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MULTIPLE POSITIVE SOLUTIONS FOR KIRCHHOFF PROBLEMS WITH SIGN-CHANGING POTENTIAL

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Abstract. In this article, we study the existence and multiplicity of positive solutions for a class of Kirchhoff type equations with sign-changing potential. Using the Nehari manifold, we obtain two positive solutions.

1. Introduction and statement of main result

Consider the Kirchhoff type problems with Dirichlet boundary value conditions

$$\begin{gathered}-(a+b\int_\Omega(|\nabla u|^2+v(x)u^2)\,dx)(\Delta u-v(x)u)=h(x)u^p+\lambda f(x,u)\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega,\end{gathered}\tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $a>0$, $b>0$, $\lambda>0$, $3<p<5$, $h\in C(\bar\Omega)$, with $h^+=\max\{h,0\}\neq 0$, $v\in C(\bar\Omega)$ is a bounded function with $\|v\|_\infty>0$, and $f(x,u)$ satisfies the following two conditions:

- (F1) $f(x,u)\in C^1(\Omega\times\mathbb{R})$ with $f(x,0)\geq 0$, and $f(x,0)\neq 0$. There exists a constant $c_1>0$, such that $f(x,u)\leq c_1(1+u^q)$ for $0<q<1$ and $(x,u)\in\Omega\times\mathbb{R}^+$.
- (F2) $f_u(x,u)\in L^\infty(\Omega\times\mathbb{R})$ and for all $u\in H_0^1(\Omega)$, $\int_{\partial\Omega}\frac{\partial}{\partial u}f(x,t|u|)u^2$ has the same sign for every $t\in(0,+\infty)$.

Remark 1.1. Note that under assumptions (F1) and (F2) hold, we have:

- (F3) there exists a constant $c_2>0$, such that $pf(x,u)-uf_u(x,u)\leq c_2(1+u)$, for all $(x,u)\in\Omega\times\mathbb{R}^+$.
- (F4) $F(x,u)-\frac{1}{p+1}f(x,u)u\leq c_2(1+u^2)$, for all $(x,u)\in\Omega\times\mathbb{R}^+$, where $F(x,u)$ is defined by $F(x,u)=\int_0^u f(x,s)ds$ for $x\in\Omega$, $u\in\mathbb{R}$.

In recent years, the existence and multiplicity of solutions to the nonlocal problem

$$\begin{gathered}-\Big(a+b\int_\Omega|\nabla u|^2\,dx\Big)\Delta u=g(x,u)\quad\text{in }\Omega,\\ u=0,\quad\text{on }\partial\Omega,\end{gathered}\tag{1.2}$$

have been studied by various researchers and many interesting and important results can be found. For instance, positive solutions could be obtained in [3, 5, 13].

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Especially, Chen et al [4] discussed a Kirchhoff type problem when $g(x,u) = f(x)u^{p-2}u + \lambda g(x)|u|^{q-2}u$, where $1 < q < 2 < p < 2^*(2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2)$, $f(x)$ and $g(x)$ with some proper conditions are sign-changing weight functions. And they have obtained the existence of two positive solutions if $p > 4$, $0 < \lambda < \lambda_0(a)$. Researchers, such as Mao and Zhang [2], Mao and Luan [1], found sign-changing solutions. As for infinitely many solutions, we refer readers to [11, 12]. He and Zou [14] considered the class of Kirchhoff type problem when $g(x,u) = \lambda f(x,u)$ with some conditions and proved a sequence of a.e. positive weak solutions tending to zero in $L^\infty(\Omega)$. In addition, problems on unbounded domains have been studied by researchers, such as Figueiredo and Santos Junior [9], Li et al. [15], Li and Ye [8].

Our main result read as follows.

Theorem 1.2. *Assume that conditions* (F1) *and* (F2) *hold. Then there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$*, problem* (1.1) *has at least two positive solutions.*

The article is organized as following: Section 2 contains notation and preliminaries. Section 3 contains the proof of Theorem 1.2.

