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A SHORT PROOF OF INCREASED PARABOLIC REGULARITY

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Abstract. We present a short proof of the increased regularity obtained by solutions to uniformly parabolic partial differential equations. Though this setting is fairly introductory, our new method of proof, which uses *a priori* estimates and an inductive method, can be extended to prove analogous results for problems with time-dependent coefficients, advection-diffusion or reaction diffusion equations, and nonlinear PDEs even when other tools, such as semigroup methods or the use of explicit fundamental solutions, are unavailable.

1. Introduction

It is well-known that solutions of uniformly parabolic partial differential equations possess a smoothing property in time. That is, beginning with initial data which may fail to be even weakly differentiable, the solution becomes extremely smooth, gaining spatial derivatives at any later time $t > 0$. Though this property is well-established, such a result is not contained within many standard texts in PDEs [3, 4, 6, 7, 8, 13]. Of course, these works contain theorems demonstrating the regularity of solutions to initial value or initial-boundary value problems, but the same degree of regularity is assumed for the initial data as is demonstrated for the solution. Notable exceptions are [2, Thm 10.1], in which a gain of regularity theorem is proved, specifically for the heat equation with initial data in $L^2(\Omega)$ using semigroup methods, and [10, Thm 6.6], in which the Steklov average is used to prove that solutions of general, uniformly parabolic equations possess two weak derivatives in $L^2_{t,x}$. Even more concentrated works on the subject of parabolic PDEs, including [5] and the classical monograph [9], do not contain such results regarding increased regularity of solutions to these equations. We note, however, that for initial-boundary value problems these works do investigate the propensity of solutions to become smoother within the interior of the spatial domain. For instance, both [5] and [10] study solutions of the initial-value Dirichlet problem with only continuous boundary data and show that they are twice continuously differentiable on the interior.

In the current paper, we present a few results highlighting the increased regularity that solutions of these equations enjoy. Though we also include theorems regarding existence and uniqueness (Theorems 2.3 and 2.5) in order to work with

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a constructed solution, the fundamental aim of the paper is to establish a new and abbreviated technique for proving additional regularity properties of solutions for $t > 0$ (Theorems 2.1, 2.4, and 2.6). While these regularity properties are generally known, the associated proofs presented within the next section utilize different methods from previously-established theorems [2, 10] and these may be extended to situations in which the previous proofs do not apply. In particular, our method of proof is novel, straightforward, and brief, and relies solely on *a priori* estimates. Hence, for problems in which traditional analytic tools like semigroup methods or explicit fundamental solutions cannot be utilized, the new approach contained within may still be effective. Finally, we note that our techniques also provide a sharp approximation for the rate of blowup of derivative estimates of solutions as $t \to 0^+$.

2. Main results

Let $n \in \mathbb{N}$ be given. We consider the Cauchy problem

$$\begin{gathered} \partial_t u - \nabla \cdot (D(x)\nabla u) = f(x), \quad x \in \mathbb{R}^n,\ t > 0 \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n \end{gathered} \tag{2.1}$$

where D, the diffusion matrix, and f, a forcing function, are both given. Equations like (2.1) arise within countless applications as diffusion is of fundamental importance to physics, chemistry, and biology, especially for problems in thermodynamics, neuroscience, cell biology, and chemical kinetics. As we are interested in displaying the utility of our method of proof, we wish to keep the framework of the current problem relatively straightforward. Thus, we will assume throughout that the diffusion matrix $D = D(x)$ satisfies the uniform ellipticity condition

$$w \cdot D(x)w \geq \theta |w|^2 \tag{2.2}$$

for some $\theta > 0$ and all $x, w \in \mathbb{R}^n$. We note that under suitable conditions on the spatial decay of u, our method may also be altered to allow for diffusion coefficients that are not *uniformly* elliptic (see [11]). Additionally, we will impose different regularity assumptions on D and f to arrive at different conclusions regarding the regularity of the solution u.

