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APPROXIMATING SOLUTIONS OF NONLINEAR PBVPS OF SECOND-ORDER DIFFERENTIAL EQUATIONS VIA HYBRID FIXED POINT THEORY

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Abstract. In this article we prove the existence and approximations of solutions of periodic boundary-value problems of second-order ordinary nonlinear hybrid differential equations. We rely our results on Dhage iteration principle or method embodied in a recent hybrid fixed point theorem of Dhage (2014) in partially ordered normed linear spaces. Our resutls are proved under weaker continuity and Lipschitz conditions. An example illustrates the theory developed in this article.

1. Statement of the problem

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ for some $T > 0$, consider the periodic boundary value problem (in short PBVP) of second-order ordinary nonlinear hybrid differential equation (in short HDE),

$$\begin{gathered} x''(t) = f(t, x(t)) + g(t, x(t)), \\ x(0) = x(T), \quad x'(0) = x'(T), \end{gathered} \tag{1.1}$$

for all $t \in J$, where $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *solution* of the HDE (1.1) we mean a function $x \in C^2(J, \mathbb{R})$ that satisfies (1.1), where $C^2(J, \mathbb{R})$ is the space of twice continuously differentiable real-valued functions defined on J.

The HDE (1.1) is a hybrid differential equation with a linear perturbation of first type and can be tackled with the hybrid fixed point theory (cf. Dhage [1, 2]). The existence theorems proved via classical fixed point theorems on the lines of Krasnoselskii [10] requires the condition that the nonlinearities involved in (1.1) to satisfy a strong Lipschitz and compactness type conditions and do not yield any algorithm to find the numerical solutions. Very recently, Dhage and Dhage [7] relaxed the above conditions and proved the existence as well as algorithms for the initial and periodic boundary value problems of nonlinear second order differential equations. The similar study is continued in Dhage et al [8] for the initial value problems of hybrid differential equations. However, we do not find any work in the literature for hybrid PBVPs along this line. This is the main motivation of this

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article and it is proved that the existence as well as algorithm of the solutions may be proved for periodic boundary value problems of nonlinear second-order ordinary differential equations under weaker partially continuity and partially compactness type conditions.

The article is organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we establish the main existence result and we provide an example to illustrate our main result.

2. Auxiliary results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is *regular* if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [9] and the references therein. We need the following definitions in the sequel.

Definition 2.1. A mapping $\mathcal{T} : E \to E$ is called *isotone* or *monotone nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called *monotone nonincreasing* if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called *monotonic* or simply *monotone* if it is either monotone nondecreasing or monotone nonincreasing on E..

An operator $\mathcal{T}$ on a normed linear space E into itself is called *compact* if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called *totally bounded* if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called *completely continuous* on E.

Definition 2.2 (Dhage [3])**.** A mapping $\mathcal{T} : E \to E$ is called *partially continuous* at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3 (Dhage [2, 3])**.** An operator $\mathcal{T}$ on a partially normed linear space E into itself is called *partially bounded* if $T(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called *uniformly partially bounded* if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called *partially compact* if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called *partially totally bounded* if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on E.

Remark 2.4. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.5 (Dhage [2]). The order relation $\preceq$ and the metric d on a non-empty set E are said to be *compatible* if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-Lipschitz if there exists an upper semi-continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x-y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $\psi(0) = 0$. If $\psi(r) = kr$, $k > 0$, then $\mathcal{T}$is called a partially Lipschitz with a Lipschitz constant k. If $k < 1$, $\mathcal{T}$ is called a partially contraction with contraction constant k. Finally, $\mathcal{T}$ is called nonlinear $\mathcal{D}$-contraction if it is a nonlinear $\mathcal{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$.

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage [2, 3, 4, 5] may be formulated as *"monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation"* and which is a powerful tool in the existence theory of nonlinear analysis. It is clear that Dhage iteration method is different from the usual Picard's successive iteration method and embodied in the following applicable hybrid fixed point theorems proved in Dhage [4] which forms a useful key tool for our work contained in this paper. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [2, 3, 4, 5, 6].

Theorem 2.7 (Dhage [3]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in E. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \dots$, converges monotonically to x^*.

Remark 2.8. The conclusion of Theorem 2.7 also remains true if we replace the compatibility of E with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ by a weaker condition of the compatibility of every compact chain C in E with respect to the order relation $\preceq$ and the norm $\|\cdot\|$. The later condition holds in particular if every partially compact subset of E possesses the compatibility property.