2. Preliminaries

Throughout this article, we use the following notation: The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2 = \int_\Omega (|\nabla u|^2 + v(x)|u|^2)\,dx$. Let S_r be the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^r(\Omega)$, where $1 \leq r < 6$, then

$$\frac{1}{S_{p+1}^{2(p+1)}} \leq \frac{\|u\|^{2(p+1)}}{(\int_\Omega |u|^{p+1})^2}. \tag{2.1}$$

We define a functional $I_\lambda(u)$: $H_0^1(\Omega) \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p+1}H(u) - \lambda \int_\Omega F(x,|u|)\,dx \quad \text{for } u \in H_0^1(\Omega), \tag{2.2}$$

where

$$H(u) = \int_\Omega h(x)|u|^{p+1}\,dx.$$

The weak solutions of (1.1) is the critical points of the functional I_λ. Generally speaking, a function u is called a solution of (1.1) if $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ it holds

$$(a + b\|u\|^2)\int_\Omega (\nabla u \cdot \nabla\varphi + v(x)u\varphi)\,dx = \int_\Omega h(x)|u|^{p-1}|u|\varphi\,dx + \lambda \int_\Omega f(x,|u|)\varphi\,dx.$$

As $I_\lambda(u)$ is unbounded below on $H_0^1(\Omega)$, it is useful to consider the functional on the Nehari manifold:

$$\mathcal{N}_\lambda(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} : \langle I'_\lambda(u), u\rangle = 0\}.$$

It is obvious that the Nehari manifold contains all the nontrivial critical points of I_λ, thus, for $u \in \mathcal{N}_\lambda(\Omega)$, if and only if

$$(a + b\|u\|^2)\|u\|^2 - \int_\Omega h(x)|u|^{p+1}\,dx - \lambda \int_\Omega f(x,|u|)|u|\,dx = 0. \tag{2.3}$$

Define

$$\psi_\lambda(u) = \langle I'_\lambda(u), u\rangle,$$

then it follows that

$$I_\lambda(tu) = \frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|u\|^4 - \frac{t^{p+1}}{p+1}\int_\Omega h(x)|u|^{p+1}\,dx - \lambda\int_\Omega F(x,|tu|)\,dx, \tag{2.4}$$

$$\psi_\lambda(tu) = at^2\|u\|^2 + bt^4\|u\|^4 - t^{p+1}\int_\Omega h(x)|u|^{p+1}\,dx - \lambda\int_\Omega f(x,|tu|)|tu|\,dx, \tag{2.5}$$

$$\begin{aligned}\langle\psi_\lambda'(tu), tu\rangle &= 2at^2\|u\|^2 + 4bt^4\|u\|^4 - (p+1)t^{p+1}\int_\Omega h(x)|u|^{p+1}\,dx \\ &\quad - \lambda\int_\Omega f_u(x,|tu|)|tu|^2\,dx - \lambda\int_\Omega f(x,|tu|)|tu|\,dx.\end{aligned} \tag{2.6}$$

Notice that $\psi_\lambda(tu) = 0$ if and only if $tu \in \mathcal{N}_\lambda(\Omega)$. And we divide $\mathcal{N}_\lambda(\Omega)$ into three parts:

$$\begin{aligned}\mathcal{N}_\lambda^-(\Omega) &= \{u \in \mathcal{N}_\lambda(\Omega) : \langle\psi_\lambda'(u), u\rangle < 0\},\\ \mathcal{N}_\lambda^+(\Omega) &= \{u \in \mathcal{N}_\lambda(\Omega) : \langle\psi_\lambda'(u), u\rangle > 0\},\\ \mathcal{N}_\lambda^0(\Omega) &= \{u \in \mathcal{N}_\lambda(\Omega) : \langle\psi_\lambda'(u), u\rangle = 0\}.\end{aligned}$$

Then we have the following results.

Lemma 2.1. *There exists a constant $\lambda_1 > 0$, for $0 < \lambda < \lambda_1$, such that $\mathcal{N}_\lambda^0(\Omega) = \emptyset$.*

Proof. By contradiction, suppose $u \in \mathcal{N}_\lambda^0(\Omega)$, we obtain

$$\begin{aligned}\langle\psi_\lambda'(u), u\rangle &= 2a\|u\|^2 + 4b\|u\|^4 - (p+1)\int_\Omega h(x)|u|^{p+1}\,dx \\ &\quad - \lambda\int_\Omega f_u(x,|u|)|u|^2\,dx - \lambda\int_\Omega f(x,|u|)|u|\,dx = 0.\end{aligned}$$

