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NONTRIVIAL PERIODIC SOLUTIONS TO SECOND-ORDER IMPULSIVE HAMILTONIAN SYSTEMS

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Abstract. Based on variational methods and critical point theory, we study the existence of nontrivial periodic solutions to a class of second-order impulsive Hamiltonian systems. A unique feature of the approach used here is that we use a combination of techniques to obtain the existence of multiple solutions.

1. Introduction

We wish to give sufficient conditions for the existence of nontrivial periodic solutions to the second-order impulsive Hamiltonian system

$$\begin{gathered} -\ddot{u}(t)+A(t)u(t)=\lambda\nabla F(t,u(t))+\nabla H(u(t)), \quad \text{a.e. } t\in[0,T],\\ \Delta(\dot{u}_i(t_j))=I_{ij}(u_i(t_j)), \quad i=1,2,\dots,N,\ j=1,2,\dots,p,\\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{gathered} \tag{1.1}$$

where $N\geq 1$, $p\geq 2$, $u=(u_1,\dots,u_N)$, $T>0$, $\lambda>0$ is a parameter, $A:[0,T]\to\mathbb{R}^{N\times N}$ is a continuous map from the interval $[0,T]$ to the set of $N\times N$ symmetric matrices, t_j, $j=1,2,\dots,p$, are the instants at which the impulses occur, $0=t_0<t_1<\dots<t_p<t_{p+1}=T$, and $\Delta(\dot{u}_i(t_j))=\dot{u}_i(t_j^+)-\dot{u}_i(t_j^-)=\lim_{t\to t_j^+}\dot{u}_i(t)-\lim_{t\to t_j^-}\dot{u}_i(t)$. Without further mention, the following conditions are assumed to hold throughout the remainder of this article. The functions $I_{ij}:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous with the Lipschitz constants $L_{ij}>0$, i.e.,

$$|I_{ij}(s_1)-I_{ij}(s_2)|\leq L_{ij}|s_1-s_2| \tag{1.2}$$

for every s_1, $s_2\in\mathbb{R}$, and $I_{ij}(0)=0$ for $i=1,2,\dots,N$, $j=1,2,\dots,p$. In addition, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is measurable with respect to t for all $u\in\mathbb{R}^N$, continuously differentiable in u for almost every $t\in[0,T]$, $F(t,0,\dots,0)=0$ for $t\in[0,T]$, and satisfies the standard summability condition

$$\sup_{|\xi|\leq a}\max\{|F(\cdot,\xi)|,\ |\nabla F(\cdot,\xi)|\}\in L^1([0,T]) \tag{1.3}$$

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for any $a > 0$. Also, the function $H : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, ∇H is Lipschitz continuous with the Lipschitz constant $L > 0$, i.e.,

$$|\nabla H(\xi_1) - \nabla H(\xi_2)| \leq L|\xi_1 - \xi_2| \tag{1.4}$$

for every ξ_1, $\xi_2 \in \mathbb{R}^N$,

$$H(0, \dots, 0) = 0, \quad \text{and} \quad \nabla H(0, \dots, 0) = 0. \tag{1.5}$$

Assuming that $\nabla F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is continuous implies that condition (1.3) is satisfied.

The study of multiplicity of solutions of Hamiltonian systems, as a special case of dynamical systems, is important mathematically as well as being interesting from a practical view point since these systems form a natural framework for mathematical models of many natural phenomena in fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, etc. For background, theory, and applications of Hamiltonian systems, we refer the reader to [9, 28, 35]. Inspired by the monographs [22, 30], the existence and multiplicity of periodic solutions for Hamiltonian systems have been investigated using variational methods by many authors; for example, see [7, 8, 10, 12, 16, 18, 23, 24, 37, 40, 41, 43, 44] and the references therein.

In recent years, investigating the existence of solutions to impulsive boundary value problems has become increasingly important due to their role in models of such things as spacecraft control, impact mechanics, physics, chemistry, chemical engineering, population dynamics, biotechnology, economics, and inspection process in operations research. We refer the reader to [3, 4, 15, 19, 32] for a general discussion of impulsive differential equations and their applications. There have been many approaches used to study existence of solutions of impulsive differential equations, such as fixed point theory, topological degree, continuation methods, coincidence degree theory, upper and lower solution methods, and the monotone iterative method; see, for example, [1, 13, 21] and references contained therein. Recently, in [2, 26, 36, 39], the authors used critical point theory to study the existence and multiplicity of solutions of impulsive problems.

Very recently, a great deal of work has been done on the existence of multiple solutions to second-order impulsive Hamiltonian systems. In [6, 34], based on variational methods and critical point theory, the existence of multiple solutions to second-order impulsive Hamiltonian systems was established. We also refer the interested reader to [20, 33, 42] in which second order Hamiltonian systems with impulsive effects have been examined. In [14], the present authors used variational methods and critical point theory to study the existence of infinitely many periodic solutions to a class of perturbed second-order impulsive Hamiltonian systems. For a discussion of multiple solutions to boundary value problems via variational methods and critical point theory, we refer the reader to [11, 17].

Our results here are motivated by the recent papers [6, 11, 34]. We begin by obtaining the existence of a nontrivial periodic solution by combining algebraic conditions on F and H. Another result, Theorem 3.14 below, is concerned with the existence of three periodic solutions. We obtain it by combining the use of algebraic conditions on the functions F and H to obtain the existence of two distinct solutions and then applying the mountain pass theorem of Pucci and Serrin to obtain the third solution. This approach of combining techniques to obtain multiple solutions of boundary value problems is somewhat unique.

2. Preliminaries

For a given non-empty set X and two functionals Φ, $\Psi : X \to \mathbb{R}$, we define the functions

$$\vartheta(r_1, r_2) = \inf_{v\in\Phi^{-1}(r_1,r_2)} \frac{\sup_{u\in\Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_1(r_1, r_2) = \sup_{v\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u)}{\Phi(v) - r_1}$$

for all r_1, $r_2 \in \mathbb{R}$ with $r_1 < r_2$, and

$$\rho_2(r) = \sup_{v\in\Phi^{-1}(r,\infty)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{\Phi(v) - r}$$

for all $r \in \mathbb{R}$. We also define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda : \Phi - \lambda\Psi.$$

The following two results are due to Bonanno; they will be used in the proofs of our main results.