3. Main results

The equivalent integral formulation of the HDE (1.1) is considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)|, \tag{3.1}$$

$$x \le y \iff x(t) \le y(t) \tag{3.2}$$

for all $t\in J$. Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered with respect to the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ has some nice properties with respect to the above order relation in it. The following lemma follows by an application of Arzela-Ascolli theorem.

Lemma 3.1. *Let $(C(J,\mathbb{R}),\le,\|\cdot\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by* (3.1) *and* (3.2) *respectively. Then $\|\cdot\|$ and $\le$ are compatible in every partially compact subset of $C(J,\mathbb{R})$.*

Proof. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \dots \le x_n(t) \le \cdots, \tag{$*$}$$

for each $t\in\mathbb{R}_+$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t\in\mathbb{R}_+$. This shows that the sequence $\{x_n(t)\}$ converges to $x(t)$ point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\le$ are compatible in S. This completes the proof. □

Definition 3.2. A function $u\in C^2(J,\mathbb{R})$ is said to be a lower solution of the HDE (1.1) if it satisfies

$$\begin{gathered} u''(t) \le f(t,u(t)) + g(t,u(t)),\\ u(0)\le u(T), \quad u'(0)\le u'(T), \end{gathered} \tag{3.3}$$

for all $t\in J$. Similarly, an upper solution $v\in C^2(J,\mathbb{R})$ for the HDE (1.1) is defined on J.

We consider the following set of assumptions:

(A1) There exist constants $\lambda>0$ and $\mu>0$, with $\lambda\ge\mu$, such that
$$0\le [f(t,x)+\lambda x]-[f(t,y)+\lambda y]\le \mu(x-y),$$
for all $t\in J$ and $x,y\in\mathbb{R}$, $x\ge y$.

(B1) There exists a constant $k_2>0$ such that $|g(t,x)|\le k_2$ for all $t\in J$ and $x\in\mathbb{R}$.

(B2) $g(t,x)$ is nondecreasing in x for all $t\in J$.

(B3) The HDE (1.1) has a lower solution $u\in C^2(J,\mathbb{R})$.

Consider the PBVP of the HDE

$$\begin{gathered} x''(t) + \lambda x(t) = \tilde{f}(t, x(t)) + g(t, x(t)), \\ x(0) = x(T), \quad x'(0) = x'(T), \end{gathered} \tag{3.4}$$

for all $t \in J$, where $\tilde{f}, g : J \times \mathbb{R} \to \mathbb{R}$ and

$$\tilde{f}(t, x) = f(t, x) + \lambda x. \tag{3.5}$$

Remark 3.3. A function $u \in C^2(J, \mathbb{R})$ is a solution of the HDE (3.4) if and only if it is a solution of the HDE (1.1) defined on J.

Consider the following assumption.

(A2) There exists a constant $k_1 > 0$ such that $|\tilde{f}(t, x)| \leq k_1$ for all $t \in J$ and $x \in \mathbb{R}$.

The following useful lemma may be found in Torres [11].

Lemma 3.4. *For any $h \in L^1(J, \mathbb{R}^+)$ and $\sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation*

$$\begin{gathered} x''(t) + h(t)x(t) = \sigma(t), \quad t \in J, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{gathered} \tag{3.6}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_h(t, s)\sigma(s)\, ds, \tag{3.7}$$

where, $G_h(t, s)$ is a Green's function associated with the homogeneous PBVP

$$\begin{gathered} x''(t) + h(t)x(t) = 0, \quad t \in J, \\ x(0) = x(T), \quad x'(0) = x'(T). \end{gathered} \tag{3.8}$$

Notice that the Green's function G_h is continuous an nonnegative on $J \times J$ and therefore, the number

$$M_h := \max\{|G_h(t, s)| : t, s \in [0, T]\}$$

exists for all $h \in L^1(J, \mathbb{R}^+)$.

As an application of Lemma 3.4 we obtain the following result.

Lemma 3.5. *Suppose that hypotheses* (A2) *and* (B1) *hold. Then a function $u \in C(J, \mathbb{R})$ is a solution of the HDE* (3.4) *if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \int_0^T G(t, s)\tilde{f}(s, x(s))\, ds + \int_0^T G(t, s)g(s, x(s))\, ds \tag{3.9}$$

for all $t \in J$, where $G(t, s)$ is a Green's function associated with the homogeneous PBVP

$$\begin{gathered} x''(t) + \lambda x(t) = 0, \quad t \in J, \\ x(0) = x(T), \quad x'(0) = x'(T). \end{gathered} \tag{3.10}$$