On one hand, from (2.1), (2.3), (2.6) and (F2), one deduces that

$$\begin{aligned}a\|u\|^2 + 3b\|u\|^4 &= p\int_\Omega h(x)|u|^{p+1}\,dx + \lambda\int_\Omega f_u(x,|u|)u^2\,dx \\ &\le L\|u\|^{p+1} + \lambda L'\|u\|^2,\end{aligned}$$

where $L = p\|h\|_\infty S_{p+1}^{p+1}$, $L' = \|f_u(x,|u|)\|_{L^\infty} S_2^2$, then

$$L\|u\|^{p+1} \ge (a - \lambda L')\|u\|^2 + 3b\|u\|^4 \ge (a-\lambda L')\|u\|^2,$$

consequently,

$$\|u\|^2 \ge \Big(\frac{a-\lambda L'}{L}\Big)^{\frac{2}{p-1}}. \tag{2.7}$$

On the other hand, by (2.1), (2.3), (2.6) and (F3), we obtain

$$\begin{aligned}a(p-1)\|u\|^2 + (bp-3)\|u\|^4 &\le \lambda\Big(\int_\Omega (pf(x,|u|) - f_u(x,|u|)|u|)|u|\,dx\Big) \\ &\le c_2\lambda\int_\Omega (|u| + |u|^2)\,dx \\ &\le \lambda c_2|\Omega|^{\frac12}S_1\|u\| + \lambda c_2 S_2^2\|u\|^2,\end{aligned}$$

then

$$\lambda c_2|\Omega|^{\frac12}S_1\|u\| + \lambda c_2 S_2^2\|u\|^2 \ge a(p-1)\|u\|^2,$$

thus one has

$$\|u\|^2 \le \Big(\frac{\lambda c_2 S_1|\Omega|^{1/2}}{a(p-1) - c_2\lambda S_2^2}\Big)^2. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\Big(\frac{a-\lambda L'}{L}\Big)^{\frac{2}{p-1}} \le \|u\|^2 \le \Big(\frac{\lambda c_2 S_1 |\Omega|^{1/2}}{a(p-1)-c_2\lambda S_2^2}\Big)^2,$$

which is a contradiction when λ is small enough. So there exists a constant $\lambda_1 > 0$ such that $\mathcal{N}_\lambda^0(\Omega) = \emptyset$. The proof is complete. □

Lemma 2.2. *There exists a constant $\lambda_2 > 0$, for $0 < \lambda < \lambda_2$, such that $\mathcal{N}_\lambda^\pm(\Omega) \neq \emptyset$.*

Proof. For $u \in H_0^1(\Omega)$, $u \neq 0$, let

$$A_u(t) = \frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|u\|^4 - \frac{t^{p+1}}{p+1}\int_\Omega h(x)|u|^{p+1}\,dx,$$

$$K_u(t) = \int_\Omega F(x,|tu|)\,dx,$$

then $I_\lambda(tu) = A_u(t) - \lambda K_u(t)$, hence if $\psi_\lambda(tu) = \langle I_\lambda'(tu), tu\rangle = 0$, then $A_u'(t) - \lambda K_u'(t) = 0$, where

$$A_u'(t) = at^2\|u\|^2 + bt^3\|u\|^4 - t^p\int_\Omega h(x)|u|^{p+1}\,dx,$$

$$K_u'(t) = \int_\Omega f(x,|tu|)|u|\,dx.$$

By (F1), one obtains

$$K_u'(t) = \int_\Omega f(x,|tu|)|u|\,dx \le \int_\Omega c_2(1+|tu|^q)|u|\,dx. \tag{2.9}$$

We consider the following two cases:

Case 1. When $H(u) \le 0$ and $\int_\Omega f(x,t|u|)u^2\,dx > 0$, we have $A_u'(t) > 0$, $A_u(0) = 0$ and $A_u(t)$ increases sharply when $t \to \infty$. At the same time, $K_u'(t) > 0$, $K_u(0)$ is a positive constant and $K_u(t)$ increases relatively slowly when $t \to \infty$ since (2.9). When $H(u) \le 0$ and $\int_\Omega f(x,t|u|)u^2\,dx \le 0$, we have $K_u'(t) \le 0$, $K_u(0)$ is a positive constant and $K_u(t)$ decreases slowly when $t \to \infty$ since (2.9).