Throughout the paper we will only assume that the initial data u_0 is square integrable. Hence, even though $u_0(x)$ may fail to be even weakly differentiable, we will show that $u(t,x)$ gains spatial derivatives in $L_t^\infty L_x^2$ on $(0,\infty) \times \mathbb{R}^n$. Hence, by the Sobolev Embedding Theorem, solutions may be classically differentiable in x assuming suitable regularity of the coefficients. In addition, we will show that u is continuous in time at any instant after the initial time $t = 0$. Though the setting (2.1) is fairly introductory and the assumptions on D and f are not fully relaxed, the new method of proof can be adapted to extend the results to problems with time-dependent terms, reaction-diffusion or advection-diffusion equations, systems of parabolic PDEs [12], different spatial settings such as a bounded or semi-infinite domain or manifold, and nonlinear equations, including quasilinear parabolic PDEs, nonlinear Fokker-Planck equations, and nonlinear transport problems arising in Kinetic Theory [1, 11, 14].

For the proofs, we will rely on *a priori* estimation and the standard Galerkin approximation to obtain regularity of the approximating sequence and then pass to the limit in order to obtain increased regularity of the solution. Hence, we focus on deriving the appropriate estimates as the remaining machinery is standard (cf.

[3, 6, 9]). In what follows, $C > 0$ will represent a constant that may change from line to line, and for derivatives we will use the notation

$$\|\nabla_x^k u(t)\|_2^2 := \sum_{|\alpha|=k} \|\partial_x^\alpha u(t)\|_2^2$$

to sum over all multi-indices of order $k \in \mathbb{N}$. When necessary, we will specify parameters on which constants may depend by using a subscript (e.g., C_T). Our first result establishes the main idea for low regularity of D and f.

Theorem 2.1 (Lower-order regularity)**.** *Assume $f \in H^1(\mathbb{R}^n)$, $u_0 \in L^2(\mathbb{R}^n)$, and $D \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^{n\times n})$. Then, for any $T > 0$ and $t \in (0, T]$, any solution of (2.1) satisfies*

$$\sup_{0\le t\le T} \|u(t)\|_2^2 \le C_T(\|u_0\|_2^2 + \|f\|_2^2) \quad \textit{and} \quad \|\nabla_x u(t)\|_2^2 \le \frac{C_T}{t}(\|u_0\|_2^2 + \|f\|_{H^1}^2).$$

Proof. We first prove two standard estimates of (2.1). First, we multiply by u, integrate the equation in x, integrate by parts and use Cauchy's Inequality to find

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \int_{\mathbb{R}^n} \nabla_x u \cdot D\nabla_x u \, dx \le \frac{1}{2}\left(\|f\|_2^2 + \|u(t)\|_2^2\right).$$

Then, using (2.2) and the assumption on f, we find

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 \le C\left(\|f\|_2^2 + \|u(t)\|_2^2\right) - \theta\|\nabla_x u(t)\|_2^2. \tag{2.3}$$

Next, we take any first-order derivative with respect to x (denoted by ∂_x) of the equation, multiply by $\partial_x u$, and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x u(t)\|_2^2 + \int_{\mathbb{R}^n} \nabla_x\partial_x u\cdot D\nabla_x\partial_x u \, dx + \int_{\mathbb{R}^n} \nabla_x\partial_x u\cdot\partial_x D\nabla_x u \, dx = \int_{\mathbb{R}^n} \partial_x f \partial_x u \, dx.$$