Theorem 2.1 ([5, Theorem 5.1])**.** *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume there are constants r_1, $r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\vartheta(r_1, r_2) < \rho_1(r_1, r_2).$$

Then, for each $\lambda \in (\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\vartheta(r_1,r_2)})$ there exists $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([5, Theorem 5.3])**.** *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Choose r so that $\inf_X \Phi < r < \sup_X \Phi$, $\rho_2(r) > 0$, and for each $\lambda > \frac{1}{\rho_2(r)}$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then, for each $\lambda \in (\frac{1}{\rho_2(r)}, +\infty)$, there exists $u_{0,\lambda} \in \Phi^{-1}(r, +\infty)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r, +\infty)$ and $I'_\lambda(u_{0,\lambda}) = 0$.*

We assume throughout that the matrix A satisfies the following conditions:

- (M1) $A(t) = (a_{kl}(t))$, $k = 1, \dots, N$, $l = 1, \dots, N$, is a symmetric matrix with $a_{kl} \in L^\infty[0, T]$ for any $t \in [0, T]$;
- (M2) There exists $\delta > 0$ such that $(A(t)\xi, \xi) \ge \delta|\xi|^2$ for any $\xi \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$.

Next, we recall some basic concepts that will be used in what follows. Set

$$E = \Big\{u : [0, T] \to \mathbb{R}^N : u \text{ is absolutely continuous}, \\ u(0) = u(T),\ \dot{u} \in L^2([0, T], \mathbb{R}^N)\Big\}$$

with the inner product

$$\prec u, v \succ_E = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))]dt.$$

The corresponding norm is defined by

$$\|u\|_E = \int_0^T (|\dot{u}(t)|^2 + |u(t)|^2)dt \quad \text{for all } u \in E.$$

For every u, $v \in E$, we define

$$\prec u, v \succ = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))]dt,$$

and we observe that conditions (M1) and (M2) ensure that this defines an inner product in E. Then E is a separable and reflexive Banach space with the norm

$$\|u\| = \prec u, u \succ^{1/2} \quad \text{for all } u \in E.$$

Clearly, E is an uniformly convex Banach space.

A simple computation shows that

$$(A(t)\xi, \xi) = \sum_{k,l=1}^{N} a_{kl}(t)\xi_k\xi_l \le \sum_{k,l=1}^{N} \|a_{kl}\|_{L^\infty}|\xi|^2$$

for every $t \in [0, T]$ and $\xi \in \mathbb{R}^N$. Along with condition (A2), this implies

$$\sqrt{m}\|u\|_E \le \|u\| \le \sqrt{M}\|u\|_E, \tag{2.1}$$

where $m = \min\{1, \delta\}$ and $M = \max\{1, \sum_{k,l=1}^{N} \|a_{kl}\|_\infty\}$, which means the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_E$. Since $(E, \|\cdot\|)$ is compactly embedded in $C([0, T], \mathbb{R}^N)$ (see [22]), there exists a positive constant c such that

$$\|u\|_\infty \le c\|u\|, \tag{2.2}$$

where $\|u\|_\infty = \max_{t\in[0,T]} | u(t) |$ and $c = \sqrt{\frac{2}{m}} \max\{\frac{1}{\sqrt{T}}, \sqrt{T}\}$ (see [6]).

For $u \in E$, $\Delta\dot{u}(t) = \dot{u}(t^+) - \dot{u}(t^-) = 0$ does not necessarily hold for every $t \in (0, T)$, and the derivative $\dot{u}$ may possess discontinuities. This leads to the impulsive effects.

Next, we define what we mean by a solution of (1.1).

Definition 2.3. A function $u \in \{u \in E : \dot{u} \in (W^{1,2}(t_j, t_{j+1}))^N,\ j = 0, 1, 2, \dots, p\}$ is said to be a classical solution of the problem (1.1) if u satisfies the differential equation, the impulse relations, and the boundary conditions given in problem (1.1).

Definition 2.4. By a weak solution of the problem (1.1), we mean any $u \in E$ such that

$$\begin{aligned} &\int_0^T \Big[(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))\Big]dt \\ &+ \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) - \lambda \int_0^T (\nabla F(t, u(t)), v(t))dt = 0 \end{aligned} \tag{2.3}$$

for every $v \in E$.

An important relationship between a weak solution and a classical solution of (1.1) is given in the next lemma.

Lemma 2.5 ([14, Lemma 2.2])**.** *If $u \in E$ is a weak solution of* (1.1)*, then u is a classical solution of* (1.1).

In what follows, unless stated otherwise, by a solution of (1.1) we will always mean a classical solution. Without further mention, we will assume throughout that

$$K := c^2(2LT + \sum_{j=1}^{p}\sum_{i=1}^{N} L_{ij}) < 1.$$

The following proposition is needed in the proofs of our main results.

Proposition 2.6. *Let $J : E \to E^*$ be the operator defined by*

$$\begin{aligned} J(u)v = \int_0^T &\big[(\dot u(t), \dot v(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))\big]dt \\ &+ \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j) \end{aligned}$$

for every u, $v \in E$. Then J admits a continuous inverse on E^*.

