

$$\|e^{-tA}u - u\|_X \le \|e^{-tA}u\|_X + \|u\|_X \le C_0 t^{(\beta-1)/\alpha}\|u\|_X + \|u\|_X. \tag{2.7}$$

Interpolating (2.6) and (2.7) yields that there exists a constant C such that for $u \in (X, D(A))_{\theta,\infty}$,

$$\|e^{-tA}u - u\|_X \le C t^{(\beta-1)/\alpha+\theta}\|u\|_{(X,D(A))_{\theta,\infty}}, \quad 0 < t \le 1. \tag{2.8}$$

Since
$$\frac{\beta-1}{\alpha}+\theta-\frac{\alpha+\beta+\theta-2}{\alpha}=\frac{(1-\theta)(1-\alpha)}{\alpha}\geq 0,$$
one has
$$\frac{\beta-1}{\alpha}+\theta\geq\frac{\alpha+\beta+\theta-2}{\alpha}>0.$$
Hence the first assertion follows. Suppose $v\in D(A)$ and $\phi\in Av\cap(X,D(A))_{\theta,\infty}$. Then by the first assertion
$$\frac{d}{dt}e^{-tA}v=\frac{d}{dt}e^{-tA}A^{-1}\phi=-e^{-tA}\phi\to-\phi$$
as $t\to 0$. □

Lemma 2.4. *For $u\in X$, $t>0$, the following equality holds,*
$$\int_0^\infty e^{-t\tau}e^{-\tau A}ud\tau=(A+t)^{-1}u.$$

Proof. With the aid of (2.3) and integration by parts one deduces
$$\begin{aligned}\int_0^\infty e^{-t\tau}e^{-\tau A}ud\tau&=-\int_0^\infty e^{-t\tau}\frac{d}{d\tau}e^{-\tau A}A^{-1}ud\tau\\&=A^{-1}u-t\int_0^\infty e^{-t\tau}e^{-\tau A}A^{-1}ud\tau\\&=A^{-1}\Big(u-t\int_0^\infty e^{-t\tau}e^{-\tau A}ud\tau\Big).\end{aligned}$$
Hence $\int_0^\infty e^{-t\tau}e^{-\tau A}ud\tau\in D(A)$ and
$$A\int_0^\infty e^{-t\tau}e^{-\tau A}ud\tau\ni u-t\int_0^\infty e^{-t\tau}e^{-\tau A}ud\tau.$$
The assertion of the lemma readily follows. □

Lemma 2.5. *Suppose $\alpha+\beta+\theta>2$. If $u\in\widetilde{X}_A^\theta$, then $e^{-tA}u$ converges as $t\to 0$. If the limit is equal to u: $\lim_{t\to 0}e^{-tA}u=u$, then $u\in X_A^{(\alpha+\beta-2+\theta)/\alpha}$ and*
$$|u|_{X_A^{(\alpha+\beta-2+\theta)/\alpha}}\leq\Gamma((\alpha+\beta-2+\theta)/\alpha)|u|_{\widetilde{X}_A^\theta}.$$

Proof. For $0<s<t$, one has
$$\begin{aligned}\|e^{-tA}u-e^{-sA}u\|_X&=\|\int_s^t\frac{d}{d\tau}e^{-\tau A}ud\tau\|_X\leq\int_s^t\tau^{(\beta-2+\theta)/\alpha}|u|_{\widetilde{X}_A^\theta}d\tau\\&=\frac{t^{(\alpha+\beta-2+\theta)/\alpha}-s^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha}|u|_{\widetilde{X}_A^\theta}.\end{aligned}$$
Hence $e^{-tA}u$ converges as $t\to 0$. If $e^{-tA}u\to u$, then with the aid of integration by parts and Lemma 2.4 one gets for $t>0$
$$\int_0^\infty e^{-t\tau}\frac{d}{d\tau}e^{-\tau A}ud\tau=-u+t\int_0^\infty e^{-t\tau}e^{-\tau A}ud\tau=-u+t(t+A)^{-1}u.$$
Hence
$$\begin{aligned}\|u-t(t+A)^{-1}u\|_X&=\|-\int_0^\infty e^{-t\tau}\frac{d}{d\tau}e^{-\tau A}ud\tau\|_X\\&\leq\int_0^\infty e^{-t\tau}\tau^{(\beta-2+\theta)/\alpha}|u|_{\widetilde{X}_A^\theta}d\tau\end{aligned}$$

$$= t^{(2-\alpha-\beta-\theta)/\alpha}\Gamma((\alpha+\beta-2+\theta)/\alpha)|u|_{\widetilde{X}_A^\theta}.$$

□

Lemma 2.6. *Let $\alpha+\beta+\theta>2$. If $v\in D(A)$ and $Av\cap\widetilde{X}_A^\theta\neq\emptyset$, then $\lim_{t\to 0}\frac{d}{dt}e^{-tA}v$ exists.*

Proof. Let $\phi\in Av\cap\widetilde{X}_A^\theta$. Then

$$\lim_{t\to 0}\frac{d}{dt}e^{-tA}v=\lim_{t\to 0}\frac{d}{dt}e^{-tA}A^{-1}\phi=-\lim_{t\to 0}e^{-tA}\phi$$

exists by Lemma 2.5. □

Remark 2.7. The limit $\lim_{t\to 0}e^{-tA}u$ is not necessarily equal to u if A is multi-valued. If $0\neq u\in A0$, then

$$e^{-tA}u=-\frac{d}{dt}e^{-tA}A^{-1}u=-\frac{d}{dt}e^{-tA}0=0.$$

Therefore $u\in\widetilde{X}_A^\theta$ for any $\theta\in(0,1)$ and $\lim_{t\to 0}e^{-tA}u=0\neq u$. If A is single valued, then

$$A^{-1}\lim_{t\to 0}e^{-tA}u=\lim_{t\to 0}e^{-tA}A^{-1}u=A^{-1}u.$$

Hence $\lim_{t\to 0}e^{-tA}u=u$ whenever the limit of the left hand side exists.

3. Main result concerning problem (1.1)

Theorem 3.1. *Suppose that $2\alpha+\beta+\theta>3$, $y_0\in D(A)$, $Ay_0\cap(X,D(A))_{\theta,\infty}\neq\emptyset$ and $f\in C([0,T];X)\cap B([0,T];(X,D(A))_{\theta,\infty})$. Then problem (1.1) admits a unique solution y such that*

$$\begin{gathered} y\in C^1([0,T];X),\\ y'-f\in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T];X)\cap B([0,T];X_A^{(2\alpha+\beta-3+\theta)/\alpha}). \end{gathered}\tag{3.1}$$

Here $B([0,T];Y)$ stands for the set of all bounded (not necessarily measurable) functions with values in Y defined in $[0,T]$.

Proof. Note that $\alpha+\beta-2+\theta\geq 2\alpha+\beta-3+\theta>0$. Set

$$\begin{gathered} y(t)=e^{-tA}y_0+\int_0^t e^{-(t-s)A}f(s)ds,\\ y_1(t)=e^{-tA}y_0,\quad y_2(t)=\int_0^t e^{-(t-s)A}f(s)ds. \end{gathered}\tag{3.2}$$

In view of Lemma 2.3, $Ay_0\cap(X,D(A))_{\theta,\infty}$ consists of a single element $\phi=-\lim_{t\to 0}\frac{d}{dt}e^{-tA}y_0$. Hence

$$y_1'(t)=\frac{d}{dt}e^{-tA}y_0=\frac{d}{dt}e^{-tA}A^{-1}\phi=-e^{-tA}\phi.\tag{3.3}$$

In view of (2.5)

$$\|\frac{d}{dt}e^{-tA}\phi\|_X\leq C_0t^{(\beta-2+\theta)/\alpha}\|\phi\|_{(X,D(A))_{\theta,\infty}}.\tag{3.4}$$

By of (3.3) and Lemma 2.3

$$\lim_{t\to 0}y_1'(t)=-\phi.\tag{3.5}$$

With the aid of (3.4), for $0 \le s < t \le T$ one obtains

$$\begin{aligned}\|y_1'(t) - y_1'(s)\|_X &= \| - e^{-tA}\phi + e^{-sA}\phi\|_X = \| \int_s^t \frac{d}{d\sigma} e^{-\sigma A}\phi \, d\sigma\|_X \\ &\le C_0 \int_s^t \sigma^{(\beta-2+\theta)/\alpha} \|\phi\|_{(X,D(A))_{\theta,\infty}} \, d\sigma \\ &\le C_0 \frac{(t-s)^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha} \|\phi\|_{(X,D(A))_{\theta,\infty}}.\end{aligned}$$

Therefore $y_1' \in C^{(\alpha+\beta-2+\theta)/\alpha}([0,T];X)$ and

$$|y_1'|_{C^{(\alpha+\beta-2+\theta)/\alpha}([0,T];X)} \le \frac{C_0}{(\alpha+\beta-2+\theta)/\alpha} \|\phi\|_{(X,D(A))_{\theta,\infty}}. \tag{3.6}$$