Thus, using Cauchy's inequality with the ellipticity and regularity assumptions, we find, for any $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_x u(t)\|_2^2 &\le -\int_{\mathbb{R}^n} \nabla_x\partial_x u \cdot D\nabla_x\partial_x u dx \\
&\quad - \int_{\mathbb{R}^n} \nabla_x\partial_x u \cdot \partial_x D\nabla_x u \, dx + \int_{\mathbb{R}^n} \partial_x f \partial_x u \, dx \\
&\le -\theta\|\nabla_x\partial_x u(t)\|_2^2 + \|D\|_{W^{1,\infty}}\Big(\varepsilon\|\nabla_x\partial_x u(t)\|_2^2 + \frac{1}{\varepsilon}\|\nabla_x u(t)\|_2^2\Big) \\
&\quad + \frac{1}{2}\Big(\|\partial_x f\|_2^2 + \|\partial_x u(t)\|_2^2\Big).
\end{aligned}$$

Choosing $\varepsilon = \theta(2\|D\|_{W^{1,\infty}})^{-1}$ and summing over all first-order spatial derivatives, we finally arrive at the estimate

$$\frac{1}{2}\frac{d}{dt}\|\nabla_x u(t)\|_2^2 \le C\left(\|f\|_{H^1}^2 + \|\nabla_x u(t)\|_2^2\right) - \frac{\theta}{2}\|\nabla_x^2 u(t)\|_2^2. \tag{2.4}$$

Now, we utilize a linear expansion in t to prove the theorem. Let $T > 0$ be given. Consider $t \in [0, T]$ and define

$$M_1(t) = \|u(t)\|_2^2 + \frac{\theta t}{2}\|\nabla_x u(t)\|_2^2.$$

We differentiate this quantity, and use the estimates (2.3) and (2.4) to find

$$\begin{aligned} M_1'(t) &= \frac{d}{dt}\|u(t)\|_2^2 + \frac{\theta}{2}\|\nabla_x u(t)\|_2^2 + \frac{\theta t}{2}\frac{d}{dt}\|\nabla_x u(t)\|_2^2 \\ &\le 2C\left(\|f\|_2^2 + \|u(t)\|_2^2\right) - 2\theta\|\nabla_x u(t)\|_2^2 + \frac{\theta}{2}\|\nabla_x u(t)\|_2^2 \\ &\quad + \frac{\theta t}{2}\left[2C\left(\|f\|_{H^1}^2 + \|\nabla_x u(t)\|_2^2\right) - \theta\|\nabla_x^2 u(t)\|_2^2\right] \\ &\le C_T\left(\|f\|_{H^1}^2 + M_1(t)\right) \end{aligned}$$

A straightforward application of Gronwall's inequality (cf. [3]) then implies

$$M_1(t) \le C_T(M_1(0) + \|f\|_{H^1}^2) = C_T(\|u_0\|_2^2 + \|f\|_{H^1}^2).$$

Finally, the bound on $M_1(t)$ yields

$$\|u(t)\|_2^2 \le C_T, \quad \|\nabla_x u(t)\|_2^2 \le \frac{C_T}{\theta t}$$

and the second estimate holds on the interval $(0, T]$. As $T > 0$ is arbitrary, the result follows. □

Next, we formulate the existence of weak solutions for our lower-order regularity setting.

Definition 2.2. We say that $u \in L^2([0,T]; H^1(\mathbb{R}^n))$, with $\partial_t u \in L^2([0,T]; H^{-1}(\mathbb{R}^n)$ is a weak solution of (2.1) if

$$\begin{gathered} \langle \partial_t u, v\rangle + \langle D(x)\nabla u, \nabla v\rangle = \langle f, v\rangle \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n \end{gathered} \tag{2.5}$$

for every $v \in H^1(\mathbb{R}^n)$ and $t \in [0,T]$.

Theorem 2.3 (Existence and uniqueness of weak solutions)**.** *Given $u_0 \in L^2(\mathbb{R}^n)$ and $f \in H^1(\mathbb{R}^n)$ and $T > 0$ arbitrary, there exists a unique*

$$u \in C((0,T]; H^1(\mathbb{R}^n)) \cap C([0,T]; L^2(\mathbb{R}^n))$$

and $\partial_t u \in L^2(([0,T]; H^{-1}(\mathbb{R}^n))$ that solves (2.5).