Proof. Since $-L|\xi|^2 \le (\nabla H(\xi), \xi) \le L|\xi|^2$ for every $\xi \in \mathbb{R}^N$, and $-L_{ij}|s|^2 \le I_{ij}(s)s \le L_{ij}|s|^2$ for every $s \in \mathbb{R}$ and all $i = 1, 2, \dots, N$, $j = 1, 2, \dots, p$, in view of (2.2), we have

$$\begin{aligned} J(u)u &= \int_0^T \Big[(\dot u(t), \dot u(t)) + (A(t)u(t), u(t)) - (\nabla H(u(t)), u(t))\Big]dt \\ &\quad + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))u_i(t_j) \\ &\ge \Big(1 - c^2LT - c^2\sum_{j=1}^{p}\sum_{i=1}^{N} L_{ij}\Big)\|u\|^2 \\ &> \Big(1 - K\Big)\|u\|^2. \end{aligned}$$

Since $K < 1$, J is coercive. Now for any u, $v \in E$,

$$\begin{aligned} \langle J(u) - J(v), u - v\rangle &= \int_0^T (\dot u(t) - \dot v(t), \dot u(t) - \dot v(t))dt \\ &\quad + \sum_{j=1}^{p}\sum_{i=1}^{N}(I_{ij}(u_i(t_j)) - I_{ij}(v_i(t_j)))(u_i(t_j) - v_i(t_j)) \\ &\quad - \int_0^T (\nabla H(u(t)) - \nabla H(v(t)), u(t) - v(t))dt \\ &\ge \Big(1 - c^2LT - c^2\sum_{j=1}^{p}\sum_{i=1}^{N} L_{ij}\Big)\|u - v\|^2 \\ &> (1 - K)\|u - v\|^2, \end{aligned}$$

so J is uniformly monotone. By [38, Theorem 26.A (d)], J^{-1} exists and is continuous on E^*. $\square$

3. Main results

Our first existence result is contained in the following theorem. For a given function $w \in E$ and a given nonnegative constant r with

$$r \neq \frac{1}{2}(1+K)\|w\|^2,$$

we set

$$a_w(r) := \frac{\int_0^T \max_{|\xi| \le c(\frac{2r}{1-K})^{1/2}} F(t,\xi)dt - \int_0^T F(t,w(t))dt}{r - \frac{1}{2}(1+K)\|w\|^2}.$$

Theorem 3.1. *Assume that there exist constants $r_1 \ge 0$ and $r_2 > 0$, and a function $w \in E$ such that*

(A1) $\left(\frac{2r_1}{1-K}\right)^{1/2} < \|w\| < \left(\frac{2r_2}{1+K}\right)^{1/2}$,

(A2) $a_w(r_2) < a_w(r_1)$.

Then, for each $\lambda \in \left(\frac{1}{a_w(r_1)}, \frac{1}{a_w(r_2)}\right)$, problem (1.1) has a non-trivial periodic solution $u^* \in E$ such that

$$r_1 < \frac{1}{2}\|u^*\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i^*(t_j)} I_{ij}(s)ds - \int_0^T H(u^*(t))dt < r_2.$$

Remark 3.2. In the above theorem, and in the results below, by u^* we mean the vector $(u_1^*, u_2^*, \dots, u_N^*)$.

Proof. Choose λ as in the conclusion of the theorem. To apply Theorem 2.1 to our problem, we take $X = E$ and define the functionals Φ, Ψ, $I_\lambda : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i(t_j)} I_{ij}(s)ds - \int_0^T H(u(t))dt,$$
$$\Psi(u) = \int_0^T F(t,u(t))dt,$$
$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for every $u \in X$. It is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$ given by

$$\Psi'(u)v = \int_0^T (\nabla F(t,u(t)), v(t))dt \tag{3.1}$$

for every $v \in X$, and that $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$ given by

$$\begin{aligned}\Phi'(u)v = &\int_0^T \Big[(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))\Big]dt \\ &+ \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j)\end{aligned} \tag{3.2}$$

for every $v = (v_1, v_2, \dots, v_N) \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous (see [14]). From (1.4) and (1.5), we have $|H(\xi)| \le L|\xi|^2$ for all

$\xi \in \mathbb{R}^N$. From (1.2), (2.2), and the fact that $I_{ij}(0) = 0$, we have

$$\frac{1}{2}(1-K)\|u\|^2 \le \Phi(u) \le \frac{1}{2}(1+K)\|u\|^2 \tag{3.3}$$

for $u \in X$. Condition (A1) together with (3.3) implies

$$r_1 < \Phi(w) < r_2.$$

From (2.2) and (3.3), for each $u \in X$,

$$\begin{aligned}\Phi^{-1}(-\infty, r_2) &= \{u \in X : \Phi(u) < r_2\} \\ &\subseteq \big\{u \in X : \frac{1}{2}(1-K)\|u\|^2 < r_2\big\} \\ &\subseteq \big\{u \in X : |u(t)| \le c\big(\frac{2r_2}{1-K}\big)^{1/2} \text{ for each } t \in [0,T]\big\},\end{aligned}$$

and it follows that

$$\sup_{u \in \Phi^{-1}(-\infty,r_2)} \Psi(u) = \sup_{u \in \Phi^{-1}(-\infty,r_2)} \int_0^T F(t,u(t))dt \le \int_0^T \max_{|\xi| \le c(\frac{2r_2}{1-K})^{1/2}} F(t,\xi)dt.$$

Therefore,

$$\begin{aligned}\vartheta(r_1,r_2) &\le \frac{\sup_{u \in \Phi^{-1}(-\infty,r_2)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_0^T \max_{|\xi| \le c\big(\frac{2r_2}{1-K}\big)^{1/2}} F(t,\xi)dt - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_0^T \max_{|\xi| \le c\big(\frac{2r_2}{1-K}\big)^{1/2}} F(t,\xi)dt - \int_0^T F(t,w(t))dt}{r_2 - \frac{1}{2}(1+K)\|w\|^2} \\ &= a_w(r_2).\end{aligned}$$

On the other hand, arguing as before,

$$\begin{aligned}\rho(r_1,r_2) &\ge \frac{\Psi(w) - \sup_{u \in \Phi^{-1}(-\infty,r_1)} \Psi(u)}{\Phi(w) - r_1} \\ &\ge \frac{\Psi(w) - \int_0^T \max_{|\xi| \le c\big(\frac{2r_1}{1-K}\big)^{1/2}} F(t,\xi)dt}{\Phi(w) - r_1} \\ &\ge \frac{\int_0^T F(t,w(t))dt - \int_0^T \max_{|\xi| \le c\big(\frac{2r_1}{1-K}\big)^{1/2}} F(t,\xi)dt}{\frac{1}{2}(1+K)\|w\|^2 - r_1} \\ &= a_w(r_1).\end{aligned}$$

Hence, from condition (A1), we have $\vartheta(r_1,r_2) < \rho(r_1,r_2)$. Therefore, by Theorem 2.1, for each $\lambda \in \big(\frac{1}{a_w(r_1)}, \frac{1}{a_w(r_2)}\big)$, the functional I_λ admits at least one critical point $u^* \in X$ such that $r_1 < \Phi(u^*) < r_2$, that is, u^* is a nontrivial local minimum for I_λ in X.