Through the above discussion, we obtain there exists t_1 such that $t_1 u \in \mathcal{N}_\lambda(\Omega)$ to every situation. When $0 < t < t_1$, one gets $\psi_\lambda(tu) < 0$ and when $t > t_1$, we have $\psi_\lambda(tu) > 0$, then $t_1 u$ is the local minimizer of $I_\lambda(u)$, so $t_1 u \in \mathcal{N}_\lambda^+(\Omega)$. In conclusion, when $H(u) \le 0$, one has $\mathcal{N}_\lambda^+(\Omega) \neq \emptyset$.

Case 2. When $H(u) > 0$ and $\int_\Omega f(x,t|u|)u^2\,dx > 0$, we have $A_u'(t) > 0$ as $t \to 0$ and $A_u'(t) < 0$ for $t \to \infty$, so $A_u(t)$ increases as $t \to 0$ and then decreases as $t \to \infty$. At the same time, $K_u'(t) > 0$, $K_u(0)$ is a positive constant and $K_u(t)$ increases relatively slowly when $t \to \infty$ since (2.9). When $H(u) > 0$ and $\int_\Omega f(x,t|u|)u^2\,dx < 0$, we have $A_u'(t) > 0$ as $t \to 0$ and $A_u'(t) < 0$ for $t \to \infty$, so $A_u(t)$ increases as $t \to 0$ and then decreases as $t \to \infty$. At the same time, $K_u'(t) < 0$, $K_u(0)$ is a positive constant and $K_u(t)$ decreases slowly when $t \to \infty$ since (2.9).

Through the above discussion, if λ is small enough, there exists $t_1 < t_2$, such that $\psi_\lambda(tu) = 0$, for $0 < t < t_1$, $\psi_\lambda(tu) < 0$, for $t_1 < t < t_2$, $\psi_\lambda(tu) > 0$, and for $t > t_2$, $\psi_\lambda(tu) < 0$. Thus $t_1 u$ is the local minimizer of $I_\lambda(u)$ and $t_2 u$ is the local maximizer of $I_\lambda(u)$. So there exists $\lambda_2 > 0$, when $\lambda < \lambda_2$, one gets $t_1 u \in \mathcal{N}_\lambda^+(\Omega)$ and $t_2 u \in \mathcal{N}_\lambda^-(\Omega)$. Therefore one concludes that when $H(u) > 0$ and λ is small enough, $\mathcal{N}_\lambda^\pm(\Omega) \neq \emptyset$. This completes the proof. □

Lemma 2.3. *Operator I_λ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$.*

Proof. From (2.1), (2.2), (2.3) and (F4), one has

$$\begin{aligned} I_\lambda(u) &= a\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\|u\|^2 + b\Big(\frac{1}{4}-\frac{1}{p+1}\Big)\|u\|^4 \\ &\quad - \lambda\int_\Omega (F(x,|u| - \frac{1}{p+1}f(x,|u|)|u|)\,dx \\ &\ge a\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\|u\|^2 + b\Big(\frac{1}{4}-\frac{1}{p+1}\Big)\|u\|^4 - \lambda c_3\int_\Omega (1+|u|^2)\,dx \\ &\ge a\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\|u\|^2 + b\Big(\frac{1}{4}-\frac{1}{p+1}\Big)\|u\|^4 - \lambda c_3\Big(|\Omega| + S_2^2\|u\|^2\Big) \\ &\ge \Big(\frac{a(p-1)}{2(p+1)} - \lambda c_3 S_2^2\Big)\|u\|^2 + b\Big(\frac{1}{4}-\frac{1}{p+1}\Big)\|u\|^4 - \lambda c_3|\Omega|. \end{aligned}$$

By $3 < p < 5$, it follows that $I_\lambda(u)$ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$. The proof is complete. □

Remark 2.4. From Lemmas 2.1 and 2.2, one has $\mathcal{N}_\lambda(\Omega) = \mathcal{N}_\lambda^+(\Omega)\cup\mathcal{N}_\lambda^-(\Omega)$ for all $0 < \lambda < \min\{\lambda_1,\lambda_2\}$. Furthermore, we obtain $\mathcal{N}_\lambda^+(\Omega)$ and $\mathcal{N}_\lambda^-(\Omega)$ are non-empty, thus, we may define

$$\alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} I_\lambda(u), \quad \alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-(\Omega)} I_\lambda(u).$$