Proof. Though the proof is well-known (cf. [9, Ch. III, Thm 4.1]) and follows a standard Galerkin approach, we include it for completeness. Take $\{w_k(x)\}_{k=0}^\infty$ to be an orthonormal basis for L^2 with $w_k \in H^s$ for $s \ge 0$. Consider functions of the form $u_m(x,t) = \sum_{k=0}^m d_k^m(t) w_k(x)$ with $d_k^m(t)$ a smooth function of t. Then the equations

$$\begin{gathered} \langle \partial_t u_m, w_k\rangle + \langle D(x)\nabla u_m, \nabla w_k\rangle = \langle f, w_k\rangle \\ \langle u_m(0,\cdot), w_k\rangle = \langle u_0, w_k\rangle \end{gathered}$$

for $k = 1, 2, \dots m$ reduce to a constant coefficient first order system of ODE's for $d_k^m(t)$, and hence existence of approximate solutions is readily established.

For these solutions, $u_m(t)$, we may repeat the proof of our *a priori* estimates verbatim. Thus we can conclude that

$$\sup_{0\le t\le T}\left(\|u_m(t)\|_2^2 + \frac{\theta t}{2}\|\nabla_x u_m(t)\|_2^2\right) \le C_T\|u_0\|_2^2.$$

From the proof, we also have the inequality

$$\int_0^T \frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2\,dt + \int_0^T \theta\|\nabla_x u_m\|_2^2\,dt \le C_T\int_0^T \Big(\|f\|_2^2 + \|u_0\|_2^2\Big)\,dt.$$

Using the above control of $\sup_{0\le t\le T}\|u_m(t)\|_2^2$, we find that

$$\int_0^T \|u_m\|_{H^1(\mathbb{R}^n)}^2 \le C_T\Big(\|f\|_2^2 + \|u_0\|_2^2\Big).$$

Finally, fix $v \in H^1(\mathbb{R}^n)$ with $\|v\|_{H^1} \le 1$ and consider

$$\langle \partial_t u_m, v\rangle = -\langle D(x)\nabla_x u_m, \nabla_x v\rangle + \langle f, v\rangle.$$

Using Cauchy-Schwartz and taking the supremum over $v \in H^1$ with $\|v\|_{H^1} \le 1$ we find $\|\partial_t u_m(t)\|_{H^{-1}} \le C_T\Big(\|f\|_2 + \|u_0\|_2\Big)$ and

$$\int_0^T \|\partial_t u_m(t)\|_{H^{-1}}^2\,dt \le C_T\Big(\|f\|_2^2 + \|u_0\|_2^2\Big).$$

Thus, u_m is a bounded sequence in $L^2([0,T];H^1(\mathbb{R}^n))$ and $\partial_t u_m$ is a bounded sequence in $L^2([0,T];H^{-1}(\mathbb{R}^n))$, so we may extract a subsequence m_j so that $u_{m_j} \to u$ in $L^2([0,T];H^1(\mathbb{R}^n))$ and $\partial_t u_{m_j} \to \partial_t u$ in $L^2([0,T];H^{-1}(\mathbb{R}^n))$.

Now fix an integer N and consider $v(t) = \sum_{k=0}^N d_k(t)w_k(x)$, where $d_k(t)$ are fixed smooth functions. Then for $m_j > N$, we have

$$\int_0^T \langle \partial_t u_{m_j}, v\rangle + \langle D(x)\nabla u_{m_j}, \nabla v\rangle\,dt = \int_0^T \langle f, v\rangle\,dt.$$

Thus passing to the limit

$$\int_0^T \langle \partial_t u, v\rangle + \langle D(x)\nabla u, \nabla v\rangle\,dt = \int_0^T \langle f, v\rangle\,dt.$$

Since v given above are dense in $L^2([0,T];H^1(\mathbb{R}^n))$, the equality holds for any v in this space. Since $u \in L^2([0,T];H^1(\mathbb{R}^n))$ and $\partial_t u \in L^2([0,T];H^{-1}(\mathbb{R}^n))$ we have that $u \in C([0,T];L^2(\mathbb{R}^n)$ by [3, Thm. 3 §5.9.2].