Since weak solutions of problem (1.1) are precisely the solutions of the equation $I'_\lambda(u) = 0$ (see (2.3), (3.1), and (3.2)), u^* is a weak solution of problem (1.1). In view of Lemma 2.5, this completes the proof of the theorem. $\square$

The following corollary provides a sufficient condition for applying Theorem 3.1 that does not require knowledge of two constants r_1, r_2 and a test function w satisfying (A1) and (a_2).

Let

$$D = \frac{(T-t_p)^2}{t_1 t_p^2} + \frac{t_1}{3t_p^2}(t_p^2 + t_p T + T^2) + (t_p - t_1) + \frac{T-t_p}{t_p^2} + \frac{1}{3t_p^2}(T^3 - t_p^3) > 0,$$

and for a given nonnegative constant θ and a positive constant η, with

$$(1-K)\theta^2 \neq c^2(1+K)DM\eta^2,$$

let

$$b_\eta(\theta) := \frac{\int_0^T \max_{|\xi|\le\theta} F(t,\xi)dt - \int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\frac{1}{2}(1-K)\theta^2 - \frac{1}{2}c^2(1+K)DM\eta^2}$$

where $\varepsilon = (1,0,\dots,0) \in \mathbb{R}^N$.

Corollary 3.3. *Assume that there exist constants $\theta_1 \ge 0$, $\theta_2 > 0$, and $\eta > 0$ with $\frac{\theta_1}{c\sqrt{Dm}} < \eta < \frac{\theta_2}{c}\sqrt{\frac{1-K}{DM(1+K)}}$ such that*

- (A3) *$F(t,\xi) \ge 0$ for each $t \in [0,t_1] \cup [t_p,T]$ and $|\xi| \le \frac{\eta T}{t_p}$,*
- (A4) *$b_\eta(\theta_2) < b_\eta(\theta_1)$.*

Then, for each $\lambda \in \big(\frac{1}{c^2}\frac{1}{b_\eta(\theta_1)}, \frac{1}{c^2}\frac{1}{b_\eta(\theta_2)}\big)$, the problem (1.1) has a non-trivial periodic solution $u^* \in E$ such that

$$\begin{aligned}\frac{1}{2}(1-K)\big(\frac{\theta_1}{c}\big)^2 < \frac{1}{2}\|u^*\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i^*(t_j)} I_{ij}(s)ds \\ - \int_0^T H(u^*(t))dt < \frac{1}{2}(1-K)\left(\frac{\theta_2}{c}\right)^2.\end{aligned}$$

Proof. Choose $r_1 = \frac{1}{2}(1-K)(\frac{\theta_1}{c})^2$, $r_2 = \frac{1}{2}(1-K)(\frac{\theta_2}{c})^2$, and

$$w(t) = \begin{cases} (T + \frac{t_p - T}{t_1}t)\frac{\eta\varepsilon}{t_p}, & t \in [0,t_1), \\ \eta\varepsilon, & t \in [t_1,t_p], \\ \frac{\eta\varepsilon}{t_p}t, & t \in (t_p,T]. \end{cases} \tag{3.4}$$

Then $w \in E$ and $\|w\|_E^2 = D\eta^2$. By (2.1),

$$Dm\eta^2 \le \|w\|^2 \le DM\eta^2, \tag{3.5}$$

and this together with the condition on η implies (A1) is satisfied. Moreover, since $0 \le w(t) \le \frac{\eta T}{t_p}$ for each $t \in [0,T]$ and (A3) holds, we have

$$\int_0^{t_1} F\Big(t, \Big(T + \frac{t_1 - T}{t_p}t\Big)\frac{\eta\varepsilon}{t_p}\Big)dt + \int_{t_p}^T F\Big(t, \frac{\eta\varepsilon}{t_p}t\Big)dt \ge 0. \tag{3.6}$$

Therefore, from (3.5) and (3.6) it follows that

$$a_w(r_2) = \frac{\int_0^T \max_{|\xi|\le c(\frac{2r_2}{1-K})^{1/2}} F(t,\xi)dt - \int_0^T F(t,w(t))dt}{r_2 - \frac{1}{2}(1+K)\|w\|^2} \le c^2 b_\eta(\theta_2),$$

$$c^2 b_\eta(\theta_1) \le \frac{\int_0^T F(t,w(t))dt - \int_0^T \max_{|\xi|\le c(\frac{2r_1}{1-K})^{1/2}} F(t,\xi)dt}{\frac{1}{2}(1+K)\|w\|^2 - r_1} = a_w(r_1).$$

Therefore, (A4) implies that (A2) is satisfied. Hence, by Theorem 3.1,the conclusion of the corollary follows. □

An easy consequence of Corollary 3.3 is the following existence result.

Corollary 3.4. *In addition to* (A3)*, assume there exist* $\theta > 0$ *and* $\eta > 0$ *with* $\eta < \frac{\theta}{c}\sqrt{\frac{1-K}{DM(1+K)}}$ *such that*

(A5)
$$\frac{\int_0^T \max_{|\xi|\le\theta} F(t,\xi)dt}{\theta^2} < \frac{1-K}{c^2(1+K)DM}\frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\eta^2},$$

where $\varepsilon = (1,0,\dots,0) \in \mathbb{R}^N$.

