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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. This article shows the existence of solutions by the least action principle, for semilinear elliptic equations with Neumann boundary conditions, under critical growth and local coercive conditions. In the subcritical growth and local coercive case, multiplicity results are established by using the minimax methods together with a standard eigenspace decomposition.

1. Introduction and statement of main results

Since the 70s, several authors have studied the existence and multiplicity of solutions for the Neumann boundary-value problem

$$\begin{gathered} -\Delta u = f(x,u) + h(x) \quad \text{for a.e. } x \in \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary and outer normal vector $n = n(x)$, $\partial u/\partial n = n(x) \cdot \nabla u$. The function $f : \bar{\Omega} \times R \longrightarrow R$ is a Caratheodory function with $F(x,u) = \int_0^u f(x,s)ds$ as its primitive. And then, for (1.1), a vast of literature related to the solvability conditions has been published. It has been showed that there is at least one solution for (1.1) under the assumptions of the periodicity condition, see[13], or the monotonicity condition, see[10, 11], or the sign condition, see[3, 5], or the Landesman-Lazer type condition, see[6, 7], or a new Landesman-Lazer type condition and sublinear condition, see[14, 15]. At the same time, some authors studied multiplicity of solutions for (1.1), see[2, 16, 17], some authors obtained sign-changing solutions, see[8, 9]. In either case, existence or multiplicity of solutions, even sign-changing solutions, the main methods are the dual least action principle and the minimax methods respectively.

In this paper, under the critical growth and local coercive condition, we obtain the existence theorem by the least action principle for (1.1). What's more, in the subcritical growth and local coercive case, multiplicity results are established by using the minimax methods, in particular, a three-critical-point theorem proposed by Brezis and Nirenberg [1]. A contribution in this direction is [18], where the

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authors use the local coercive condition to study the second order Hamiltonian systems by variational method. We study (1.1) under the following assumptions:

(H1) There exist a constant $C_1 > 0$ and a real function $\gamma \in L^1(\Omega)$ such that
$$|f(x,t)| \le C_1|t|^{2^*-1} + \gamma(x)$$
for all $t \in R$ and a.e. $x \in \Omega$, where
$$2^* = \begin{cases} \frac{2N}{N-2}, & N \ge 3 \\ \text{any value} & q \in (2,+\infty),\ N = 1,2 \end{cases}$$

(H1') There exist $C_2 > 0$ and $2 < p < 2^*$ such that
$$|f(x,t)| \le C_2(|t|^{p-1} + 1)$$
for all $t \in R$ and a.e. $x \in \Omega$.

(H2) There exists a subset E of Ω with $\mathrm{meas}(E) > 0$ such that $F(x,t) \to -\infty$ as $|t| \to \infty$, uniformly for a.e. $x \in E$.

(H3) There exists $g \in L^1(\Omega)$ such that $F(x,t) \le g(x)$ for all $t \in R$ and a.e. $x \in \Omega$.

(H4) There exists $h \in L^{2^{*'}}(\Omega)$ such that
$$\int_\Omega h(x)dx = 0.$$
where $2^{*'}$ is the conjugate exponent of 2^*, that is, $\frac{1}{2^{*'}} + \frac{1}{2^*} = 1$.

(H5) There exist $\delta > 0$ and an integer $m \ge 1$ such that
$$\mu_m \le \frac{f(x,t)}{t} \le \mu_{m+1}$$
for all $0 < |t| \le \delta$, and a.e. $x \in \Omega$, where
$$0 = \mu_1 < \mu_2 \le \dots \le \mu_m \le \mu_{m+1} \le \dots, \quad \mu_m \to \infty$$
is the sequence of eigenvalues in $H^1(\Omega)$ for $-\Delta$ with Neumann boundary condition.

Our main results read as follows.