To see $u(0) = u_0$, we take $v \in C^1([0,T];H^1(\mathbb{R}^n))$ with the property that $v(T) = 0$, then we find that

$$\langle u_{m_j}(0), v(0)\rangle - \int_0^T \langle u_{m_j}, \partial_t v\rangle + \langle D(x)\nabla u_{m_j}, v\rangle\,dt = \int_0^T \langle f, v\rangle\,dt.$$

Notice $u_{m_j}(0) = \sum_{k=0}^{m_j}\langle u_0, w_k\rangle w_k \to u_0$ in $L^2(\mathbb{R}^n)$ as $m_j \to \infty$. Thus passing to the limit, we find that $\langle u(0), v(0)\rangle = \langle u_0, v(0)\rangle$ for $v(0)$ arbitrary. Hence $u(0) = u_0$.

To prove uniqueness, notice that for any two solutions u and $\tilde{u}$ the difference $u - \tilde{u}$ satisfies our equation with $u_0 = 0$ and $f = 0$. Thus our a priori estimate gives that $\sup_{0\le t\le T}\|u(t) - \tilde{u}(t)\|_2^2 \le 0$ and uniqueness follows immediately.

Finally, to show that $u \in C((0,T];H^1(\mathbb{R}^n))$ we consider $w_s(t) = u(t+s) - u(t)$. Then $w_s(t)$ satisfies our equation with $f = 0$ and $w(0) = u_0 - u(s)$. From our *a priori* estimate

$$\|w_s(t)\| + \frac{\theta t}{2}\|\nabla_x w_s(t)\|_2^2 \le C_T(\|u_0 - u(s)\|_2^2).$$

From the fact that $u \in C([0,T];L^2(\mathbb{R}^n))$, we have for $t > 0$ that $\lim_{s\to 0}\|u(t+s) - u(t)\| = 0$ and $\lim_{s\to 0}\|\nabla_x u(t+s) - \nabla_x u(t)\| = 0$, whence the result follows. □

Next, we use induction to extend the previous estimate to higher regularity assuming that D and f possess additional weak derivatives.

Theorem 2.4 (Higher-order regularity)**.** *For every $m \in \mathbb{N}$, if $f \in H^m(\mathbb{R}^n)$, $D \in W^{m,\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n})$, and $u_0 \in L^2(\mathbb{R}^n)$, then for any $T > 0$ and $t \in (0,T]$ the previously derived solution of* (2.1) *satisfies*

$$\|\nabla_x^k u(t)\|_2^2 \le \frac{C_T}{t^k}\Big(\|u_0\|_2^2 + \|f\|_{H^k}^2\Big) \quad \text{for } k = 0,1,\dots,m.$$

Proof. We will prove the result by induction on m. The base case ($m = 1$) follows immediately from Theorem 2.1. Prior to the inductive step, we first prove a useful estimate for solutions of (2.1). For the estimate, assume D and f possess $k \in \mathbb{N}$ derivatives in L^∞ and L^2, respectively. Take any kth-order derivative with respect to x (denoted by ∂_x^α) of the equation, multiply by $\partial_x^\alpha u$, and integrate using integration by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u(t)\|_2^2 + \int_{\mathbb{R}^n} \nabla_x \partial_x^\alpha u \cdot \sum_{j=0}^k \sum_{\substack{|\beta|=j\\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \partial_x^\beta D \nabla_x \partial_x^\gamma u\, dx = \int_{\mathbb{R}^n} \partial_x^\alpha f \partial_x^\alpha u\, dx$$

and thus

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u(t)\|_2^2 = -\int_{\mathbb{R}^n} \nabla_x \partial_x^\alpha u \cdot \sum_{j=0}^k \sum_{\substack{|\beta|=j\\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \partial_x^\beta D \nabla_x \partial_x^\gamma u\, dx + \int_{\mathbb{R}^n} \partial_x^\alpha f \partial_x^\alpha u\, dx.$$