Lemma 2.5. *If $u \in H_0^1(\Omega)\backslash\{0\}$, there exists a constant $\lambda_3 > 0$, such that $I_\lambda(tu) > 0$, for $\lambda < \lambda_3$.*

Proof. For every $u \in H_0^1(\Omega)$, $u \neq 0$, if $H(u) \le 0$, by (2.4), we obtain $I_\lambda(tu) > 0$ when t is large enough. Assume $H(u) > 0$, and let

$$\phi_1(t) = \frac{a}{2}t^2\|u\|^2 - \frac{t^{p+1}}{p+1}H(u).$$

Through calculations, one obtains that $\phi_1(t)$ takes on a maximum at

$$t_{\max} = \Big(\frac{a\|u\|^2}{H(u)}\Big)^{\frac{1}{p-1}}.$$

It follows that

$$\begin{aligned} \phi_1(t_{\max}) &= \frac{p-1}{2(p+1)}\Big(\frac{(a\|u\|^2)^{p+1}}{(\int_\Omega h(x)|u|^{p+1}\,dx)^2}\Big)^{\frac{1}{p-1}} \\ &\ge \frac{p-1}{2(p+1)}\Big(\frac{a^{p+1}}{\|h^+\|_\infty^2 S_{p+1}^{2(p+1)}}\Big)^{\frac{1}{p-1}} := \delta_1. \end{aligned}$$

When $1 \le r < 6$, one has

$$\begin{aligned} (t_{\max})^r\int_\Omega |u|^r\,dx &\le S_r^r\Big(\frac{a\|u\|^2}{H(u)}\Big)^{\frac{r}{p-1}}(\|u\|^2)^{r/2} \\ &= S_r^r a^{-\frac{r}{2}}\Big(\frac{(a\|u\|^2)^{p+1}}{(H(u))^2}\Big)^{\frac{r}{2(p-1)}} \\ &= S_r^r a^{-\frac{r}{2}}\Big(\frac{2(p+1)}{p-1}\Big)^{r/2}\Big(\phi_1(t_{\max})\Big)^{r/2} \\ &= c\big(\phi_1(t_{\max})\big)^{r/2}. \end{aligned} \tag{2.10}$$

Then by (F1) and (F4), we deduce that

$$\begin{aligned} &\int_\Omega F(x, t_{\max}|u|)\,dx \\ &\le \frac{1}{p+1}\int_\Omega c_4(2+|t_{\max}u|^2)\,dx + \int_\Omega c_1(|t_{\max}u| + |t_{\max}u|^{q+1}) \\ &\le B_0 + B_1\phi_1(t_{\max}) + B_2(\phi_1(t_{\max}))^{1/2} + B_3\phi_1(t_{\max})^{\frac{q+1}{2}}. \end{aligned} \tag{2.11}$$

Since

$$I_\lambda(t_{\max}u) = A_u(t_{\max}) - \lambda K_u(t_{\max}) \ge \phi_1(t_{\max}) - \lambda\int_\Omega F(x, t_{\max}|u|)\,dx,$$

according to (2.4), (2.10) and (2.11), one obtains

$$\begin{aligned} I_\lambda(t_{\max}u) &\ge \phi_1(t_{\max}) - \lambda\int_\Omega F(x, t_{\max}|u|)\,dx \\ &\ge \phi_1(t_{\max}) - \lambda\Big[B_0 + B_1\phi_1(t_{\max}) + B_2(\phi_1(t_{\max}))^{1/2} + B_3\phi_1(t_{\max})^{\frac{q+1}{2}}\Big] \\ &\ge \delta_1\Big[1 - \lambda\Big(B_0\delta^{-1} + B_1 + B_2\delta^{-\frac12} + B_3\delta^{\frac{q-1}{2}}\Big)\Big]. \end{aligned}$$

So, if $\lambda < \lambda_3 = (2(B_0\delta^{-1} + B_1 + B_2\delta^{-\frac12} + B_3\delta^{\frac{q-1}{2}}))^{-1}$, we obtain $I_\lambda(t_{\max}u) > 0$. □

Remark 2.6. If $\lambda < \lambda_3$ and $u \in \mathcal{N}_\lambda^-(\Omega)$, by (F2), we conclude that there is a global maximum on u for $I_\lambda(u)$, then $I_\lambda(u) > I_\lambda(t_{\max}u) > 0$.