Theorem 1.1. *Under hypotheses* (H1)–(H4)*, Problem* (1.1) *has at least one solution in the Sobolev space $H^1(\Omega)$.*

Theorem 1.2. *If $h = 0$, under hypotheses* (H1'), (H2), (H3), (H5)*, Problem* (1.1) *has at least two nonzero solutions in $H^1(\Omega)$.*

Remark 1.3. Theorem 1.1 generalizes [16, Theorem 1] because that conditions (H2) and (H3) are weaker than [16, condition (3)]. There are functions $f(x,t)$ and $h(x)$ satisfying our Theorem 1.1 and not satisfying the corresponding results in [2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17]. In fact, let
$$f(x,t) = -(x-x_0)\frac{2t}{1+t^2} + 2^*|t|^{2^*-2}t\cos|t|^{2^*}$$
and $h \in L^{2^{*'}}(\Omega)$ satisfying (H4), where $x_0 \in \bar{\Omega}$. A direct computation shows that
$$F(x,t) = -(x-x_0)\ln(1+t^2) + \sin|t|^{2^*}$$
satisfies (H1), (H2) and (H3). But $f(x,t)$ does not satisfy the conditions in [2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17].

Remark 1.4. Obviously, Theorem 1.2 generalizes [16, Theorem 2] because the local coercive condition (H2) and (H3) are weaker than [16, condition (3)] (2.2), and condition (H5) is weaker than [16, condition (7)]. Hence, we solve the open question posed in [16, Remark 4]. There are functions $f(x,t)$ satisfying our Theorem 1.2 and not satisfying the conditions in [2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17]. For example,

$$f(x,t)=\begin{cases}-(x-x_0)\frac{2t}{1+t^2}+C_3p|t|^{p-2}t\cos|t|^p, & |t|\geq\delta\\ [\mu_m\sin^2 t^{-2}+\mu_{m+1}(1-\sin^2 t^{-2})]t, & |t|\leq\delta\\ 0, & t=0\end{cases}$$

where $x_0\in\bar\Omega$, $C_3>0$ and $2<p<2^*$.

2. Proof of main results

The methods to prove the theorems are variational basically based upon minimization of coercive lower semicontinuous functionals for Theorem 1.1, and minmax methods together with a standard eigenspace decomposition for Theorem 1.2.

To make the statements precise, let us introduce some notation. The Sobolev space $H^1(\Omega)$ is the usual space of $L^2(\Omega)$ functions with weak derivative in $L^2(\Omega)$, endowed with the norm

$$\|u\|_*=(|\bar u|^2+\int_\Omega|\nabla u(x)|^2dx)^{1/2}$$

where

$$\bar u=(\operatorname{meas}\Omega)^{-1}\int_\Omega u(x)dx,$$

or the norm defined by

$$\|u\|=\Big(\int_\Omega|u(x)|^2dx+\int_\Omega|\nabla u(x)|^2dx\Big)^{1/2}$$

for all $u\in H^1(\Omega)$. The two norms $\|u\|$ and $\|u\|_*$ are equivalent. In fact, Poincaré-Wirtinger's inequality asserts that

$$\int_\Omega|u-\bar u|^2dx\leq c_1\int_\Omega|\nabla u|^2dx$$

for some constant $c_1>0$. Hence, one has

$$\int_\Omega|u|^2dx\leq c_2(|\bar u|^2+\int_\Omega|\nabla u|^2dx)$$

for some constant $c_2>0$, which implies $\|u\|\leq c_3\|u\|_*$ for some constant $c_3>0$. On the other hand, Hölder inequality leads to

$$\bar u=(\operatorname{meas}\Omega)^{-1}\int_\Omega u(x)dx\leq\|u\|_{L^2}$$

Thus, we obtain $\|u\|_*\leq c_4\|u\|$ for some constant $c_4>0$. That is, the two norms $\|u\|$ and $\|u\|_*$ are equivalent.

It is well known that, by Sobolev's inequality, there exists a constant $C>0$ such that

$$\|u\|_{L^1(\Omega)}\leq C\|u\|,\quad \|u\|_{L^{2^*}(\Omega)}\leq C\|u\|,\quad \|u\|_{L^p(\Omega)}\leq C\|u\| \tag{2.1}$$

where p is the same as in Theorem 1.2. Now, the functional φ on $H^1(\Omega)$ is given by

$$\varphi(u)=\frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx-\int_\Omega F(x,u(x))dx-\int_\Omega hu\,dx$$

for all $u\in H^1(\Omega)$. By the critical growth conditions (H1) or subcritical growth condition (H1'), we can easy prove that φ is continuously differentiable in $H^1(\Omega)$, in a way similar to [12, Theorem 1.4]. It is well known that finding solutions of (1.1) is equivalent to finding critical points of φ in $H^1(\Omega)$.