Labeling the first term on the right side A, we use (2.2) and the regularity of D with Cauchy's inequality (with $\varepsilon > 0$) to find

$$\begin{aligned} A &= -\int_{\mathbb{R}^n} \nabla_x \partial_x^\alpha u \cdot D \nabla_x \partial_x^\alpha u\, dx - \int_{\mathbb{R}^n} \nabla_x \partial_x^k u \cdot \sum_{j=1}^k \sum_{\substack{|\beta|=j\\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \partial_x^\beta D \nabla_x \partial_x^\gamma u\, dx \\ &\le -\theta \|\nabla_x \partial_x^\alpha u(t)\|_2^2 + C\|D\|_{W^{k,\infty}}^2 \Big(\varepsilon \|\nabla_x \partial_x^\alpha u(t)\|_2^2 + \frac{1}{\varepsilon}\|u(t)\|_{H^{k-1}}^2\Big) \\ &\le -\frac{\theta}{2}\|\nabla_x \partial_x^\alpha u(t)\|_2^2 + C\|u(t)\|_{H^{k-1}}^2 \end{aligned}$$

where we have chosen $\varepsilon = \theta\left(2C\|D\|_{W^{k,\infty}}^2\right)^{-1}$ in the third line. Inserting this into the above equality and using Cauchy's inequality again we find

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u(t)\|_2^2 \le -\frac{\theta}{2}\|\nabla_x \partial_x^\alpha u(t)\|_2^2 + C\|u(t)\|_{H^{k-1}}^2 + \frac{1}{2}\|\partial_x^\alpha f\|_2^2 + \frac{1}{2}\|\partial_x^\alpha u(t)\|_2^2.$$

Finally, summing over all first-order derivatives and using the regularity of f yields the estimate

$$\frac{1}{2}\frac{d}{dt}\|\nabla_x^k u(t)\|_2^2 \le -\frac{\theta}{2}\|\nabla_x^{k+1} u(t)\|_2^2 + C\left(\|f\|_{H^k}^2 + \|u(t)\|_{H^k}^2\right). \tag{2.6}$$

Now, we prove the theorem utilizing this estimate for $k = 0,1,\dots,m$. Assume $f \in H^m(\mathbb{R}^n)$ and $D \in W^{m,\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n})$. Since this implies that $f \in H^{m-1}(\mathbb{R}^n)$ and $D \in W^{m-1,\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n})$, we find that $u \in C((0,\infty);H^{m-1}(\mathbb{R}^n))$ by the induction hypothesis. Let $T > 0$ be given. Consider $t \in [0,T]$ and define

$$M(t) = \sum_{k=0}^m \frac{(\theta t)^k}{2^k k!}\|\nabla_x^k u(t)\|_2^2.$$

We differentiate to find

$$M'(t) = \sum_{k=1}^{m} \frac{\theta^k t^{k-1}}{2^k (k-1)!} \|\nabla_x^k u(t)\|_2^2 + \sum_{k=0}^{m} \frac{(\theta t)^k}{2^k k!} \frac{d}{dt} \|\nabla_x^k u(t)\|_2^2 =: I + II.$$