Then, for each
$$\lambda \in \Big(\frac{(1+K)DM\eta^2}{2\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt},\ \frac{(1-K)\theta^2}{2c^2\int_0^T \max_{|\xi|\le\theta} F(t,\xi)dt}\Big),$$
problem (1.1) *has a non-trivial periodic solution* $u^* \in E$ *such that*
$$0 < \frac{1}{2}\|u^*\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i^*(t_j)} I_{ij}(s)ds - \int_0^T H(u^*(t))dt < \frac{1}{2}(1-K)\big(\frac{\theta}{c}\big)^2.$$

Proof. Choosing $\theta_1 = 0$ and $\theta_2 = \theta$, we have
$$\begin{aligned} b_\eta(\theta) &< \frac{\big(1-\frac{c^2(1+K)DM\eta^2}{(1-K)\theta^2}\big)\int_0^T \max_{|\xi|\le\theta} F(t,\xi)dt}{\frac{1}{2}(1-K)\theta^2 - \frac{1}{2}c^2(1+K)DM\eta^2} \\ &= \frac{\int_0^T \max_{|\xi|\le\theta} F(t,\xi)dt}{\frac{1}{2}(1-K)\theta^2} \\ &< \frac{1}{\frac{1}{2}c^2(1+K)DM}\frac{\int_{t_1}^{t_p} F(t,\eta\varepsilon)dt}{\eta^2} = b_\eta(0). \end{aligned}$$
In particular,
$$b_\eta(\theta) < \frac{\int_0^T \max_{|\xi|\le\theta} F(t,\xi)dt}{\frac{1}{2}(1-K)\theta^2}.$$
The conclusion then follows from Corollary 3.3. □

Next, we present an application of Theorem 2.2 that will be used to obtain multiple solutions to problem (1.1).

Theorem 3.5. *Assume there exist a constant* $\bar{r} > 0$ *and a function* $\bar{w}$ *with* $\frac{2\bar{r}}{1-K} < \|\bar{w}\|^2$ *such that*

(B1) $\int_0^T \max_{|\xi|\le c(\frac{2\bar{r}}{1-K})^{1/2}} F(t,\xi)dt < \int_0^T F(t,\bar{w}(t))dt$;

(B2) $\limsup_{|\xi|\to+\infty} \frac{F(t,\xi)}{|\xi|^2} \le 0$ *uniformly for* $t \in [0,T]$.

Then, for each $\lambda \in (\bar{\lambda}, +\infty)$*, where*
$$\bar{\lambda} := \frac{\frac{1}{2}(1+K)\|\bar{w}\|^2 - \bar{r}}{\int_0^T F(t,\bar{w}(t))dt - \int_0^T \max_{|\xi|\le c(\frac{2\bar{r}}{1-K})^{1/2}} F(t,\xi)dt},$$

problem (1.1) *has at least one non-trivial periodic solution* $\bar{u} \in E$ *such that*

$$\frac{1}{2}\|\bar{u}\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{\bar{u}_i(t_j)} I_{ij}(s)ds - \int_0^T H(\bar{u}(t))dt > \bar{r}.$$

Proof. Choose λ as in the conclusion of the theorem. Taking X and the functionals Φ and Ψ as in the proof of Theorem 3.1, we see that all the regularity assumptions required in Theorem 2.2 are satisfied. By (B2), there is a constant ϵ and a function $h_\epsilon \in L^1([0,T])$ with $0 < \epsilon < \frac{1-K}{2\lambda c^2}$ such that

$$F(t,\xi) \le \epsilon|\xi|^2 + h_\epsilon(t) \quad \text{for all } t \in [0,T],\ \xi \in \mathbb{R}^N. \tag{3.7}$$

From the definitions of Φ and Ψ, (2.2), (3.3) and (3.7), we obtain

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{2}(1-K)\|u\|^2 - \lambda\epsilon\int_0^T |u(t)|^2dt - \lambda\int_0^T h_\epsilon(t)dt \\ &\ge \frac{1}{2}\Big(1-K-\lambda\epsilon c^2\Big)\|u\|^2 - \lambda\|h_\epsilon\|_{L^1([0,T])}. \end{aligned}$$

Since $1-K-\lambda\epsilon c^2 > 0$, the functional I_λ is coercive. Arguing as in the proof of Theorem 3.1 shows that

$$\rho_2(\bar{r}) \ge \frac{\int_0^T F(t,\bar{w}(t))dt - \int_0^T \max_{|\xi|\le c(\frac{2\bar{r}}{1-K})^{1/2}} F(t,\xi)dt}{\frac{1}{2}(1+K)\|\bar{w}\|^2 - \bar{r}} > 0$$

by (B1) and (B2). By Theorem 2.2, the functional I_λ admits at least one local minimum $\bar{u} \in X$ such that $\frac{1}{2}\|\bar{u}\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{\bar{u}_i(t_j)} I_{ij}(s)ds - \int_0^T H(\bar{u}(t))dt > \bar{r}$, and the conclusion follows. □

The following corollary provides a sufficient condition for applying Theorem 3.5 that does not require knowledge of a constant $\bar{r}$ and a test function $\overline{w}$ satisfying (B1) and (B2).

Corollary 3.6. *Assume that* (A3)*, and* (B2) *hold and there exist positive constants* $\bar{\theta}$ *and* $\bar{\eta}$ *with* $\frac{\bar{\theta}}{c\sqrt{Dm}} < \bar{\eta}$ *such that*

(B3)
$$\int_0^T \max_{|\xi|\le\bar{\theta}} F(t,\xi)dt < \int_{t_1}^{t_p} F(t,\bar{\eta}\varepsilon)dt$$

where $\varepsilon = (1,0,\dots,0) \in \mathbb{R}^N$.