For the sake of convenience, we show C_i $(i=1,2,\dots,8)$ be positive constants. Before giving the proof of Theorem 1.1, we show the following lemmas.

Lemma 2.1 (The least action principle, [12, Theorem 1.1])**.** *Suppose that X is a reflexive Banach space and $\varphi:X\to R$ is weakly lower semi-continuous. Assume that φ is coercive; that is, $\varphi(u)\to+\infty$ as $\|u\|\to\infty$ for $u\in X$. Then φ has at least one minimum.*

Lemma 2.2. *Suppose that F satisfies assumption* (H1) *and* (H2)*. Then there exist a real function $\beta\in L^1(\Omega)$, and $G\in C(R,R)$ which is subadditive, that is,*

$$G(s+t)\le G(s)+G(t)$$

for all $s,t\in R$, and coercive, that is, $G(t)\to+\infty$ as $|t|\to\infty$ and satisfies

$$G(t)\le |t|+4$$

for all $t\in R$, such that

$$F(x,t)\le -G(t)+\beta(x)$$

for all $t\in R$ and a.e. $t\in E$.

The proof of Lemma 2.2 is essentially the same one as the introductory part of the proof of [16, Theorem 1].

Proof of Theorem 1.1. First, we prove that the functional φ is coercive. By Lemma 2.2, (H3) and (2.1) we obtain

$$\begin{aligned}
\int_\Omega F(x,u)dx&=\int_E F(x,u)dx+\int_{\Omega\setminus E}F(x,u)dx\\
&\le-\int_E G(u)dx+\int_E\beta(x)dx+\int_{\Omega\setminus E}g(x)dx\\
&\le-\int_E G(\bar u)dx+\int_E G(-\tilde u)dx+\int_E\beta(x)dx+\int_{\Omega\setminus E}g(x)dx\\
&\le-\operatorname{meas}E\cdot G(\bar u)+\int_E G(-\tilde u)dx+\int_\Omega|\beta(x)|dx+\int_\Omega|g(x)|dx\\
&\le-\operatorname{meas}EG(\bar u)+\int_E(|\tilde u|+4)dx+C_4\\
&\le-\operatorname{meas}EG(\bar u)+\|\tilde u\|_{L^1(\Omega)}+4\operatorname{meas}E+C_4\\
&\le\operatorname{meas}E(4-G(\bar u))+C\|\tilde u\|+C_4
\end{aligned}$$

for all $u\in H^1(\Omega)$, where $C_4=\int_\Omega|\beta(x)|dx+\int_\Omega|g(x)|dx$ and

$$\tilde u(x)=u(x)-\bar u.$$

Hence by the inequality above, Hölder inequality and (2.1) we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx - \int_\Omega hudx \\
&\geq \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2 dx + \operatorname{meas} E(G(\bar{u})-4) - C\|\tilde{u}\| - C_4 - \int_\Omega h\tilde{u}dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2 dx + (G(\bar{u})-4)\operatorname{meas} E - C\|\tilde{u}\| - C_4 - \|h\|_{L^{2^*\prime}(\Omega)}\|\tilde{u}\|_{L^{2^*}(\Omega)} \\
&\geq \frac{1}{2}\|\tilde{u}\|^2 + (G(\bar{u})-4)\operatorname{meas} E - C(1+\|h\|_{L^{2^*\prime}(\Omega)})\|\tilde{u}\| - C_4
\end{aligned}$$

for all $u \in H^1(\Omega)$. By Lemma 2.2, we know that $G(t) \to +\infty$ as $|t| \to \infty$, together with the fact that

$$\|\tilde{u}\|^2 + \|\bar{u}\|^2 = \|u\|^2,$$

it is easy to obtain φ is coercive.

Next, by (H3), in a way similar to the first part of the proof of [4, Theorem 1] or the part of the proof of [16, Theorem 1], we can easily prove the functional φ is weakly lower semicontinuous. Derived by the least action principle (see, Lemma 2.1), φ has a minimum. Hence (1.1) has at least one solution, which completes the proof. □

Next, we prove Theorem 1.2 by using the following three-critical-point theorem proposed by Brezis-Nirenberg [1].