For $s > 0$, in view of (3.3), (3.4) and $0 < (2-\beta-\theta)/\alpha < 1$, one deduces

$$\begin{aligned}& s^{(2-\beta-\theta)/\alpha} \| \frac{d}{ds} e^{-sA} y_1'(t)\|_X \\ &= s^{(2-\beta-\theta)/\alpha} \| \frac{d}{ds} e^{-sA} e^{-tA}\phi\|_X \\ &= s^{(2-\beta-\theta)/\alpha} \| \frac{\partial}{\partial s} e^{-(s+t)A}\phi\|_X \\ &\le C_0 s^{(2-\beta-\theta)/\alpha} (s+t)^{(\beta-2+\theta)/\alpha} \|\phi\|_{(X,D(A))_{\theta,\infty}} \\ &= C_0 \left(\frac{s}{s+t}\right)^{(2-\beta-\theta)/\alpha} \|\phi\|_{(X,D(A))_{\theta,\infty}} \le C_0 \|\phi\|_{(X,D(A))_{\theta,\infty}}.\end{aligned}$$

Hence $y_1'(t) \in \widetilde{X}_A^\theta$ and

$$|y_1'(t)|_{\widetilde{X}_A^\theta} \le C_0 \|\phi\|_{(X,D(A))_{\theta,\infty}}. \tag{3.7}$$

Using (3.3) one observes that

$$e^{-\tau A} y_1'(t) = -e^{-\tau A} e^{-tA}\phi = -e^{-(t+\tau)A}\phi \to -e^{-tA}\phi = y_1'(t)$$

as $\tau \to 0$. Hence in view of Lemma 2.5 $y_1'(t) \in X_A^{(\alpha+\beta-2+\theta)/\alpha}$ and

$$|y_1'(t)|_{X_A^{(\alpha+\beta-2+\theta)/\alpha}} \le \Gamma((\alpha+\beta-2+\theta)/\alpha) |y_1'(t)|_{\widetilde{X}_A^\theta}. \tag{3.8}$$

This inequality, (3.7) and (3.3) yield

$$\|y_1'(t)\|_{X_A^{(\alpha+\beta-2+\theta)/\alpha}} \le C_0 \Gamma((\alpha+\beta-2+\theta)/\alpha) \|\phi\|_{(X,D(A))_{\theta,\infty}} + \|e^{-tA}\phi\|_X.$$

Hence $y_1' \in B([0,T]; X_A^{(\alpha+\beta-2+\theta)/\alpha})$ and

$$\begin{aligned}& \|y_1'\|_{B([0,T];X_A^{(\alpha+\beta-2+\theta)/\alpha})} \\ &\le C_0 \Gamma((\alpha+\beta-2+\theta)/\alpha) \|\phi\|_{(X,D(A))_{\theta,\infty}} + \sup_{0\le t\le T} \|e^{-tA}\phi\|_X.\end{aligned} \tag{3.9}$$

The second term of the right hand side of (3.9) is finite by (3.3) and (3.5). In view of (2.5)

$$\begin{aligned}\| \frac{\partial}{\partial t} e^{-(t-s)A} f(s)\|_X &\le C_0 (t-s)^{(\beta-2+\theta)/\alpha} \|f(s)\|_{(X,D(A))_{\theta,\infty}} \\ &\le C_0 (t-s)^{(\beta-2+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}.\end{aligned} \tag{3.10}$$

Hence

$$\|\int_0^t \frac{\partial}{\partial t}e^{-(t-s)A}f(s)ds\|_X \le C_0\int_0^t (t-s)^{(\beta-2+\theta)/\alpha}\|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}ds = C_0\frac{t^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha}\|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \tag{3.11}$$

For $0 \le t < t' \le T$ one has with the aid of the change of the order of integration

$$\begin{aligned}&\int_t^{t'}\int_0^\tau \frac{\partial}{\partial\tau}e^{-(\tau-s)A}f(s)dsd\tau\\ &= \int_0^t\int_t^{t'}\frac{\partial}{\partial\tau}e^{-(\tau-s)A}f(s)d\tau ds + \int_t^{t'}\int_s^{t'}\frac{\partial}{\partial\tau}e^{-(\tau-s)A}f(s)d\tau ds.\end{aligned}$$

It is obvious that if $s < t < t'$, then

$$\int_t^{t'}\frac{\partial}{\partial\tau}e^{-(\tau-s)A}f(s)d\tau = e^{-(t'-s)A}f(s) - e^{-(t-s)A}f(s).$$

Since $f(s) \in (X, D(A))_{\theta,\infty}$, in view of Lemma 2.3, $e^{-(\tau-s)A}f(s) \to f(s)$ as $\tau \to s$. Hence

$$\int_s^{t'}\frac{\partial}{\partial\tau}e^{-(\tau-s)A}f(s)d\tau = e^{-(t'-s)A}f(s) - f(s). \tag{3.12}$$

Therefore

$$\begin{aligned}&\int_t^{t'}\int_0^\tau \frac{\partial}{\partial\tau}e^{-(\tau-s)A}f(s)dsd\tau\\ &= \int_0^t \big[e^{-(t'-s)A}f(s) - e^{-(t-s)A}f(s)\big]ds + \int_t^{t'}\big[e^{-(t'-s)A}f(s) - f(s)\big]ds\\ &= \int_0^{t'} e^{-(t'-s)A}f(s)ds - \int_0^t e^{-(t-s)A}f(s)ds - \int_t^{t'} f(s)ds\\ &= y_2(t') - y_2(t) - \int_t^{t'} f(s)ds.\end{aligned}$$

This means that $y_2(t)$ is differentiable and

$$y_2'(t) = f(t) + \int_0^t \frac{\partial}{\partial t}e^{-(t-\sigma)A}f(\sigma)\,d\sigma. \tag{3.13}$$

With the aid of (3.13), for $0 \le s < t \le T$, one has

$$\begin{aligned}&(y_2'(t) - f(t)) - (y_2'(s) - f(s))\\ &= \int_0^t \frac{\partial}{\partial t}e^{-(t-\sigma)A}f(\sigma)\,d\sigma - \int_0^s \frac{\partial}{\partial s}e^{-(s-\sigma)A}f(\sigma)\,d\sigma\\ &= \int_0^s \Big[\frac{\partial}{\partial t}e^{-(t-\sigma)A} - \frac{\partial}{\partial s}e^{-(s-\sigma)A}\Big]f(\sigma)\,d\sigma + \int_s^t \frac{\partial}{\partial t}e^{-(t-\sigma)A}f(\sigma)\,d\sigma\\ &= F_1(s,t) + F_2(s,t).\end{aligned} \tag{3.14}$$

Write $F_1(s,t)$ as

$$F_1(s,t) = \int_0^s\int_s^t \frac{\partial^2}{\partial\tau^2}e^{-(\tau-\sigma)A}d\tau f(\sigma)\,d\sigma. \tag{3.15}$$

Similarl to (2.5) it can be shown that for $u \in (X, D(A))_{\theta,\infty}$,

$$\|\frac{d^2}{dt^2} e^{-tA} u\|_X \le C_0 t^{(\beta-3+\theta)/\alpha} \|u\|_{(X,D(A))_{\theta,\infty}}.$$

Hence

$$\begin{aligned} \|\frac{\partial^2}{\partial\tau^2} e^{-(\tau-\sigma)A} f(\sigma)\|_X &\le C_0(\tau-\sigma)^{(\beta-3+\theta)/\alpha} \|f(\sigma)\|_{(X,D(A))_{\theta,\infty}} \\ &\le C_0(\tau-\sigma)^{(\beta-3+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \end{aligned} \tag{3.16}$$

Therefore

$$\begin{aligned} &\|F_1(s,t)\|_X \\ &\le C_0 \int_0^s \int_s^t (\tau-\sigma)^{(\beta-3+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})} d\tau\, d\sigma \\ &\le \frac{C_0(t-s)^{(2\alpha+\beta-3+\theta)/\alpha}}{(3-\alpha-\beta-\theta)/\alpha \cdot (2\alpha+\beta-3+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \end{aligned} \tag{3.17}$$

In view of (3.10) one obtains

$$\begin{aligned} \|F_2(s,t)\|_X &\le C_0 \int_s^t (t-\sigma)^{(\beta-2+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}\, d\sigma \\ &\le C_0 \frac{(t-s)^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \end{aligned} \tag{3.18}$$

From (3.14), (3.17) and (3.18) it follows that

$$y_2' - f \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T];X).$$

This and (3.6) yield

$$y' - f \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T];X). \tag{3.19}$$

From (3.13) one deduces that for $\tau > 0$,

$$\begin{aligned} \frac{d}{d\tau} e^{-\tau A}(y_2'(t) - f(t)) &= \frac{d}{d\tau} e^{-\tau A} \int_0^t \frac{\partial}{\partial t} e^{-(t-s)A} f(s) ds \\ &= \int_0^t \frac{d}{d\tau} e^{-\tau A} \frac{\partial}{\partial t} e^{-(t-s)A} f(s) ds \\ &= \int_0^t \frac{\partial^2}{\partial t^2} e^{-(t-s+\tau)A} f(s) ds. \end{aligned}$$