Then, for each $\lambda \in (\bar{\lambda}', +\infty)$*, where*

$$\bar{\lambda}' := \frac{\frac{1}{2}(1+K)DM\bar{\eta}^2 - \frac{1}{2}(1-K)\big(\frac{\bar{\theta}}{c}\big)^2}{\int_{t_1}^{t_p} F(t,\bar{\eta}\varepsilon)dt - \int_0^T \max_{|\xi|\le\bar{\theta}} F(t,\xi)dt},$$

problem (1.1) *has at least one non-trivial periodic solution* $\bar{u} \in E$ *such that*

$$\frac{1}{2}\|\bar{u}\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{\bar{u}_i(t_j)} I_{ij}(s)ds - \int_0^T H(\bar{u}(t))dt > \frac{1}{2}(1-K)\big(\frac{\bar{\theta}}{c}\big)^2.$$

Proof. Choose $\bar{r} = \frac{1}{2}(1-K)(\frac{\bar{\theta}}{c})^2$ and let $\bar{w}$ be as in (3.4) with η replaced by $\bar{\eta}$. The conclusion follows from an application of Theorem 3.5. □

Next, we point out some results for which the function F is in factored form. To be precise, consider the problem

$$\begin{gathered} -\ddot{u}(t)+A(t)u(t)=\lambda b(t)\nabla G(u(t))+\nabla H(u(t)), \quad a.e. t\in[0,T], \\ \Delta(\dot{u}_i(t_j))=I_{ij}(u_i(t_j)), \quad i=1,2,\dots,N,\ j=1,2,\dots,p, \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{gathered} \tag{3.8}$$

where $b\in L^1([0,T])$, $b(t)\geq 0$ a.e. $t\in[0,T]$, $b\not\equiv 0$, $G\in C^1(\mathbb{R}^N,\mathbb{R})$, $G(0,\dots,0)=0$, and each component of the vector $\nabla G:\mathbb{R}^N\to\mathbb{R}^N$ is a non-negative continuous function.

Remark 3.7. Since the first term on the right hand side of the equation in (3.8) is nonnegative, any weak solution of (3.8) is nonnegative. To see this, assume that the set $\mathcal{A}=\big\{t\in[0,T]:u_0(t)<0\big\}$ is non-empty and of positive measure. Let $\bar{v}(t)=\min\{0,u_0(t)\}$ for all $t\in[0,T]$. Clearly, $\bar{v}\in E$. Using the fact that u_0 is a weak solution of (3.8), we have

$$\begin{aligned} &\int_0^T\Big((\dot{u}_0(t),\dot{\bar{v}}(t))+(A(t)u_0(t),\bar{v}(t))-(\nabla H(u_0(t)),\bar{v}(t))\Big)dt \\ &+\sum_{j=1}^p\sum_{i=1}^N I_{ij}(u_{0i}(t_j))\bar{v}_i(t_j) \\ &=\lambda\int_0^T(b(t)\nabla G(u_0(t)),\bar{v}(t))dt. \end{aligned}$$

Thus,

$$\begin{aligned} 0&\leq\Big(1-c^2LT-c^2\sum_{j=1}^p\sum_{i=1}^N L_{ij}\Big)\int_{\mathcal{A}}[(\dot{u}_0(t),\dot{u}_0(t))+(A(t)u_0(t),u_0(t))]dt \\ &\leq\int_{\mathcal{A}}\Big((\dot{u}_0(t),\dot{u}_0(t))+(A(t)u_0(t),u_0(t))-(\nabla H(u_0(t)),u_0(t))\Big)dt \\ &\quad+\sum_{j=1}^p\sum_{i=1}^N I_{ij}(u_{0i}(t_j))u_{0i}(t_j)\leq 0. \end{aligned}$$

Now $c^2LT+c^2\sum_{j=1}^p\sum_{i=1}^N L_{ij}<1$, so $u_0=0$ is in $\mathcal{A}$, which is a contradiction.

We will now present some existence results that are consequences of Corollaries 3.3, 3.4, and 3.6, respectively. For a given nonnegative constant θ and a positive constant η, with

$$(1-K)\theta^2\neq c^2(1+K)DM\eta^2,$$

define

$$c_\eta(\theta):=\frac{\max_{|\xi|\leq\theta}G(\xi)\int_0^T b(t)dt-G(\eta\varepsilon)\int_{t_1}^{t_p}b(t)dt}{\frac{1}{2}(1-K)\theta^2-\frac{1}{2}c^2(1+K)DM\eta^2},$$

where $\varepsilon=(1,0,\dots,0)\in\mathbb{R}^N$.

Corollary 3.8. *Assume that there exist constants $\theta_1\geq 0$, $\theta_2>0$, and $\eta>0$ with $\frac{\theta_1}{c\sqrt{Dm}}<\eta<\frac{\theta_2}{c}\sqrt{\frac{1-K}{DM(1+K)}}$ such that*

(A6) $c_\eta(\theta_2)<c_\eta(\theta_1)$.

Then, for each $\lambda \in \big(\frac{1}{c^2}\frac{1}{c_\eta(\theta_1)},\ \frac{1}{c^2}\frac{1}{c_\eta(\theta_2)}\big)$, *problem* (3.8) *has a positive periodic solution* $u^* \in E$ *such that*

$$\begin{aligned}\frac{1}{2}(1-K)\big(\frac{\theta_1}{c}\big)^2 &< \frac{1}{2}\|u^*\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i^*(t_j)} I_{ij}(s)ds - \int_0^T H(u^*(t))dt \\ &< \frac{1}{2}(1-K)\big(\frac{\theta_2}{c}\big)^2.\end{aligned}$$

Corollary 3.9. *Assume that there exist constants* $\theta > 0$ *and* $\eta > 0$ *with* $\eta < \frac{\theta}{c}\sqrt{\frac{1-K}{DM(1+K)}}$ *such that*

(A7)
$$\frac{\max_{|\xi|\le\theta} G(\xi)\int_0^T b(t)dt}{\theta^2} < \frac{1-K}{c^2(1+K)DM}\frac{G(\eta\varepsilon)\int_{t_1}^{t_p} b(t)dt}{\eta^2},$$
where $\varepsilon = (1,0,\dots,0) \in \mathbb{R}^N$.