Lemma 2.7. *If $u \in H_0^1(\Omega)\backslash\{0\}$, there exists a constant $\lambda_4 > 0$ such that $\psi_\lambda(tu) = \langle I_\lambda'(tu), tu\rangle > 0$ when $\lambda < \lambda_4$.*

Proof. For every $u \in H_0^1(\Omega)$, $u \ne 0$, if $H(u) \le 0$, by (2.5), we get $\psi_\lambda(tu) > 0$ when t is large enough. Assume $H(u) > 0$, and let

$$\psi_1(t) = at^2\|u\|^2 - t^{p+1}H(u).$$

Through calculations, we obtain that $\psi_1(t)$ takes on a maximum at

$$\tilde t_{\max} = \Big(\frac{2a\|u\|^2}{(p+1)H(u)}\Big)^{\frac{1}{p-1}}.$$

It follows that

$$\begin{aligned} \psi_1(\tilde t_{\max}) &= \Big(\frac{2a}{p+1}\Big)^{\frac{2}{p-1}}\Big(\frac{p-1}{p+1}\Big)\Big(\frac{(\|u\|^2)^{p+1}}{(\int_\Omega h(x)|u|^{p+1}\,dx)^2}\Big)^{\frac{1}{p-1}} \\ &\ge \Big(\frac{2a}{p+1}\Big)^{\frac{2}{p-1}}\Big(\frac{p-1}{p+1}\Big)\Big(\frac{1}{\|h^+\|_\infty^2 S_{p+1}^{2(p+1)}}\Big)^{\frac{1}{p-1}} := \delta_2. \end{aligned}$$

Similar to the proof of Lemma 2.5, when $1 \le r < 6$, one obtains

$$(\tilde t_{\max})^r\int_\Omega |u|^r\,dx \le \tilde c\,\big(\psi_1(\tilde t_{\max})\big)^{r/2}. \tag{2.12}$$

According to (F1), we deduce that

$$\begin{aligned} \int_\Omega f(x, \tilde t_{\max}|u|)|\tilde t_{\max}u|\,dx &\le c_1\int_\Omega \big(|\tilde t_{\max}u| + |\tilde t_{\max}u|^{q+1}\big)\,dx \\ &\le b_0\,\big(\psi_1(\tilde t_{\max})\big)^{1/2} + b_1\,\big(\psi_1(\tilde t_{\max})\big)^{\frac{q+1}{2}}, \end{aligned} \tag{2.13}$$

then, by (2.5), (2.12) and (2.13), it follows that

$$\begin{aligned}\psi_\lambda(\tilde{t}_{\max}u) &\geq \psi_1(\tilde{t}_{\max}) - \lambda \int_\Omega f(x, \tilde{t}_{\max}|u|)|\tilde{t}_{\max}u|\,dx \\ &\geq (\psi_1(\tilde{t}_{\max}))^{\frac{1+q}{2}} \Big(\psi_1(\tilde{t}_{\max}))^{\frac{1-q}{2}} - \lambda(b_0(\psi_1(\tilde{t}_{\max}))^{-\frac{q}{2}} + b_1)\Big) \\ &\geq \delta_2^{\frac{1+q}{2}} \Big(\delta_2^{\frac{1-q}{2}} - \lambda(b_0\delta_2^{-\frac{q}{2}} + b_1)\Big),\end{aligned}$$

consequently, when $\lambda < \lambda_4 = \delta_2^{\frac{1-q}{2}}/2(b_0\delta_2^{-\frac{q}{2}} + b_1)$, we obtain $\psi_\lambda(\tilde{t}_{\max}u) > 0$. □

Remark 2.8. We claim that: (1) If $H(u) \leq 0$ for every $u \in H_0^1(\Omega)\backslash\{0\}$, there exists t_1 such that $I_\lambda(t_1u) < 0$ for $t_1u \in \mathcal{N}_\lambda^+(\Omega)$. Indeed, obviously, in this condition, $\psi_\lambda(0) < 0$ and $\lim_{t\to\infty}\psi_\lambda(tu) = +\infty$, therefore, there exists $t_1 > 0$ such that $\psi_\lambda(tu) = 0$. Because of $\psi_\lambda(tu) < 0$ for $0 < t < t_1$ and $\psi_\lambda(tu) > 0$ for $t > t_1$, we obtain that $t_1u \in \mathcal{N}_\lambda^+(\Omega)$ and $I_\lambda(t_1u) < I_\lambda(0) = 0$.