With the aid of (3.16) one obtains

$$\begin{aligned} &\|\frac{d}{d\tau} e^{-\tau A}(y_2'(t) - f(t))\|_X \\ &= \|\int_0^t \frac{\partial^2}{\partial t^2} e^{-(t-s+\tau)A} f(s) ds\|_X \\ &\le C_0 \int_0^t (t-s+\tau)^{(\beta-3+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})} ds \\ &= C_0 \frac{\tau^{(\alpha+\beta-3+\theta)/\alpha} - (t+\tau)^{(\alpha+\beta-3+\theta)/\alpha}}{(3-\alpha-\beta+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})} \\ &\le \frac{C_0 \tau^{(\alpha+\beta-3+\theta)/\alpha}}{(3-\alpha-\beta+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \end{aligned}$$

This means that $y_2'(t) - f(t) \in \widetilde{X}_A^{\alpha+\theta-1}$ and

$$|y_2'(t) - f(t)|_{\widetilde{X}_A^{\alpha+\theta-1}} \le \frac{C_0}{(3-\alpha-\beta-\theta)/\alpha}\|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \tag{3.20}$$

For $\tau > 0$,

$$\begin{aligned} e^{-\tau A}\big[y_2'(t) - f(t)\big] &= e^{-\tau A}\int_0^t \frac{\partial}{\partial t} e^{-(t-\sigma)A} f(\sigma)\, d\sigma \\ &= \int_0^t \frac{\partial}{\partial t} e^{-(t-\sigma+\tau)A} f(\sigma)\, d\sigma. \end{aligned} \tag{3.21}$$

As $\tau \to 0$ for $0 < \sigma < t$,

$$\|\frac{\partial}{\partial t} e^{-(t-\sigma+\tau)A} f(\sigma) - \frac{\partial}{\partial t} e^{-(t-\sigma)A} f(\sigma)\|_X \to 0, \tag{3.22}$$

in view of (3.10),

$$\begin{aligned} \|\frac{\partial}{\partial t} e^{-(t-\sigma+\tau)A} f(\sigma)\|_X &\le C_0(t-\sigma+\tau)^{(\beta-2+\theta)/\alpha}\|f\|_{B([0,T];(X,D(A))_{\theta,\infty})} \\ &\le C_0(t-\sigma)^{(\beta-2+\theta)/\alpha}\|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}, \end{aligned} \tag{3.23}$$

and

$$\int_0^t (t-\sigma)^{(\beta-2+\theta)/\alpha}\, d\sigma = \frac{t^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha} < \infty. \tag{3.24}$$

It follows from (3.21), (3.22), (3.23) and (3.24) that as $\tau \to 0$,

$$e^{-\tau A}\big[y_2'(t) - f(t)\big] \to \int_0^t \frac{\partial}{\partial t} e^{-(t-\sigma)A} f(\sigma)\, d\sigma = y_2'(t) - f(t). \tag{3.25}$$

Therefore, noting $\alpha + \beta + (\alpha + \theta - 1) = 2\alpha + \beta + \theta - 1 > 2$, $0 < \alpha + \theta - 1 < 1$ we can apply Lemma 2.5 and (3.20) to obtain that $y_2'(t) - f(t) \in X_A^{(2\alpha+\beta-3+\theta)/\alpha}$ and

$$\begin{aligned} |y_2'(t) - f(t)|_{X_A^{(2\alpha+\beta-3+\theta)/\alpha}} &\le \Gamma((2\alpha+\beta-3+\theta)/\alpha)|y_2'(t) - f(t)|_{\widetilde{X}_A^{\alpha+\theta-1}} \\ &\le \frac{C_0\Gamma((2\alpha+\beta-3+\theta)/\alpha)}{(3-\alpha-\beta-\theta)/\alpha}\|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \end{aligned}$$

From this inequality and the boundedness of $\|y_2'(t) - f(t)\|_X$, which follows from (3.11), and (3.13) one concludes that $y_2' - f \in B([0,T]; X_A^{(2\alpha+\beta-3+\theta)/\alpha})$. This and (3.9) imply $y' - f \in B([0,T]; X_A^{(2\alpha+\beta-3+\theta)/\alpha})$. Finally we show that y defined by (3.2) satisfies (1.1). By virtue of (3.13) one has

$$y'(t) = y_1'(t) + y_2'(t) = \frac{d}{dt} e^{-tA} y_0 + f(t) + \int_0^t \frac{\partial}{\partial t} e^{-(t-s)A} f(s) ds.$$

Hence

$$\begin{aligned} A^{-1}y'(t) &= A^{-1}\frac{d}{dt} e^{-tA} y_0 + A^{-1} f(t) + \int_0^t A^{-1}\frac{\partial}{\partial t} e^{-(t-s)A} f(s) ds \\ &= -e^{-tA} y_0 + A^{-1} f(t) - \int_0^t e^{-(t-s)A} f(s) ds. \end{aligned}$$

Therefore, one obtains

$$A^{-1}(y'(t) - f(t)) = -e^{-tA} y_0 - \int_0^t e^{-(t-s)A} f(s) ds = -y(t).$$

From this the assertion (1.1) readily follows. Thus the proof complete. $\square$

Theorem 3.2. *Suppose that $\alpha+\beta > 3/2$, $\theta > 2(2-\alpha-\beta)$, $y_0 \in D(A)$, $Ay_0 \cap \widetilde{X}^\theta_A \neq \emptyset$, $f \in C([0,T];X) \cap B([0,T];\widetilde{X}^\theta_A)$ and $\lim_{\tau\to 0} e^{-\tau A} f(t) = f(t)$ for every $t \in [0,T]$. Then problem (1.1) admits a unique solution y such that $y \in C^1([0,T];X)$ and*

$$y' - f \in C^{(2\alpha+2\beta-4+\theta)/\alpha}([0,T];X) \cap B([0,T];X_A^{(2\alpha+2\beta-4+\theta)/\alpha}). \tag{3.26}$$

Proof. Define functions y_1 and y_2 by (3.2). Let $\phi \in Ay_0 \cap \widetilde{X}^\theta_A$. Since

$$\alpha+\beta-2+\theta > \alpha+\beta-2+2(2-\alpha-\beta) = 2-\alpha-\beta \geq 0,$$

we can show as for (3.6) and (3.7) that

$$\begin{gathered} |y_1'|_{C^{(\alpha+\beta-2+\theta)/\alpha}([0,T];X)} \leq \frac{|\phi|_{\widetilde{X}^\theta_A}}{(\alpha+\beta-2+\theta)/\alpha}, \\ |y_1'(t)|_{\widetilde{X}^\theta_A} \leq |\phi|_{\widetilde{X}^\theta_A}, \end{gathered} \tag{3.27}$$

and furthermore (3.8) holds. Consequently the following statements are obtained as (3.9) was

$$y_1' \in B([0,T];X_A^{(\alpha+\beta-2+\theta)/\alpha}), \tag{3.28}$$

$$\|y_1'\|_{B([0,T];X_A^{(\alpha+\beta-2+\theta)/\alpha})} \leq \Gamma((\alpha+\beta-2+\theta)/\alpha)|\phi|_{\widetilde{X}^\theta_A} + \sup_{0\leq t\leq T} \|e^{-tA}\phi\|_X. \tag{3.29}$$

The second term in the right-hand side of (3.29) is finite since $\lim_{t\to 0} e^{-tA}\phi$ exists by Lemma 2.5.

In the present case the following inequality holds instead of (3.10),

$$\begin{aligned} \|\frac{\partial}{\partial t} e^{-(t-s)A} f(s)\|_X &\leq (t-s)^{(\beta-2+\theta)/\alpha} |f(s)|_{\widetilde{X}^\theta_A} \\ &\leq (t-s)^{(\beta-2+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}^\theta_A)}. \end{aligned} \tag{3.30}$$

Noting that (3.12) holds since $e^{-(\tau-s)A} f(s) \to f(s)$ as $\tau \to s$ by assumption, it is possible to show that $y_2(t) = \int_0^t e^{-(t-s)A} f(s) ds$ is differentiable and (3.13) holds. Let $F_1(s,t)$ and $F_2(s,t)$ be functions defined by (3.14). We have for $u \in \widetilde{X}^\theta_A$,

$$\begin{aligned} \|\frac{d^2}{dt^2} e^{-tA} u\|_X &= 4\|\big(\frac{d}{dt} e^{-(t/2)A}\big)^2 u\|_X \leq 4\|\frac{d}{dt} e^{-(t/2)A}\|_{\mathcal{L}(X)} \|\frac{d}{dt} e^{-(t/2)A} u\|_X \\ &\leq 4\frac{C_0}{2}\big(\frac{t}{2}\big)^{(\beta-2)/\alpha} \frac{1}{2}\left(\frac{t}{2}\right)^{(\beta-2+\theta)/\alpha} |u|_{\widetilde{X}^\theta_A} \\ &= C_1 t^{(2\beta-4+\theta)/\alpha} |u|_{\widetilde{X}^\theta_A}, \end{aligned}$$

where $C_1 = C_0 2^{(4-2\beta-\theta)/\alpha}$. Hence

$$\begin{aligned} \|\frac{d^2}{d\tau^2} e^{-(\tau-\sigma)A} f(\sigma)\|_X &\leq C_1(\tau-\sigma)^{(2\beta-4+\theta)/\alpha} |f(\sigma)|_{\widetilde{X}^\theta_A} \\ &\leq C_1(\tau-\sigma)^{(2\beta-4+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}^\theta_A)}. \end{aligned} \tag{3.31}$$