Then, for each

$$\lambda \in \Big(\frac{(1+K)DM\eta^2}{2G(\eta\varepsilon)\int_{t_1}^{t_p} b(t)dt},\ \frac{(1-K)\theta^2}{2c^2\max_{|\xi|\le\theta} G(\xi)\int_0^T b(t)dt}\Big),$$

problem (3.8) *has a positive periodic solution* $u^* \in E$ *such that*

$$\frac{1}{2}\|u^*\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i^*(t_j)} I_{ij}(s)ds - \int_0^T H(u^*(t))dt < \frac{1}{2}(1-K)\big(\frac{\theta}{c}\big)^2.$$

Corollary 3.10. *Assume there exist constants* $\bar\theta > 0$ *and* $\bar\eta > 0$ *with* $\frac{\bar\theta}{c\sqrt{Dm}} < \bar\eta$ *such that*

(B4)
$$\max_{|\xi|\le\bar\theta} G(\xi)\int_0^T b(t)dt < G(\bar\eta\varepsilon)\int_{t_1}^{t_p} b(t)dt,$$
where $\varepsilon = (1,0,\dots,0) \in \mathbb{R}^N$;

(B5) $\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} \le 0$.

Then, for each $\lambda \in (\hat\lambda, +\infty)$, *where*

$$\hat\lambda := \frac{\frac{1}{2}(1+K)DM\bar\eta^2 - \frac{1}{2}(1-K)(\frac{\bar\theta}{c})^2}{G(\bar\eta\varepsilon)\int_{t_1}^{t_p} b(t)dt - \max_{|\xi|\le\bar\theta} G(\xi)\int_0^T b(t)dt}, \tag{3.9}$$

problem (3.7) *has at least one positive periodic solution* $\bar u \in E$ *such that*

$$\frac{1}{2}\|\bar u\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{\bar u_i(t_j)} I_{ij}(s)ds - \int_0^T H(\bar u(t))dt > \frac{1}{2}(1-K)\big(\frac{\bar\theta}{c}\big)^2.$$

One consequence of Corollary 3.9 is the following existence result.

Theorem 3.11. *Assume that*

$$\lim_{x\to0^+}\frac{\max_{|\xi|\le x} G(\xi)}{|x|^2} = +\infty. \tag{3.10}$$

Then, for each $\lambda \in (0, \lambda^*)$, where

$$\lambda^* := \frac{1-K}{2c^2 \int_0^T b(t)dt} \sup_{\theta>0} \frac{\theta^2}{\max_{|\xi|\le\theta} G(\xi)},$$

problem (3.8) *has a positive periodic solution.*

Proof. For fixed $\lambda \in (0, \lambda^*)$, there exists a positive constant θ such that

$$\lambda < \frac{1-K}{2c^2 \int_0^T b(t)dt} \frac{\theta^2}{\max_{|\xi|\le\theta} G(\xi)}.$$

Moreover, by (3.10), we can choose $\eta > 0$ satisfying $\eta < \frac{\theta}{c}\sqrt{\frac{1-K}{DM(1+K)}}$ such that

$$\frac{(1+K)DM}{2\lambda \int_{t_1}^{t_p} b(t)dt} < \frac{G(\eta\varepsilon)}{\eta^2},$$

where $\varepsilon = (1, 0, \dots, 0) \in \mathbb{R}^N$. The conclusion then follows from Corollary 3.9. □

The following examples illustrates some of our results.

Example 3.12. Take $N = 1$ and consider the problem

$$\begin{gathered} -u''(t) + u(t) = \lambda b(t) g(u(t)) + h(u(t)), \quad \text{a.e. } t \in [0,3], \\ \Delta(u'(t_j)) = I_j(u(t_j)), \quad j = 1,2, \\ u(0) - u(3) = u'(0) - u'(3) = 0, \end{gathered} \tag{3.11}$$

where $b(t) = e^t$ for every $t \in [0,3]$, $t_1 = 1$, $t_2 = 2$, $g(x) = 1 + 3x|x|e^{x^4} + 4x|x|^5 e^{x^4}$, $h(x) = \frac{1}{12}x^+$, $x^+ = \max\{x, 0\}$, and $I_j(x) = \frac{1}{8}x$ for $j = 1,2$ for every $x \in \mathbb{R}$. It is easy to see that $G(x) = x + |x|^3 e^{x^4}$, and

$$\lim_{x\to 0^+} \frac{\max_{|\xi|\le x} G(\xi)}{x^2} = +\infty.$$

Moreover, since $c = \sqrt{6}$, $L = \frac{1}{72}$, and $L_{1j} = \frac{1}{48}$ for $j = 1,2$, we see that $K = \frac{3}{4} < 1$. Hence, applying Theorem 3.11, for each $\lambda \in \Big(0, \frac{1}{48(e^3-1)(1+e)}\Big)$, problem (3.11) has a positive periodic solution.