(2) If $H(u) > 0$ for $0 < \lambda < \lambda_1$, there exists $t_1 < t_2$, such that $t_1u \in \mathcal{N}_\lambda^+(\Omega)$, $t_2u \in \mathcal{N}_\lambda^-(\Omega)$ and $I_\lambda(t_1u) < 0$. Indeed, in this condition, one gets $\psi_\lambda(0) < 0$ and $\lim_{t\to\infty}\psi_\lambda(tu) = -\infty$. By Lemma 2.7, there exists $T > 0$ such that $\psi_\lambda(Tu) > 0$, therefore, we could obtain there exists $0 < t_1 < T < t_2$, such that $\psi_\lambda(t_1u) = \psi_\lambda(t_2u) = 0$, $t_1u \in \mathcal{N}_\lambda^+(\Omega)$, $t_2u \in \mathcal{N}_\lambda^-(\Omega)$ and $I_\lambda(t_1u) < I_\lambda(0) = 0$.

Lemma 2.9. *Suppose $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$ sequence for $I_\lambda(u)$, then $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be such that

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty.$$

We claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Otherwise, we can suppose that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from (2.1), (2.4), (2.5) and (F4) that

$$\begin{aligned}&1 + c + o(1)\|u_n\| \\ &\geq I_\lambda(u_n) - \frac{1}{p+1}\langle I'_\lambda(u_n), u_n\rangle \\ &\geq a\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_n\|^2 + b\Big(\frac{1}{4} - \frac{1}{p+1}\Big)\|u_n\|^4 \\ &\quad - \lambda\int_\Omega [F(x,|u_n|) - \frac{1}{p+1}f(x,|u_n|)|u_n|]\,dx \\ &\geq a\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_n\|^2 + b\Big(\frac{1}{4} - \frac{1}{p+1}\Big)\|u_n\|^4 - \lambda c_3\int_\Omega (1 + |u_n|^2)\,dx \\ &\geq a\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_n\|^2 + b\Big(\frac{1}{4} - \frac{1}{p+1}\Big)\|u_n\|^4 - \lambda c_3\left(|\Omega| + S_2^2\|u_n\|^2\right) \\ &\geq \Big(\frac{a(p-1)}{2(p+1)} - \lambda c_3S_2^2\Big)\|u_n\|^2 + b\Big(\frac{1}{4} - \frac{1}{p+1}\Big)\|u_n\|^4 - \lambda c_3|\Omega|.\end{aligned}$$

Since $3 < p < 5$, it follows that the last inequality is an absurd. Therefore, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. So Lemma 2.9 holds. □

3. Proof of Theorem 1.2

Let $\lambda^* = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, then Lemmas 2.1–2.9 hold for every $\lambda \in (0, \lambda^*)$. We prove Theorem 1.2 by three steps.

Step 1. We claim that $I_\lambda(u)$ has a minimizer on $\mathcal{N}_\lambda^+(\Omega)$. Indeed, from Remark 2.8, there exists $u \in \mathcal{N}_\lambda^+(\Omega)$ such that $I_\lambda(u) < 0$, so it follows that $\inf_{u\in\mathcal{N}_\lambda^+(\Omega)} I_\lambda(u) < 0$. By Lemma 2.3, let $\{u_n\}$ be a sequence minimizing for $I_\lambda(u)$ on $\mathcal{N}_\lambda^+(\Omega)$. Clearly, this minimizing sequence is of course bounded, up to a subsequence (still denoted $\{u_n\}$), there exists $u_1 \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_1, \quad \text{weakly in } H_0^1(\Omega),\\ u_n \to u_1, \quad \text{strongly in } L^p(\Omega)\ (1 \le p < 6),\\ u_n(x) \to u_1, \quad \text{a.e. in } \Omega. \end{gathered}$$