With the aid of (3.15) and (3.31) we obtain

$$\begin{aligned} &\|F_1(s,t)\|_X \\ &\leq C_1 \int_0^s \int_s^t (\tau-\sigma)^{(2\beta-4+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}^\theta_A)} d\tau\, d\sigma \\ &\leq \frac{C_1\alpha^2}{(4-\alpha-2\beta-\theta)(2\alpha+2\beta-4+\theta)} (t-s)^{(2\alpha+2\beta-4+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}^\theta_A)}. \end{aligned} \tag{3.32}$$

Using (3.30), one obtains

$$\begin{aligned} \|F_2(s,t)\|_X &\le \int_s^t (t-\sigma)^{(\beta-2+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)}\, d\sigma \\ &= \frac{(t-s)^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)}. \end{aligned} \tag{3.33}$$

Inequalities (3.32) and (3.33) imply that $y_2' - f \in C^{(2\alpha+2\beta-4+\theta)/\alpha}([0,T];X)$. This and (3.27) yield $y' - f \in C^{(2\alpha+2\beta-4+\theta)/\alpha}([0,T];X)$. With the aid of (3.31) one observes that

$$\begin{aligned} \|\frac{d}{d\tau} e^{-\tau A}(y_2'(t) - f(t))\|_X &= \|\int_0^t \frac{\partial^2}{\partial t^2} e^{-(t-s+\tau)A} f(s) ds\|_X \\ &\le C_1 \int_0^t (t-s+\tau)^{(2\beta-4+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)} ds \\ &= C_1 \frac{\tau^{(\alpha+2\beta-4+\theta)/\alpha} - (t+\tau)^{(\alpha+2\beta-4+\theta)/\alpha}}{(4-\alpha-2\beta+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)} \\ &\le \frac{C_1 \tau^{(\alpha+2\beta-4+\theta)/\alpha}}{(4-\alpha-2\beta+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)}. \end{aligned}$$

This means that $y_2'(t) - f(t) \in \widetilde{X}_A^{\alpha+\beta+\theta-2}$ and

$$|y_2'(t) - f(t)|_{\widetilde{X}_A^{\alpha+\beta+\theta-2}} \le \frac{C_1}{(4-\alpha-2\beta+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)}. \tag{3.34}$$

From (3.21), (3.22), (3.24) and the inequality

$$\begin{aligned} \|\frac{\partial}{\partial t} e^{-(t-\sigma+\tau)A} f(\sigma)\|_X &\le (t-\sigma+\tau)^{(\beta-2+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)} \\ &\le C_1 (t-\sigma)^{(\beta-2+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)}, \end{aligned}$$

which follows from (3.30). Then (3.25) holds. Therefore noting $\alpha+\beta+(\alpha+\beta+\theta-2) = 2\alpha+2\beta+\theta-2 > 2$ we may apply Lemma 2.5 and (3.34) to obtain that $y_2'(t) - f(t) \in X_A^{(2\alpha+2\beta-4+\theta)/\alpha}$ and

$$\begin{aligned} |y_2'(t) - f(t)|_{X_A^{(2\alpha+2\beta-4+\theta)/\alpha}} &\le \Gamma((2\alpha+2\beta-4+\theta)/\alpha) |y_2'(t) - f(t)|_{\widetilde{X}_A^{\alpha+\beta+\theta-2}} \\ &\le \frac{C_1 \Gamma((2\alpha+2\beta-4+\theta)/\alpha)}{(4-\alpha-2\beta+\theta)/\alpha} \|f\|_{B([0,T];\widetilde{X}_A^\theta)}. \end{aligned}$$

From this inequality, (3.13) and (3.30) one concludes that

$$y_2' - f \in B([0,T]; X_A^{(2\alpha+2\beta-4+\theta)/\alpha}).$$

This and (3.28) imply $y' - f \in B([0,T]; X_A^{(2\alpha+2\beta-4+\theta)/\alpha})$. The proof that y defined by (3.2) satisfies (1.1) is the same as that in the proof of Theorem 3.1. □

Remark 3.3. Suppose that the assumptions of Theorem 3.1 are satisfied, and let $k \in \mathbb{R}$. Consider the problem

$$\begin{gathered} \frac{d}{dt} y(t) + Ay(t) + ky(t) \ni f(t), \quad 0 \le t \le T, \\ y(0) = y_0. \end{gathered} \tag{3.35}$$

The operator $-(A+k)$ generates a differentiable semigroup $e^{-t(A+k)} = e^{-kt}e^{-tA}$. Then define $C_0' = C_0 e^{\max\{k,0\}T}$, so that for $0 < t \le T$ one gets

$$\|\frac{d^i}{dt^i} e^{-t(A+k)}\|_{\mathcal{L}(X)} \le C_0' t^{(\beta-i-1)/\alpha}, \quad i = 0,1,2.$$

The unique solution to (3.35) is given by

$$y(t) = e^{-t(A+k)} y_0 + \int_0^t e^{-(t-s)(A+k)} f(s) ds.$$

Since $e^{kt}y(t)$ is a solution to (1.1) with $f(t)$ replaced by $e^{kt}f(t)$, y satisfies $y \in C^1([0,T];X)$ and

$$y' + ky - f \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T];X) \cap B([0,T]; X_A^{(2\alpha+\beta-3+\theta)/\alpha}).$$

Furthermore, the following results hold just as (3.13) and (3.11):

$$\frac{d}{dt}\int_0^t e^{-(t-s)(A+k)} f(s) ds = f(t) + \int_0^t \frac{\partial}{\partial t} e^{-(t-s)(A+k)} f(s) ds,$$

$$\|\int_0^t \frac{\partial}{\partial t} e^{-(t-s)(A+k)} f(s) ds\|_X \le \frac{C_0' t^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha} \|f\|_{B([0,T];(X,D(A))_{\theta,\infty})}.$$

An analogous remark holds also for Theorem 3.2.

4. An identification problem for the differential equation

In this section we consider problem (1.4):

$$\frac{d}{dt} y(t) + Ay(t) \ni f(t)z + h(t), \quad t \in [0,T], \tag{4.1}$$

$$y(0) = y_0, \tag{4.2}$$

$$\Phi[y(t)] = g(t), \quad t \in [0,T], \tag{4.3}$$

Theorem 4.1. *Suppose that* $2\alpha+\beta+\theta > 3$, $y_0 \in D(A)$, $Ay_0 \cap (X,D(A))_{\theta,\infty} \neq \emptyset$, $z \in (X,D(A))_{\theta,\infty}$, $h \in C([0,T];X) \cap B([0,T];(X,D(A))_{\theta,\infty})$, $g \in C^1([0,T];\mathbb{C})$, $\Phi \in X^*$, $\Phi[y_0] = g(0)$, $\Phi[z] \neq 0$. *Then problem* (4.1)-(4.3) *admits a unique solution* (y,f) *such that*

$$\begin{gathered} y \in C^1([0,T];X), \quad f \in C([0,T];\mathbb{C}), \\ y' - f(\cdot)z - h \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T];X) \cap B([0,T]; X_A^{(2\alpha+\beta-3+\theta)/\alpha}). \end{gathered} \tag{4.4}$$

Proof. Supposing that $f \in C([0,T];\mathbb{C})$ is known, define a function y by

$$y(t) = e^{-tA} y_0 + \int_0^t f(s) e^{-(t-s)A} z ds + \int_0^t e^{-(t-s)A} h(s) ds. \tag{4.5}$$

Since $\alpha+\beta+\theta-2 \ge 2\alpha+\beta+\theta-3 > 0$, by Lemma 2.3 and (2.5) the following statements hold:

$$e^{-tA}z \to z \text{ as } t \to 0, \quad \|\frac{d}{dt} e^{-tA} z\|_X \le C_0 t^{(\beta-2+\theta)/\alpha} \|z\|_{(X,D(A))_{\theta,\infty}}. \tag{4.6}$$

Hence $\int_0^t f(s) e^{-(t-s)A} z ds$ is differentiable and

$$\frac{d}{dt}\int_0^t f(s) e^{-(t-s)A} z ds = f(t)z + \int_0^t f(s) \frac{\partial}{\partial t} e^{-(t-s)A} z ds. \tag{4.7}$$