Example 3.13. Let $N = 2$, $p = 2$, $T = 3$, $t_1 = 1$, and $t_2 = 2$. Let $A : [0,3] \to \mathbb{R}^{2\times 2}$ be the identity matrix, let $G(\xi_1, \xi_2) = \xi_1 + \xi_2 + \frac{1}{4}\xi_1^4 + \frac{1}{4}\xi_2^4$ for all $(\xi_1, \xi_2) \in \mathbb{R}^2$, $b \in L^1([0,3])$ be a positive function, $I_{ij}(s) = \frac{1}{96}s(1 + e^{-s})$ for all $s \in \mathbb{R}$, for $i = 1,2$ and $j = 1,2$, and $H(\xi_1, \xi_2) = \frac{1}{72\sqrt{2}}(\frac{1}{2}\xi_1^2 + \xi_1\xi_2 + \frac{1}{2}\xi_2^2)$ for all $(\xi_1, \xi_2) \in \mathbb{R}^2$. It is clear that

$$\lim_{x\to 0^+} \frac{\max_{|\xi|\le x} G(\xi)}{x^2} = +\infty.$$

Moreover, since $c = \sqrt{6}$, $L = \frac{1}{72\sqrt{2}}$ and $L_{ij} = \frac{1}{96}$ for $i = 1,2$, $j = 1,2$, we have $K = \frac{2+\sqrt{2}}{4\sqrt{2}} < 1$. Hence, applying Theorem 3.11, for each

$$\lambda \in \Big(0, \frac{1 - \frac{2+\sqrt{2}}{4\sqrt{2}}}{12\int_0^T b(t)dt} \sup_{\theta>0} \frac{\theta^2}{\max_{|\xi|\le\theta}\big(\xi_1 + \xi_2 + \frac{1}{4}\xi_1^4 + \frac{1}{4}\xi_2^4\big)}\Big),$$

problem (3.8) has a positive periodic solution.

Our next theorem is for the existence of three positive periodic solutions to problem (3.8). It is based on Corollaries 3.9 and 3.10. We use a combination of algebraic conditions on the functions G and H that give two local minimums for the functional I_λ, and then we apply the Pucci-Serrin mountain pass lemma to obtain the third solution.

Theorem 3.14. *Let* (B5) *hold and assume there exist constants $\theta > 0$, $\eta > 0$, $\bar\theta > 0$, and $\bar\eta > 0$ with*

$$c\sqrt{\frac{DM(1+K)}{1-K}}\eta < \theta \le \bar\theta < c\sqrt{Dm}\bar\eta$$

such that (A7) *and* (B4) *hold. If*

$$\frac{\max_{|\xi|\le\theta} G(\xi)\int_0^T b(t)dt}{\theta^2} < \frac{1-K}{2c^2}\frac{\max_{|\xi|\le\bar\theta} G(\xi)\int_0^T b(t)dt - G(\bar\eta\varepsilon)\int_{t_1}^{t_p} b(t)dt}{\frac12(1-K)(\frac{\bar\theta}{c})^2 - \frac12(1+K)DM\bar\eta^2}, \tag{3.12}$$

where $\varepsilon = (1,0,\dots,0)\in\mathbb{R}^N$, then, for each

$$\lambda\in\Lambda := \Big(\max\Big\{\hat\lambda, \frac{(1+K)DM\eta^2}{2G(\eta\varepsilon)\int_{t_1}^{t_p} b(t)dt}\Big\}, \frac{(1-K)(\frac{\theta}{c})^2}{2\max_{|\xi|\le\theta} G(\xi)\int_0^T b(t)dt}\Big),$$

where $\hat\lambda$ is given in (3.9), *problem* (3.8) *has at least three positive periodic solutions.*

Proof. First we observe that (3.12) implies $\Lambda\ne\emptyset$. Fix $\lambda\in\Lambda$. Using Corollary 3.9, we obtain the first positive periodic solution u^* as a local minimum of the functional I_λ with

$$\frac12\|u^*\|^2 + \sum_{j=1}^p\sum_{i=1}^N\int_0^{u_i^*(t_j)} I_{ij}(s)ds - \int_0^T H(u^*(t))dt < \frac12(1-K)\big(\frac{\theta}{c}\big)^2.$$

Corollary 3.10 guarantees a second positive periodic solution $\bar u$ with

$$\frac12\|\bar u\|^2 + \sum_{j=1}^p\sum_{i=1}^N\int_0^{\bar u_i(t_j)} I_{ij}(s)ds - \int_0^T H(\bar u(t))dt > \frac12(1-K)\big(\frac{\bar\theta}{c}\big)^2.$$

The mountain pass theorem of Pucci and Serrin ([27]) then ensures the existence of a third positive periodic solution. $\square$

As a consequence of Theorem 3.14 we have the following result.

Theorem 3.15. *Assume that*

$$\limsup_{|x|\to0^+}\frac{\max_{|\xi|\le x} G(\xi)}{|x|^2} = +\infty, \tag{3.13}$$

$$\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{|\xi|^2} = 0, \tag{3.14}$$

and there are constants $\bar\theta > 0$ and $\bar\eta > 0$ with $\frac{\bar\theta}{c\sqrt{Dm}} < \bar\eta$ such that

$$\frac{\max_{|\xi|\le\bar\theta} G(\xi)\int_0^T b(t)dt}{\bar\theta^2} < \frac{1-K}{c^2(1+K)DM}\frac{G(\bar\eta\varepsilon)\int_{t_1}^{t_p} b(t)dt}{\bar\eta^2} \tag{3.15}$$

where $\varepsilon = (1, 0, \dots, 0) \in \mathbb{R}^N$. *Then, for each*

$$\lambda \in \Big(\frac{(1+K)DM\bar{\eta}^2}{2G(\bar{\eta}\varepsilon)\int_{t_1}^{t_p} b(t)dt}, \; \frac{(1-K)\bar{\theta}^2}{2c^2 \max_{|\xi|\le\bar{\theta}} G(\xi)\int_0^T b(t)dt} \Big),$$

(3.8) *has at least three positive periodic solutions.*

Proof. We can easily observe from (3.14) that (B5) is satisfied. Moreover, by choosing η small enough and $\theta = \bar{\theta}$, we see that (3.13) implies condition (A7) holds, and (3.15) implies (B4) and (3.12) hold. We thus have the conclusion of the theorem. □

In conclusion, we would like to mention that we believe that this approach of combining techniques to obtain multiple solutions of boundary value problems, with or without impulses, will prove to be a valuable strategy.

Future research. One direction for future work would be to extend the results here to the case where the right hand side of the equation in (1.1) is not continuous. In this regard, we refer the reader to the paper of Molica Bisci and Repovš [25]. Another possibility is to consider the situation where the equation in (1.1) is replaced by an inclusion.

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