Theorem 3.6. *Assume that hypotheses* (A1)-(A2) *and* (B1)-(B3) *hold. Furthermore, if $\lambda MT < 1$, then the HDE* (1.1) *has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = \int_0^T G(t, s)\tilde{f}(s, x_n(s))\, ds + \int_0^T G(t, s)g(s, x_n(s))\, ds \tag{3.11}$$

for all $t \in J$, where $x_0 = u$ converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then by Lemma 3.1, every compact chain in E is compatible with respect to the norm $\|\cdot\|$ and order relation $\leq$. Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds, \quad t \in J, \tag{3.12}$$

$$\mathcal{B}x(t) = \int_0^T G(t,s)g(s,x(s))\,ds, \quad t \in J. \tag{3.13}$$

From the continuity of the integrals, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A}, \mathcal{B} : E \to E$. Now, by Lemma 3.5, the HDE (3.4) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \quad t \in J. \tag{3.14}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.7. This is achieved in the series of following steps.

Step I: $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E. Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A1), we obtain

$$\begin{aligned}
\mathcal{A}x(t) &= \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \\
&\geq \int_0^T G(t,s)\tilde{f}(s,y(s))\,ds \\
&= \mathcal{A}y(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E. Similarly using hypothesis (B2), it is shown that $\mathcal{B}$ is also nondecreasing on E into itself. Thus $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E into itself.

Step II: $\mathcal{A}$ is a partially bounded and partially contraction operator on E. Let $x \in E$ be arbitrary. Then by (A2),

$$\begin{aligned}
|\mathcal{A}x(t)| &\leq \Big|\int_0^T G(t,s)|\tilde{f}(s,x(s))|\,ds\Big| \\
&\leq \int_0^T G(t,s)k_1\,ds \\
&\leq Mk_1T
\end{aligned}$$

for all $t \in J$. Taking the supremum over t in above inequality, we obtain $\|\mathcal{A}x\| \leq k_1MT$, and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &= \Big|\int_0^T G(t,s)[\tilde{f}(s,x(s)) - \tilde{f}(s,y(s))]\,ds\Big| \\
&\leq \int_0^T G(t,s)\mu(x(s) - y(s))\,ds \\
&\leq \int_0^T G(t,s)\lambda|x(s) - y(s)|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^T \lambda M \|x-y\|\,ds \\
&= \lambda M T \|x-y\|,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t in above inequality, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \le \alpha \|x-y\|,$$

for all $x, y \in E$ with $x \ge y$, where $0 \le \alpha = \lambda M T < 1$. Hence $\mathcal{A}$ is a partially contraction on E which further implies that $\mathcal{A}$ is a partially continuous on E.

Step III: $\mathcal{B}$ is a partially continuous operator on E. Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_0^T G(t,s) g(s, x_n(s))\,ds \\
&= \int_0^T G(t,s) \big[\lim_{n\to\infty} g(s, x_n(s)) \big]\,ds \\
&= \int_0^T G(t,s) g(s, x(s))\,ds \\
&= \mathcal{B}x(t),
\end{aligned}
$$

for all $t \in J$. This shows that Bx_n converges to Bx pointwise on J.

Next, we show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$
\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &= \Big| \int_0^T G(t_1,s) g(s, x_n(s)) ds - \int_0^T G(t_2,s) g(s, x_n(s)) ds \Big| \\
&\le \Big| \int_0^T |G(t_1,s) - G(t_2,s)| |g(s, x_n(s))|\,ds \Big| \\
&\le \int_0^T |G(t_1,s) - G(t_2,s)| k_2\,ds \\
&\to 0 \quad \text{as } t_2 - t_1 \to 0
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $Bx_n \to Bx$ is uniformly and hence B is partially continuous on E.

Step IV: $\mathcal{B}$ is a partially compact operator on E. Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then

$$
\begin{aligned}
|\mathcal{B}x(t)| &= \Big| \int_0^T G(t,s) g(s, x(s))\,ds \Big| \\
&\le \int_0^T G(t,s) |g(s, x(s))|\,ds \\
&\le \int_0^T M k_2\,ds \\
&\le M k_2 T = r,
\end{aligned}
$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{B}x\| \le r$ for all $x \in C$. Hence $\mathcal{B}$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| &= \Big| \int_0^T [G(t_1,s) - G(t_2,s)] g(s,x(s))\,ds \Big| \\ &\le \int_0^T |G(t_1,s) - G(t_2,s)||g(s,x(s))|\,ds \\ &\le \int_{t_0}^T |G(t_1,s) - G(t_2,s)|\,k_2\,ds \\ &\to 0 \quad \text{as } t_1 \to t_2\end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{B}(C)$ is a compact subset of E and consequently $\mathcal{B}$ is a partially compact operator on E into itself.