According to the proof of Theorem 3.1 (cf. (3.13) and (3.11)) $\int_0^t e^{-(t-s)A}h(s)ds$ is differentiable and

$$\frac{d}{dt}\int_0^t e^{-(t-s)A}h(s)ds = h(t) + \int_0^t \frac{\partial}{\partial t}e^{-(t-s)A}h(s)ds, \tag{4.8}$$

$$\Big\|\int_0^t \frac{\partial}{\partial t}e^{-(t-s)A}h(s)ds\Big\|_X \le C_0\frac{t^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha}\|h\|_{B([0,T];(X,D(A))_{\theta,\infty})}. \tag{4.9}$$

Hence y is differentiable and

$$\frac{d}{dt}y(t) = \frac{d}{dt}e^{-tA}y_0 + f(t)z + \int_0^t f(s)\frac{\partial}{\partial t}e^{-(t-s)A}zds + \frac{d}{dt}\int_0^t e^{-(t-s)A}h(s)ds. \tag{4.10}$$

Assuming that $y(t)$ satisfies (4.3) one deduces from (4.10) the following identity

$$\begin{aligned} g'(t) &= \Phi\Big[\frac{d}{dt}e^{-tA}y_0\Big] + f(t)\Phi[z] + \int_0^t f(s)\Phi\Big[\frac{\partial}{\partial t}e^{-(t-s)A}z\Big]ds \\ &\quad + \Phi\Big[\frac{d}{dt}\int_0^t e^{-(t-s)A}h(s)ds\Big]. \end{aligned} \tag{4.11}$$

Rewriting (4.11) one obtains the following integral equation to be satisfied by f:

$$\begin{aligned} &f(t) + \chi\int_0^t f(s)\Phi\Big[\frac{\partial}{\partial t}e^{-(t-s)A}z\Big]ds \\ &= \chi g'(t) - \chi\Phi\Big[\frac{d}{dt}e^{-tA}y_0\Big] - \chi\Phi\Big[\frac{d}{dt}\int_0^t e^{-(t-s)A}h(s)ds\Big], \end{aligned} \tag{4.12}$$

where $\chi = \Phi[z]^{-1}$. Set

$$\begin{gathered} \kappa(t) = \chi\Phi\Big[\frac{d}{dt}e^{-tA}z\Big], \\ \psi(t) = \chi g'(t) - \chi\Phi\Big[\frac{d}{dt}e^{-tA}y_0\Big] - \chi\Phi\Big[\frac{d}{dt}\int_0^t e^{-(t-s)A}h(s)ds\Big]. \end{gathered} \tag{4.13}$$

Then (4.12) is rewritten as

$$f(t) + \int_0^t \kappa(t-s)f(s)ds = \psi(t),$$

or briefly

$$f + \kappa * f = \psi. \tag{4.14}$$

By (4.6) one has

$$|\kappa(t)| \le C_0|\chi|\|\Phi\|t^{(\beta-2+\theta)/\alpha}\|z\|_{(X,D(A))_{\theta,\infty}}.$$

In view of (3.5), (4.8) and (4.9) one observes that $\psi \in C([0,T];\mathbb{C})$. Let r be the solution to the integral equation

$$\kappa + r + r * \kappa = 0.$$

This equation is solved by successive approximations, and the solution r satisfies

$$\kappa + r + \kappa * r = 0, \quad |r(t)| \le C_2 t^{(\beta-2+\theta)/\alpha}, \quad C_2 > 0.$$

The integral equation (4.14) admits a unique solution $f \in C([0,T];\mathbb{C})$ given by $f = \psi + r * \psi$, or

$$f(t) = \psi(t) + \int_0^t r(t-s)\psi(s)ds. \tag{4.15}$$

It is easy to verify that if we define y by (4.5) with f given by (4.15), the pair (y, f) satisfies (4.1) and (4.2). From (4.10) and (4.11) it follows that

$$g'(t) = \Phi\Big[\frac{d}{dt}y(t)\Big] = \frac{d}{dt}\Phi[y(t)].$$

From this equality and the compatibility condition $\Phi[y_0] = g(0)$ (4.3) follows. Since $f(\cdot)z + h \in C([0,T];X) \cap B([0,T];(X,D(A))_{\theta,\infty})$, the assertion (4.4) follows from Theorem 3.1. □

Theorem 4.2. *Suppose that $\alpha + \beta > 3/2$, $2(2 - \alpha - \beta) < \theta < 1$, $y_0 \in D(A)$, $Ay_0 \cap \widetilde{X}^\theta_A \neq \emptyset$, $z \in \widetilde{X}^\theta_A$, $\lim_{t\to 0} e^{-tA}z = z$, $g \in C^1([0,T];\mathbb{C})$, $\Phi \in X^*$, $\Phi[y_0] = g(0)$, $\Phi[z] \neq 0$, $h \in C([0,T];X) \cap B([0,T];\widetilde{X}^\theta_A)$, $\lim_{\tau\to 0} e^{-\tau A}h(t) = h(t)$ for every $t \in [0,T]$. Then problem (4.1)-(4.3) admits a unique solution y such that $y \in C^1([0,T];X)$ and*

$$y' - f \in C^{(2\alpha+2\beta-4+\theta)/\alpha}([0,T];X) \cap B([0,T];X_A^{(2\alpha+2\beta-4+\theta)/\alpha}).$$

Proof. Supposing that $f \in C([0,T];X)$ is known, define the function y by (4.5). Let $\phi \in Ay_0 \cap \widetilde{X}^\theta_A$. Then in view of Lemma 2.5,

$$\frac{d}{dt}e^{-tA}y_0 = \frac{d}{dt}e^{-tA}A^{-1}\phi = -e^{-tA}\phi$$

converges as $t \to 0$. Hence

$$\text{the function } t \mapsto \frac{d}{dt}e^{-tA}y_0 \text{ belongs to } C([0,T];X). \tag{4.16}$$

By the definition of $\widetilde{X}^\theta_A$ one has

$$\|\frac{d}{dt}e^{-tA}z\|_X \le t^{(\beta-2+\theta)/\alpha}|z|_{\widetilde{X}^\theta_A}. \tag{4.17}$$

Hence using the assumption $\lim_{t\to 0} e^{-tA}z = z$ one observes that $\int_0^t f(s)e^{-(t-s)A}z\,ds$ is differentiable and (4.7) holds. According to the proof of Theorem 3.2 (cf. (3.30)) $\int_0^t e^{-(t-s)A}h(s)ds$ is differentiable, and equality (4.8) and the inequality

$$\|\int_0^t \frac{\partial}{\partial t}e^{-(t-s)A}h(s)ds\|_X \le \frac{t^{(\alpha+\beta-2+\theta)/\alpha}}{(\alpha+\beta-2+\theta)/\alpha}\|h\|_{B(0,T;\widetilde{X}^\theta_A)} \tag{4.18}$$

holds. Hence y is differentiable and (4.10) holds. Assuming that $y(t)$ satisfies (4.3) one deduces (4.14) from (4.10) as in the proof of Theorem 4.1, where κ and ψ are functions defined by (4.13). By virtue of (4.17) one has

$$|\kappa(t)| \le |\chi|\|\Phi\|_{X^*}t^{(\beta-2+\theta)/\alpha}|z|_{\widetilde{X}^\theta_A}.$$

In view of (4.16), (4.8) and (4.18) one observes $\psi \in C([0,T];\mathbb{C})$. The remaining part of the proof is the same as that of Theorem 4.1. □

5. Equations with several unknown scalar functions

In this section we consider the problem consisting of recovering several unknown scalar functions $f_1, \dots, f_n$ and a vector function y such that

$$\frac{d}{dt}y(t) + Ay(t) \ni \sum_{j=1}^n f_j(t)z_j + h(t), \quad t \in [0,T], \tag{5.1}$$

$$y(0) = y_0, \tag{5.2}$$

$$\Phi_j[y(t)] = g_j(t), \quad j = 1, \dots, n, \quad t \in [0, T]. \tag{5.3}$$

Theorem 5.1. *Suppose* $2\alpha + \beta + \theta > 3$, $y_0 \in D(A)$, $Ay_0 \cap (X, D(A))_{\theta,\infty} \neq \emptyset$, $z_j \in (X, D(A))_{\theta,\infty}$, $j = 1, \dots, n$, $h \in C([0, T]; X) \cap B([0, T]; (X, D(A))_{\theta,\infty})$, $g_j \in C^1([0, T]; \mathbb{C})$, $\Phi_j \in X^*$, $\Phi_j[y_0] = g_j(0)$, $j = 1, \dots, n$, *and*

$$\det \begin{pmatrix} \Phi_1[z_1] & \dots & \Phi_1[z_n] \\ \dots & & \dots \\ \Phi_n[z_1] & \dots & \Phi_n[z_n] \end{pmatrix} \neq 0. \tag{5.4}$$

Then, problem (5.1)–(5.3) *admits a unique solution* $(y, f_1, \dots, f_n)$ *such that*

$$\begin{gathered} y \in C^1([0, T]; X), \quad f_1, \dots, f_n \in C([0, T]; \mathbb{C}), \\ y' - \sum_{j=1}^n f_j(\cdot) z_j - h \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0, T]; X) \cap B([0, T]; X_A^{(2\alpha+\beta-3+\theta)/\alpha}). \end{gathered} \tag{5.5}$$