Lemma 2.3 ([1]). *Let X be a Banach space with a direct sum decomposition*

$$X = X_1 \oplus X_2$$

with $\dim X_2 < \infty$ and let φ be a C^1 function on X with $\varphi(0) = 0$, satisfying the (PS) condition. Assume that, for some $\delta_0 > 0$,

$$\begin{aligned}
&\varphi(v) \geq 0, \quad \text{for } v \in X_1 \text{ with } \|v\| \leq \delta_0, \\
&\varphi(v) \leq 0, \quad \text{for } v \in X_2 \text{ with } \|v\| \leq \delta_0\,.
\end{aligned}$$

Assume also that φ is bounded from below and $\inf_X \varphi < 0$. Then φ has at least two nonzero critical points.

Proof of Theorem 1.2. Let $X = H^1(\Omega) = X_1 \oplus X_2$, where $X_2 = \oplus_{1\leq i\leq m} \ker(\Delta + \mu_i)$ is a finite dimension subspace and $X_1 = X_2^{\perp}$.

Obviously, φ is a C^1 function on $H^1(\Omega)$ with $\varphi(0) = 0$. Similar to the proof of the coercivity of φ in Theorem 1.1, by condition (H2), (H3) and (H1'), the subcritical growth condition, we can easily obtain that φ is coercive and bounded from below. Therefore, the functional φ satisfies the (PS) condition; that is, $\{u_n\}$ possesses a convergent subsequence if $\{u_n\}$ is a sequence of X such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$.

Firstly, we obtain that

$$\varphi(u) \leq 0, \quad \text{for } u \in X_2 \text{ with } \|u\| \leq \delta_0 \tag{2.2}$$

By (H5), we have

$$\mu_m t^2 \leq tf(x,t) \leq \mu_{m+1} t^2$$

for all $|t| \leq \delta$ and a.e.$x \in \Omega$. Hence, the following inequality holds

$$\mu_m t^2 s \leq tf(x,ts) \leq \mu_{m+1} t^2 s$$

for all $0 < s \le 1$, $|t| \le \delta$ and a.e. $x \in \Omega$. It follows from the fact that $F(x,t) = \int_0^1 tf(x,st)ds$,

$$\frac{1}{2}\mu_m t^2 \le F(x,t) \le \frac{1}{2}\mu_{m+1}t^2 \tag{2.3}$$

for all $|t| \le \delta$ and a.e. $x \in \Omega$. X_2 is a finite dimensional space, hence there is a positive constant C_5 such that $\|u\|_\infty \le C_5\|u\|$ for all $u \in X_2$. Therefore, by (2.3), we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_\Omega |\nabla u(x)|^2dx - \int_\Omega F(x,u(x))dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u(x)|^2dx - \frac{1}{2}\mu_m\int_\Omega |u(x)|^2dx,\end{aligned}$$

for all $u \in X_2$ with $|u| \le \delta$, which implies that

$$\varphi(u) \le 0, \quad \text{with } \|u\| \le \frac{\delta}{C_5}\,.$$

Secondly, we prove that

$$\varphi(u) \ge 0, \quad \text{for } u \in X_1 \text{ with } \|u\| \le \delta_0\,. \tag{2.4}$$

In fact, by (H1'), one has

$$|F(x,t)| \le C_2(\frac{|t|^p}{p} + |t|)$$

for all $t \in R$ and a.e.$x \in \Omega$. Thus, we have

$$|F(x,t)| \le C_2(p^{-1} + \delta^{1-p})|t|^p = C_6|t|^p \tag{2.5}$$

for all $|t| \ge \delta$ and a.e.$x \in \Omega$, where $C_6 = C_2(p^{-1} + \delta^{1-p})$.