Step V: u satisfies the operator inequality $u \le \mathcal{A}u + \mathcal{B}u$. By hypothesis (H4), the PBVP (2.1) has a lower solution u. Then we have

$$\begin{gathered} u''(t) \le f(t,u(t)) + g(t,u(t)), \quad t \in J, \\ u(0) \le u(T), \quad u'(0) \le u'(T). \end{gathered} \tag{3.15}$$

Integrating (3.15) twice which together with the definition of the operator $\mathcal{T}$ implies that $u(t) \le \mathcal{T}u(t)$ for all $t \in J$. See Heikkilä and Lakshmikantham [9, lemma 4.5.1] and references therein. Consequently, u is a lower solution to the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.7 with $x_0 = u$ and we apply it to conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution. Consequently the integral equation and the HDE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.4) converges monotonically to x^*. This completes the proof. □

Remark 3.7. The conclusion of Theorem 3.6 also remains true if we replace the hypothesis (B3) with the following one:

(B3') The HDE (1.1) has an upper solution $v \in C^2(J, \mathbb{R})$.

Example 3.8. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the PBVP of HDE,

$$\begin{gathered} x''(t) = \tan^{-1} x(t) - x(t) + g(t,x(t)), \\ x(0) = x(1), \quad x'(0) = x'(1), \end{gathered} \tag{3.16}$$

for all $t \in J$, where $g : J \times \mathbb{R} \to \mathbb{R}$ is defined as

$$g(t,x) = \begin{cases} 1, & \text{if } x \le 1, \\ \frac{2x}{1+x}, & \text{if } x > 1. \end{cases}$$

Here, $f(t,x) = \tan^{-1} x - x$. Clearly, the functions f and g are continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (A1) with $\lambda = 1 > \mu$. To see this, we have

$$0 \le \tan^{-1} x - \tan^{-1} y \le \frac{1}{1+\xi^2}(x-y)$$

for all $x, y \in \mathbb{R}$, $x \ge y$, where $x > \xi > y$. Therefore, $\lambda = 1 > \frac{1}{1+\xi^2} = \mu$. Moreover, the function $\tilde{f}(t,x) = \tan^{-1} x$ is bounded on $J \times \mathbb{R}$ with bound $k_1 = \frac{\pi}{2}$ and so the hypothesis (A2) is satisfied.

Again, since g is bounded on $J \times \mathbb{R}$, by 1, the hypothesis (B1) holds. Furthermore, $g(t,x)$ is nondecreasing in x for all $t \in J$, and thus hypothesis (B2) is satisfied. Finally the HDE (3.16) has a lower solution

$$u(t) = -2\int_0^1 G(t,s)\,ds + \int_0^1 G(t,s)\,ds,$$

defined on J. Thus all hypotheses of Theorem 3.6 are satisfied in view of Remark 2.8. Hence we apply Theorem 3.6 and conclude that the PBVP (3.16) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \int_0^1 G(t,s)\tan^{-1} x_n(s)\,ds + \int_0^1 G(t,s)g(s,x_n(s))\,ds, \tag{3.17}$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 3.9. in view of Remark 3.7, the existence of the solutions x^* of the PBVP (3.16) may be obtained under the upper solution

$$v(t) = 2\int_0^1 G(t,s)\,ds + 2\int_0^1 G(t,s)\,ds,$$

defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \int_0^1 G(t,s)\tan^{-1} x_n(s)\,ds + \int_0^1 G(t,s)g(s,x_n(s))\,ds, \tag{3.18}$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^*.

Conclusion

From the foregoing discussion it is clear that unlike Krasnoselskii fixed point theorem, the proof of Theorem 3.6 does not invoke the construction of a non-empty, closed, convex and bounded subset of the Banach space of navigation which is mapped into itself by the operators related to the given differential equation. The convexity hypothesis is altogether omitted from the discussion and still we have proved the existence of the solutions for the differential equation considered in this article. Similarly, unlike the use of Banach fixed point theorem, Theorems 3.6 does not make any use of any type of Lipschitz condition on the nonlinearities involved in the PBVP (1.1), but even then we proved the algorithms for the solutions of the hybrid differential equation (1.1) in terms of the Picard's iteration scheme. The limitation of the our result lies in the fact that the convergence of the algorithms are not geometrical and so there is no way to obtain the rate of convergence of the algorithms to the solutions of the related problems. However, by a way we have been able to prove the existence results for the PBVP (1.1) under much weaker conditions with strong conclusion of the monotone convergence of successive approximations to the solutions than those proved in the existing literature on nonlinear hybrid differential equations.

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