The proof is performed in parallel to the proof of Theorem 4.1. If $f_1, \dots, f_n \in C([0, T]; \mathbb{C})$ are known, y is given by

$$y(t) = e^{-tA} y_0 + \int_0^t \sum_{j=1}^n f_j(s) e^{-(t-s)A} z_j ds + \int_0^t e^{-(t-s)A} h(s) ds.$$

Just as the proof of (4.10) one deduces from this equality

$$\begin{aligned} y'(t) &= \frac{d}{dt} e^{-tA} y_0 + \sum_{j=1}^n f_j(t) z_j \\ &\quad + \int_0^t \sum_{j=1}^n f_j(s) D_t e^{-(t-s)A} z_j ds + \frac{d}{dt} \int_0^t e^{-(t-s)A} h(s) ds. \end{aligned} \tag{5.6}$$

It follows from (5.3) and (5.6) that

$$\begin{aligned} g_i'(t) &= \Phi_i[y'(t)] \\ &= \Phi_i\big[D_t e^{-tA} y_0\big] + \sum_{j=1}^n f_j(t) \Phi_i[z_j] + \int_0^t \sum_{j=1}^n f_j(s) \Phi_i[D_t e^{-(t-s)A} z_j] ds \\ &\quad + \Phi_i\Big[D_t \int_0^t e^{-(t-s)A} h(s) ds\Big]. \end{aligned}$$

This is rewritten as

$$\begin{aligned} &\begin{pmatrix} \Phi_1[z_1] & \dots & \Phi_1[z_n] \\ \dots & & \dots \\ \Phi_n[z_1] & \dots & \Phi_n[z_n] \end{pmatrix} \begin{pmatrix} f_1(t) \\ \dots \\ f_n(t) \end{pmatrix} \\ &= \begin{pmatrix} g_1'(t) - \Phi_1[D_t e^{-tA} y_0] - \Phi_1\big[D_t \int_0^t e^{-(t-s)A} h(s) ds\big] \\ \dots \\ g_n'(t) - \Phi_n[D_t e^{-tA} y_0] - \Phi_n\big[D_t \int_0^t e^{-(t-s)A} h(s) ds\big] \end{pmatrix} \\ &\quad \int_0^t \begin{pmatrix} \Phi_1[D_t e^{-(t-s)A} z_1] & \dots & \Phi_1[D_t e^{-(t-s)A} z_n] \\ \dots & & \dots \\ \Phi_n[D_t e^{-(t-s)A} z_1] & \dots & \Phi_n[D_t e^{-(t-s)A} z_n] \end{pmatrix} \begin{pmatrix} f_1(s) \\ \dots \\ f_n(s) \end{pmatrix} ds. \end{aligned} \tag{5.7}$$

Set

$$\mathcal{A} = \begin{pmatrix} \Phi_1[z_1] & \dots & \Phi_1[z_n] \\ \dots & & \dots \\ \Phi_n[z_1] & \dots & \Phi_n[z_n] \end{pmatrix}.$$

Then by assumption (5.4), $\mathcal{A}^{-1}$ exists. Set

$$\Phi(t) = \mathcal{A}^{-1} \begin{pmatrix} g_1'(t) - \Phi_1[D_t e^{-tA} y_0] - \Phi_1\big[D_t \int_0^t e^{-(t-s)A} h(s) ds\big] \\ \dots \\ g_n'(t) - \Phi_n[D_t e^{-tA} y_0] - \Phi_n\big[D_t \int_0^t e^{-(t-s)A} h(s) ds\big] \end{pmatrix},$$

$$K(t) = \mathcal{A}^{-1} \begin{pmatrix} \Phi_1[D_t e^{-tA} z_1] & \dots & \Phi_1[D_t e^{-tA} z_n] \\ \dots & & \dots \\ \Phi_n[D_t e^{-tA} z_1] & \dots & \Phi_n[D_t e^{-tA} z_n] \end{pmatrix}, \quad F(t) = \begin{pmatrix} f_1(t) \\ \dots \\ f_n(t) \end{pmatrix}.$$

Tt follows from (5.7) that

$$F(t) = \Phi(t) - \int_0^t K(t-s) F(s)\, ds.$$

Since

$$\|K(t)\|_{\mathcal{L}(\mathbb{C}^n)} \le C t^{(\beta-2+\theta)/\alpha},$$

the remaining part of the proof is the same as that of Theorem 4.1. Analogously the following theorem is established.

Theorem 5.2. *Suppose* $2(2-\alpha-\beta) < \theta < 1$, $y_0 \in D(A)$, $Ay_0 \cap \widetilde{X}_A^\theta \neq \emptyset$, $z_j \in \widetilde{X}_A^\theta$, $\lim_{t\to 0} e^{-tA} z_j = z_j$, $j = 1, \dots, n$, $h \in C([0,T]; X) \cap B([0,T]; \widetilde{X}_A^\theta)$, $\lim_{\tau\to 0} e^{-\tau A} h(t) = h(t)$ *for every* $t \in [0,T]$, $g_j \in C^1([0,T]; \mathbb{C})$, $\Phi_j \in X^*$, $\Phi_j[y_0] = g_j(0)$, $j = 1, \dots, n$, *and*

$$\det \begin{pmatrix} \Phi_1[z_1] & \dots & \Phi_1[z_n] \\ \dots & & \dots \\ \Phi_n[z_1] & \dots & \Phi_n[z_n] \end{pmatrix} \neq 0.$$

Then, problem (5.1)-(5.3) *admits a unique solution* $(y, f_1, \dots, f_n)$ *such that*

$$y \in C^1([0,T]; X), \quad f_1, \dots, f_n \in C([0,T]; \mathbb{C}),$$

$$y' - \sum_{j=1}^n f_j(\cdot) z_j - h \in C^{(2\alpha+2\beta-4+\theta)/\alpha}([0,T]; X) \cap B([0,T]; X_A^{(2\alpha+2\beta-4+\theta)/\alpha}).$$

6. Problems (1.6) and (1.7)

Let L and M be two linear closed operators satisfying (1.8) and (1.9). Set $A = LM^{-1}$. Namely

$$\begin{gathered} D(A) = MD(L) = \{Mu : u \in D(L)\}, \\ Ay = \{Lu : y = Mu,\ u \in D(L)\} \text{ for } y \in D(A). \end{gathered} \tag{6.1}$$

It is shown in Favini and Yagi [6] that A satisfies (1.2) and (1.3). The graph-norm of $D(A)$ is defined by

$$\|y\|_{D(A)} = \inf\{\|Lu\|_X : y = Mu,\ u \in D(L)\} \quad \text{for} \quad y \in D(A).$$

Consider the problem

$$\frac{d}{dt} Mu(t) + Lu(t) = f(t), \quad t \in [0,T], \tag{6.2}$$

$$Mu(0) = Mu_0. \tag{6.3}$$

Theorem 6.1. *Suppose that $2\alpha + \beta + \theta > 3$, $u_0 \in D(L)$, $Lu_0 \in (X, D(A))_{\theta,\infty}$, $f \in C([0,T]; X) \cap B([0,T]; (X, D(A))_{\theta,\infty})$. Then problem (6.2)–(6.3) admits a unique solution u such that*

$$\begin{gathered} Mu \in C^1([0,T]; X), \\ Lu \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T]; X) \cap B([0,T]; X_A^{(2\alpha+\beta-3+\theta)/\alpha}). \end{gathered} \tag{6.4}$$

Proof. Set $y_0 = Mu_0$. Then $y_0 \in D(A)$, and $Ay_0 \cap (X, D(A))_{\theta,\infty}$ is not empty, since it contains Lu_0. In view of Theorem 3.1 there exists a solution y to (1.1), A being defined by (6.1). Set $u(t) = L^{-1}(f(t) - y'(t))$. Then

$$Lu(t) = f(t) - y'(t) \in Ay(t) = LM^{-1}y(t). \tag{6.5}$$

Since L is bijective, it follows from (6.5) that $u(t) \in M^{-1}y(t)$, or $y(t) = Mu(t)$. From the first equation of (6.5) equation (6.2) follows. It is obvious that u satisfies (6.3). □

Analogously, using Theorem 3.2 instead of Theorem 3.1, the following theorem is obtained.