For $u \in X_1$, let $u = v + w$, where $v \in E(\mu_{m+1}), w \in W = (X_2 + E(\mu_{m+1}))^\perp$. For $\|u\| \le \frac{\delta}{2C_5}$, and $|u(x)| > \delta$, we have

$$\begin{aligned}|w(x)| &\ge |u(x)| - |v(x)| \ge |u(x)| - \|v\|_\infty \\ &\ge |u(x)| - C_5\|v\| \ge |u(x)| - C_5\|u\| \\ &\ge \frac{1}{2}|u(x)|\end{aligned}$$

Moreover,

$$\mu_{m+2}\int_\Omega |w(x)|^2dx \le \int_\Omega |\nabla w(x)|^2dx$$

Hence, we obtain

$$\|w\|^2 = \int_\Omega |\nabla w(x)|^2dx + \int_\Omega |w(x)|^2dx \le (1 + \frac{1}{\mu_{m+2}})\int_\Omega |\nabla w(x)|^2dx\,;$$

that is,

$$\int_\Omega |\nabla w(x)|^2dx \ge \frac{\mu_{m+2}}{1+\mu_{m+2}}\|w\|^2 \tag{2.6}$$

By (2.3), (2.5), (2.1) and (2.6), one has

$$\begin{aligned}&\varphi(u)\\ &= \frac{1}{2}\int_\Omega |\nabla u(x)|^2dx - \int_\Omega F(x,u(x))dx \\ &= \frac{1}{2}\int_\Omega |\nabla u(x)|^2dx - \int_{\{x\in\Omega:|u(x)|>\delta\}} F(x,u(x))dx - \int_{\{x\in\Omega:|u(x)|\le\delta\}} F(x,u(x))dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_\Omega |\nabla u(x)|^2dx - \int_{\{x\in\Omega:|u(x)|\le\delta\}} \frac{1}{2}\mu_{m+1}|u|^2dx \\
&\quad - \int_{\{x\in\Omega:|u(x)|>\delta\}} F(x,u(x))dx - \int_{\{x\in\Omega:|u(x)|\le\delta\}} \Big(F(x,u) - \frac{1}{2}\mu_{m+1}|u|^2\Big)dx \\
&\ge \frac{1}{2}\int_\Omega |\nabla w(x)|^2dx + \frac{1}{2}\int_\Omega |\nabla v(x)|^2dx - \int_\Omega \frac{1}{2}\mu_{m+1}|u|^2dx \\
&\quad - \int_{\{x\in\Omega:|u(x)|>\delta\}} |F(x,u(x))|dx \\
&\ge \frac{1}{2}\int_\Omega |\nabla w(x)|^2dx + \frac{1}{2}\int_\Omega |\nabla v(x)|^2dx - \int_\Omega \frac{1}{2}\mu_{m+1}w^2dx \\
&\quad - \int_\Omega \frac{1}{2}\mu_{m+1}v^2dx - \int_{\{x\in\Omega:|u(x)|>\delta\}} C_6|u|^pdx \\
&\ge \frac{1}{2}\int_\Omega |\nabla w(x)|^2dx - \frac{1}{2}\int_\Omega \mu_{m+1}|w(x)|^2dx - \int_\Omega C_6|2w|^pdx \\
&= \frac{1}{2}\int_\Omega |\nabla w(x)|^2dx - \frac{1}{2}\int_\Omega \mu_{m+1}|w(x)|^2dx - C_6\|2w\|^p_{L^p(\Omega)} \\
&\ge \frac{1}{2}(1-\frac{\mu_{m+1}}{\mu_{m+2}})\int_\Omega |\nabla w(x)|^2dx - C_6C^p\|2w\|^p \\
&\ge \frac{\mu_{m+2}-\mu_{m+1}}{2(1+\mu_{m+2})}\|w\|^2 - C_7\|w\|^p = C_8\|w\|^2 - C_7\|w\|^p
\end{aligned}
$$

for all $u \in X_1$ with $\|u\| \le \frac{\delta}{2C_5}$. From the above inequality, we can conclude that

$$
\varphi(u) \ge 0, \quad \text{for } u \in X_1 \text{ with } \|u\| \le \delta_1 = \big(\frac{C_8}{C_7}\big)^{\frac{1}{p-2}}
$$

Let $\delta_0 = \min\{\frac{\delta}{2C_5}, \delta_1\}$, hence (2.2) and (2.4) hold.

In the case $\inf_X \varphi < 0$, the proof of Theorem 1.2 is complete directly by Lemma 2.3.

In the case $\inf_X \varphi \ge 0$, it follows from (2.2) that

$$
\varphi(u) = \inf_X \varphi = 0 \quad \text{for all } u \in X_2 \text{ with } \|u\| \le \delta
$$

Hence all $u \in X_2$ with $\|u\| \le \delta$ are solutions of (1.1). Therefore, Theorem 1.2 is proved. □

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