Now we claim that $u_n \to u_1$ in $H_0^1(\Omega)$. In fact, set $\lim_{n\to\infty}\|u_n\|^2 = l^2$. By the Ekeland's variational principle [7], it follows that

$$\begin{aligned} o(1) &= \langle I_\lambda'(u_n), u_1\rangle\\ &= \big(a + bl^2\big)\int_\Omega (\nabla u_n\cdot\nabla u_1 + v(x)u_n u_1)\,dx\\ &\quad - \int_\Omega h(x)|u_n|^p u_1\,dx - \lambda\int_\Omega f(x,|u_n|)|u_1|\,dx, \end{aligned}$$

thus one obtains

$$0 = (a + bl^2)\|u_1\|^2 - \int_\Omega h(x)|u_1|^{p+1}\,dx - \lambda\int_\Omega f(x,|u_1|)|u_1|\,dx. \tag{3.1}$$

Replacing u_1 with u_n, we obtain

$$\begin{aligned} o(1) &= \langle I_\lambda'(u_n), u_n\rangle\\ &= \big(a + bl^2\big)\,l^2 - \int_\Omega h(x)|u_n|^{p+1}\,dx - \lambda\int_\Omega f(x,|u_n|)|u_n|\,dx, \end{aligned}$$

consequently, one obtains

$$0 = (a + bl^2)l^2 - \int_\Omega h(x)|u_1|^{p+1}\,dx - \lambda\int_\Omega f(x,|u_1|)|u_1|\,dx. \tag{3.2}$$

According to (3.1) and (3.2), we obtain $\|u_1\|^2 = l^2 = \lim_{n\to\infty}\|u_n\|^2$, which suggests that $u_n \to u_1$ in $H_0^1(\Omega)$. Therefore, by Remark 2.8, one obtains

$$I_\lambda(u_1) = \alpha_\lambda^+ = \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} I_\lambda(u) < 0.$$

So we proved the claim.

Step 2. $I_\lambda(u)$ has a minimizer on $\mathcal{N}_\lambda^-(\Omega)$. As a matter of fact, from Remark 2.6, we have $I_\lambda(u) > 0$ for $u \in \mathcal{N}_\lambda^-(\Omega)$, so it follows that $\inf_{u\in\mathcal{N}_\lambda^-(\Omega)} I_\lambda(u) > 0$. Similarly to step 1, we define a sequence $\{u_n\}$ as a minimizing for $I_\lambda(u)$ on $\mathcal{N}_\lambda^-(\Omega)$, and there exists $u_2 \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_2, \quad \text{weakly in } H_0^1(\Omega),\\ u_n \to u_2, \quad \text{strongly in } L^p(\Omega)\ (1 \le p < 6),\\ u_n(x) \to u_2, \quad \text{a.e. in } \Omega. \end{gathered}$$

We claim that $H(u_n) > 0$. By contradiction, assume $H(u_n) \le 0$, then $-pH(u_n) \ge 0$, from $u_n \in \mathcal{N}_\lambda^-(\Omega)$, by (2.1), (2.4), (2.5) and (F2), it follows that

$$a\|u_n\|^2 < a\|u_n\|^2 + 3b\|u_n\|^4 - pH(u_n)$$

$$\begin{aligned} &< \lambda \int_\Omega f_u(x,|u_n|)|u_n|^2\,dx \\ &\le \lambda \|f_u(x,|u_n|)\|_{L^\infty} S_2^2 \|u_n\|^2, \end{aligned}$$

which is a contradiction when λ is small enough. We get $H(u_n) > 0$. Therefore $H(u_2) > 0$ as $n \to \infty$. Similar to the proof of step 1, one can get $u_n \to u_2$ in $H_0^1(\Omega)$. Therefore,

$$I_\lambda(u_2) = \alpha_\lambda^- = \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^-(\Omega)} I_\lambda(u) > 0.$$

From above discussion, we obtain that $I_\lambda(u)$ has a minimizer on $\mathcal{N}_\lambda^-(\Omega)$.

By Step 1 and Step 2, there exist $u_1 \in \mathcal{N}_\lambda^+(\Omega)$ and $u_2 \in \mathcal{N}_\lambda^-(\Omega)$ such that $I_\lambda(u_1) = \alpha_\lambda^+ < 0$ and $I_\lambda(u_2) = \alpha_\lambda^- > 0$. It follows that u_1 and u_2 are nonzero solutions of (1.1). Because of $I_\lambda(u) = I_\lambda(|u|)$, one gets $u_1, u_2 \ge 0$. Therefore, by the Harnack inequality (see [6, Theorem 8.20]), we have $u_1, u_2 > 0$ a.e. in Ω. Consequently the proof of Theorem 1.2 is complete.

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