Theorem 6.2. *Suppose that $\theta > 2(2 - \alpha - \beta)$, $u_0 \in D(L)$, $Lu_0 \in \widetilde{X}_A^\theta$, $f \in C([0,T]; X) \cap B([0,T]; \widetilde{X}_A^\theta)$, $\lim_{\tau\to 0} e^{-\tau A} f(t) = f(t)$ for every $t \in [0,T]$. Then problem (6.2)–(6.3) admits a unique solution u such that*

$$\begin{gathered} Mu \in C^1([0,T]; X), \\ Lu \in C^{(2\alpha+2\beta-4+\theta)/\alpha}([0,T]; X) \cap B([0,T]; X_A^{(2\alpha+2\beta-4+\theta)/\alpha}). \end{gathered} \tag{6.6}$$

Next, consider the problem

$$\frac{d}{dt} Mu(t) + Lu(t) = \sum_{j=1}^n f_j(t) z_j + h(t), \quad t \in [0,T], \tag{6.7}$$

$$Mu(0) = Mu_0, \tag{6.8}$$

$$\Phi_j[Mu(t)] = g_j(t), \quad j = 1, \dots, n, \quad t \in [0,T]. \tag{6.9}$$

Theorem 6.3. *Suppose $2\alpha + \beta + \theta > 3$, $u_0 \in D(L)$, $Lu_0 \in (X, D(A))_{\theta,\infty}$, $z_j \in (X, D(A))_{\theta,\infty}$, $j = 1, \dots, n$, $h \in C([0,T]; X) \cap B([0,T]; (X, D(A))_{\theta,\infty})$, $\Phi_j \in X^*$, $g_j \in C^1([0,T; \mathbb{C})$, $\Phi_j[Mu_0] = g_j(0)$, $j = 1, \dots, n$, and (5.4) holds. Then problem (6.7)–(6.9) admits a unique solution $(u, f_1, \dots, f_n)$ such that*

$$\begin{gathered} Mu \in C^1([0,T]; X), \quad f_1, \dots, f_n \in C([0,T]; \mathbb{C}), \\ Lu \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T]; X) \cap B([0,T]; X_A^{(2\alpha+\beta-3+\theta)/\alpha}). \end{gathered} \tag{6.10}$$

Proof. Let $(y, f_1, \dots, f_n)$ be a solution to problem (5.1)-(5.3) with A defined by (6.1) and define a function u by

$$u(t) = L^{-1}\Big[\sum_{j=1}^n f_j(t) z_j + h(t) - y'(t)\Big], \quad t \in [0,T].$$

Then in view of (5.1)

$$Lu(t) = \sum_{j=1}^n f_j(t) z_j + h(t) - y'(t) \in Ay(t) = LM^{-1}y(t). \tag{6.11}$$

Since L is injective, one gets $u(t) \in M^{-1}y(t)$. This implies

$$Mu(t) = y(t). \tag{6.12}$$

The first equation in (6.11) and (6.12) imply (6.7). It is obvious that (6.8) and (6.9) hold. From the first equation in (6.11) and the second equation in (5.5) it follows that

$$Lu = \sum_{j=1}^{n} f_j(\cdot)z_j + h - y' \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T];X) \cap B([0,T];X_A^{(2\alpha+\beta-3+\theta)/\alpha}).$$

Thus the regularity property (6.10) is proved. □

The following theorem is analogously established.

Theorem 6.4. *Suppose $\theta > 2(2-\alpha-\beta)$, $u_0 \in D(L)$, $Lu_0 \in \widetilde{X}_A^\theta$, $z_j \in \widetilde{X}_A^\theta$, $\lim_{t\to 0} e^{-tA}z_j = z_j$, $j = 1,\dots,n$, $h \in C([0,T];X) \cap B([0,T];\widetilde{X}_A^\theta)$, $\lim_{\tau\to 0} e^{-\tau A}h(t) = h(t)$ for every $t \in [0,T]$, $g_j \in C^1([0,T];\mathbb{C})$, $\Phi_j \in X^*$, $\Phi_j[Mu_0] = g_j(0)$, $j = 1,\dots,n$, and (5.4) holds. Then, problem (6.7)–(6.9) admits a unique solution $(u, f_1,\dots,f_n)$ such that*

$$Mu \in C^1([0,T];X), \quad f_1,\dots,f_n \in C([0,T];\mathbb{C}),$$
$$Lu \in C^{(2\alpha+2\beta-4+\theta)/\alpha}([0,T];X) \cap B([0,T];X_A^{(2\alpha+2\beta-4+\theta)/\alpha}).$$

Remark 6.5. When L is the realization in $L^2(\Omega)$ of a second order strongly elliptic linear differential operator $\mathcal{L}$ with the Dirichlet boundary condition in a bounded domain Ω and M is the multiplication operator by a function belonging to $L^\infty(\Omega)$ one has $\alpha = 1$, $\beta = 1/2$. Hence the assumption $2\alpha + \beta + \theta > 3$ of Theorems 6.1 and 6.3 is not satisfied for $\theta = 1/2$, and the assumption $\theta > 2(2-\alpha-\beta)$ of Theorems 6.2 and 6.4 is not satisfied for $\theta \in (0,1)$. A treatment of this case is given in [7]. Furthermore, owing to the inclusion relations $D(\widetilde{A}) \subset L^2(\Omega)_A^{1/2} \subset (L^2(\Omega), D(A))_{1/2,\infty}$ the assumptions are described by using a clearer space $D(\widetilde{A})$ than $\widetilde{L}^2(\Omega)_A^{1/2}$, $(L^2(\Omega), D(A))_{1/2,\infty}$ in [7], where $\widetilde{A} = \widetilde{L}M^{-1}$ and $\widetilde{L}$ is the realization of $\mathcal{L}$ in $H^{-1}(\Omega) = H_0^1(\Omega)^*$.

7. Problems for systems

Let us consider the following inverse problem: Recover y_i, f_{ij}, $i = 1,\dots,n$, $j = 1,\dots,N$, such that

$$\begin{gathered} y_1' = A_1y_1 + B_{11}y_1 + \dots + B_{1n}y_n + f_{11}(t)z_1 + \dots + f_{1N}(t)z_N + h_1(t), \\ \dots \\ y_n' = A_ny_n + B_{n1}y_1 + \dots + B_{nn}y_n + f_{n1}(t)z_1 + \dots + f_{nN}(t)z_N + h_n(t), \end{gathered} \tag{7.1}$$

$$y_1(0) = y_{10},\dots,y_n(0) = y_{n0}, \tag{7.2}$$

$$\Phi_j[y_i(t)] = g_{ji}(t), \quad i = 1,\dots,n; j = 1,\dots,N. \tag{7.3}$$

Assume $0 < \beta \le \alpha \le 1$ and $2\alpha+\beta+\theta > 3$. It is also assumed that for $i,j = 1,\dots,n$, A_i and B_{ij} satisfy

$$\|(\lambda - A_i)^{-1}\|_{\mathcal{L}(X)} \le \frac{C}{(1+|\lambda|)^\beta} \quad \text{for } \lambda \in \Sigma_\alpha = \{\lambda in\mathbb{C}; \operatorname{Re}\lambda \ge -c(1+|\operatorname{Im}\lambda|)^\alpha\},$$
$$B_{ij} \in \mathcal{L}(D(A_j), X_{A_i}^\theta).$$

Also assume that $y_{i0} \in D(A_i)$, $A_i y_{i0} \in (X, D(A_i))_{\theta,\infty}$, $z_j \in \cap_{k=1}^n (X, D(A_k))_{\theta,\infty}$, $h_i \in C([0,T]; X) \cap B([0,T]; (X, D(A_i))_{\theta,\infty}$, $g_{ji} \in C^1([0,T]; \mathbb{C})$, $\Phi_j \in X^*$, $\Phi_j[y_{i0}] = g_{ji}(0)$, $i = 1, \dots, n$, $j = 1, \dots, N$. Set

$$y = \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & A_n \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & \dots & B_{1n} \\ \dots & & \dots \\ B_{n1} & \dots & B_{nn} \end{pmatrix}.$$

Then A and $A+B$ generate infinitely differentiable semigroups in X^n. The system (7.1) is written as

$$\begin{aligned} y' = (A+B)y + f_{11}(t) \begin{pmatrix} z_1 \\ 0 \\ \dots \\ 0 \end{pmatrix} + f_{12}(t) \begin{pmatrix} z_2 \\ 0 \\ \dots \\ 0 \end{pmatrix} + \dots + f_{1N}(t) \begin{pmatrix} z_N \\ 0 \\ \dots \\ 0 \end{pmatrix} \\ + f_{21}(t) \begin{pmatrix} 0 \\ z_1 \\ \dots \\ 0 \end{pmatrix} + f_{22}(t) \begin{pmatrix} 0 \\ z_2 \\ \dots \\ 0 \end{pmatrix} + \dots + f_{2N}(t) \begin{pmatrix} 0 \\ z_N \\ \dots \\ 0 \end{pmatrix} + \dots \\ + f_{n1}(t) \begin{pmatrix} 0 \\ 0 \\ \dots \\ z_1 \end{pmatrix} + f_{n2}(t) \begin{pmatrix} 0 \\ 0 \\ \dots \\ z_2 \end{pmatrix} + \dots + f_{nN}(t) \begin{pmatrix} 0 \\ 0 \\ \dots \\ z_N \end{pmatrix} + \begin{pmatrix} h_1(t) \\ h_2(t) \\ \dots \\ h_n(t) \end{pmatrix}. \end{aligned}$$

Theorem 6.3 applies provided that

$$\det \begin{pmatrix} \Phi_1[z_1] & \dots & \Phi_1[z_N] \\ \dots & & \dots \\ \Phi_N[z_1] & \dots & \Phi_N[z_N] \end{pmatrix} \neq 0.$$

Indeed, the further information reduces to the N linear systems in the unknowns $f_{i1}, \dots, f_{iN}, i = 1, \dots, n$, whose determinant is just the indicated above. Identification problem (7.1)–(7.3) admits a unique solution $y = (y_1, \dots, y_n)^t$, f_{ij}, $i = 1, \dots, n$, $j = 1, \dots, N$ such that

$$\begin{aligned} & y \in C^1([0,T]; X^n), \quad f_{ij} \in C^1([0,T]; \mathbb{C}), \quad i = 1, \dots, n, \ j = 1, \dots, N, \\ & (A+B)y \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T]; X^n) \cap B([0,T]; X_{A+B}^{(2\alpha+\beta-3+\theta)/\alpha}) \\ & \subset C^{(2\alpha+\beta-3+\theta)/\alpha}([0,T]; X^n) \cap \big[B([0,T]; (X, D(A_1))_{(2\alpha+\beta-3+\theta)/\alpha,\infty}) \\ & \times \dots \times B([0,T]; (X, D(A_n))_{(2\alpha+\beta-3+\theta)/\alpha,\infty})\big]. \end{aligned}$$

Example. Let us consider the inverse problem

$$\frac{d}{dt}(A+1)y + Ay = \int_0^t k(t-s)Ay(s)ds + f(t)z + h(t), \quad 0 \le t \le T, \tag{7.4}$$

$$(A+1)y(0) = (A+1)y_0, \tag{7.5}$$

$$\Phi[(A+1)y(t)] = \Phi[g(t)], \quad 0 \le t \le T. \tag{7.6}$$

We suppose that -1 is a simple pole for the resolvent of A, i.e. $(A+1+\lambda)^{-1}$ exists for $0 < |\lambda| \le \varepsilon$ and

$$\|(A+1+\lambda)^{-1}\|_{\mathcal{L}(X)} \le \frac{C}{|\lambda|}.$$

A change of variable $y(t) = e^{\kappa t}x(t)$ transforms equation (7.4) into

$$\begin{aligned}&\frac{d}{dt}(A+1)x(t) + \kappa(A+1)x(t) + Ax(t)\\&= \int_0^t k(t-s)e^{-\kappa(t-s)}Ax(s)ds + f_1(t)z + e^{-\kappa t}h(t),\end{aligned}$$

where $f_1(t) = e^{-\kappa t}f(t)$. Now

$$\begin{aligned}&\lambda(A+1) + \kappa(A+1) + A = (\lambda+\kappa+1)A + \lambda + \kappa\\&= (\lambda+\kappa+1)\Big(A + \frac{\lambda+\kappa}{\lambda+\kappa+1}\Big) = (\lambda+\kappa+1)\Big(A + 1 - \frac{1}{\lambda+\kappa+1}\Big).\end{aligned}$$

Hence if $0 < |\lambda+\kappa+1|^{-1} < \varepsilon$, i.e. $|\lambda+\kappa+1| > \varepsilon^{-1}$, $\lambda(A+1)+\kappa(A+1)+A$ has a bounded inverse. Take κ so large that $\kappa+1 > \varepsilon^{-1}$. Then $(A+1-\frac{1}{\lambda+\kappa+1})^{-1} \in \mathcal{L}(X)$ exists for $\lambda \in \mathbb{C} \setminus S(-1-\kappa, \varepsilon^{-1})$ and

$$\begin{aligned}&(A+1)(\lambda(A+1) + \kappa(A+1) + A)^{-1}\\&= (A+1)(\lambda+\kappa+1)^{-1}\big(A+1-\frac{1}{\lambda+\kappa+1}\big)^{-1}\\&= (\lambda+\kappa+1)^{-1}\big\{1 + \frac{1}{\lambda+\kappa+1}\big(A+1-\frac{1}{\lambda+\kappa+1}\big)^{-1}\big\},\end{aligned}$$

so that

$$\|(A+1)(\lambda(A+1)+\kappa(A+1)+A)^{-1}\|_{\mathcal{L}(X)} \le C|\lambda+\kappa+1|^{-1}$$

for $|\lambda+\kappa+1| > \varepsilon^{-1}$. Hence the previous results (See also Favini and Tanabe [5]) apply for $\alpha = \beta = 1$.

However, this pole case allows a better treatment. First of all the change of variable $y = e^{-t}x$ transforms the given problem (7.4)–(7.6) into

$$\frac{d}{dt}(A+1)x(t) - x(t) = \int_0^t k_1(t-s)Ax(s)ds + f_1(t)z + h_1(t), \tag{7.7}$$

$$(A+1)x(0) = (A+1)y_0, \tag{7.8}$$

$$\Phi[(A+1)x(t)] = g_1(t). \tag{7.9}$$

where $k_1(t) = e^t k(t)$, $f_1(t) = e^t f(t)$, $h_1(t) = e^t h(t)$, $g_1(t) = e^t g(t)$. If -1 is a simple pole for $(A-\lambda)^{-1}$, so that

$$X = N(A+1) \oplus R(A+1),$$

and P denotes the projection onto $N(A+1)$, problem (7.7)–(7.9) reduces to

$$\begin{aligned}&\frac{d}{dt}(A+1)(1-P)x(t) - (1-P)x(t)\\&= \int_0^t k_1(t-s)A(1-P)x(s)ds + f_1(t)(1-P)z + (1-P)h_1(t),\end{aligned} \tag{7.10}$$

$$(A+1)(1-P)x(0) = (A+1)(1-P)y_0, \tag{7.11}$$

$$\Phi[(A+1)(1-P)x(t)] = g_1(t), \tag{7.12}$$

$$\begin{aligned}-Px(t) &= \int_0^t k_1(t-s)P(A+1-1)x(s)ds + f_1(t)Pz + Ph_1(t)\\&= -\int_0^t k_1(t-s)Px(s)ds + f_1(t)Pz + Ph_1(t).\end{aligned} \tag{7.13}$$

Since the restriction of $A+1$ to $R(A+1)$ is boundedly invertible, the change of variable $(A+1)(1-P)x(t) = \xi(t)$ transforms (7.10)–(7.12) into

$$\frac{d}{dt}\xi(t) - R\xi(t) = \int_0^t k_1(t-s)[1-R]\xi(s)ds + f_1(t)(1-P)z + (1-P)h_1(t), \tag{7.14}$$

$$\xi(0) = (A+1)(1-P)y_0, \tag{7.15}$$

$$\Phi[\xi(t)] = g_1(t), \tag{7.16}$$

where R indicates the inverse of the restriction of $A+1$ to $R(A+1)$. Therefore, if k is continuous in $[0,T]$, $\Phi[(1-P)z] \neq 0$, $h \in C([0,T];X)$, $(1-P)y_0 \in D(A)$, $g \in C^1([0,T];\mathbb{C})$, problem (7.14)–(7.16) admits a unique strict solution (ξ, f_1). Hence, we have a unique strict solution $((1-P)x, f_1)$ to (7.10)–(7.12). Since f_1 is now known, we only remain to solve integral equation (7.13), that is uniquely solvable. Notice that this improves the preceding result, since condition $\Phi[z] \neq 0$ is replaced by the weaker condition $\Phi[(1-P)z] \neq 0$.

References

[1] A. Favini, A. Lorenzi, G. Marinoschi, H. Tanabe; *Perturbation methods and identification problems for degenerate evolution systems*, contribution to the Seventh Congress of Romanian Mathematicians, Brasov, 2011, Eds. L. Beznea, V. Brinzanescu, M. Iosifescu, G. Marinoschi, R. Purice, D. Timotin, Publishing House of the Romanian Academy of Science, 88-96, 2013.

[2] A. Favini, A. Lorenzi, H. Tanabe: *Direct and inverse problems for systems of singular differential boundary value problems*, Electronic Journal of Diffential Equations, **2012** (2012), 1-34.

[3] A. Favini, A. Lorenzi, H. Tanabe; *Degenerate integrodifferential equations of parabolic type with Robin boundary conditions: L^p-theory*, preprint.

[4] A. Favini, A. Lorenzi, H. Tanabe, A. Yagi; *An L^p-approach to singular linear parabolic equations with lower order terms*, Discrete and Continuous Dynamical Systems, 22 No.4, 989-1008.

[5] A. Favini, H. Tanabe; *Degenerate differential equations of parabolic type and inverse problems*, Proceedings of Seminar on Partial Differential Equations in Osaka 2012, Osaka University, August 20-24, 2012, 89-100.

[6] A. Favini and A. Yagi; *Degenerate Differential Equations in Banach spaces*, Marcel Dekker, Inc., New York-Barsel-Hong Kong, 1999.

[7] H. Tanabe; *Identification problem for degenerate parabolic equations*, Proceedings of Seminar on Partial Differential Equations in Osaka 2012, Osaka University, August 20-24, 2012, 83-88.

[8] A. Prilepko, D. G. Orlovsky, I. Vasin; *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, Inc., New York-Barsel-Hong Kong, 2000.

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