We use (2.6) for any $k = 0, ..., m$ and relabel the index of the sum so that

$$\begin{aligned} II &\le \sum_{k=0}^{m} \frac{(\theta t)^k}{2^k k!} \left(-\theta \|\nabla_x^{k+1} u(t)\|_2^2 + C\left(\|f\|_{H^k}^2 + \|u(t)\|_{H^k}^2\right)\right) \\ &= -2\sum_{k=0}^{m} \frac{\theta^{k+1} t^k}{2^{k+1} k!} \|\nabla_x^{k+1} u(t)\|_2^2 + C\sum_{k=0}^{m} \frac{(\theta t)^k}{2^k k!} \left(\|f\|_{H^k}^2 + \|u(t)\|_{H^k}^2\right) \\ &\le -2I - \frac{\theta^{m+1} t^m}{2^{m+1} m!} \|\nabla_x^{m+1} u(t)\|_2^2 + C\sum_{k=0}^{m} \frac{(\theta t)^k}{2^k k!} \left(\|f\|_{H^k}^2 + \|u(t)\|_{H^k}^2\right) \end{aligned}$$

Notice the induction hypothesis gives $\|u(t)\|_{H^{k-1}}^2 \le \frac{C_T}{t^{k-1}}(\|f\|_{H^{k-1}}^2 + \|u_0\|_2^2)$. Using this bound and the previous inequality within the estimate of $M'(t)$, we find

$$\begin{aligned} M'(t) &\le C\sum_{k=0}^{m} \frac{(\theta t)^k}{2^k k!} \left(\|f\|_{H^k}^2 + \|u(t)\|_{H^k}^2\right) \\ &\le C_T\Big(\|f\|_{H^m}^2 + \sum_{k=0}^{m} \frac{(\theta t)^k}{2^k k!} \left[\|\nabla_x^k u(t)\|_2^2 + \|u(t)\|_{H^{k-1}}^2\right]\Big) \\ &\le C_T\Big(\|f\|_{H^m}^2 + M(t) + \sum_{k=0}^{m} \frac{(\theta t)^k}{2^k k!} \frac{C_T}{t^{k-1}} \left(\|f\|_{H^{k-1}}^2 + \|u_0\|_2^2\right)\Big) \\ &\le C_T\left(\|f\|_{H^m}^2 + \|u_0\|_2^2 + M(t)\right). \end{aligned}$$

Another straightforward application of Gronwall's inequality then implies

$$M(t) \le C_T\Big(\|f\|_{H^m}^2 + \|u_0\|_2^2 + M(0)\Big) \le C_T(\|f\|_{H^m}^2 + \|u_0\|_2^2).$$

Finally, the bound on $M(t)$ yields

$$\|\nabla_x^m u(t)\|_2^2 \le \frac{C_T}{(\theta t)^m}$$

which completes the inductive step and the proof. □

Theorem 2.5 (Existence and Uniqueness of Weak solutions). *Let $m \in \mathbb{N}$ be given. For any $u_0 \in L^2(\mathbb{R}^n)$, $f \in H^m(\mathbb{R}^n)$, and $T > 0$ arbitrary, there exists a unique $u \in C((0,T]; H^m(\mathbb{R}^n)) \cap C([0,T]; L^2(\mathbb{R}^n))$ and $u' \in L^2(([0,T]; H^{-1}(\mathbb{R}^n))$ that solves (2.5).*

The result in the above theorem follows by a straightforward repetition of the proof of Theorem 2.3 with the obvious modifications.

Of course, if the dimension n satisfies $n < 2m - 1$ this result implies classical differentiability of solutions by Sobolev Embedding and these functions satisfy the PDE in the classical sense. Finally, this result can be easily used to deduce infinite spatial differentiability of the solution assuming D and f satisfy the same condition.

Theorem 2.6 (Infinite Differentiability)**.** *If* $f \in H^\infty(\mathbb{R}^n)$*,* $D \in W^{\infty,\infty}(\mathbb{R}^n; \mathbb{R}^{n\times n})$*, and* $u_0 \in L^2(\mathbb{R}^n)$*, then for any* $T > 0$ *arbitrary, any solution of* (2.1) *satisfies* $u \in C^\infty((0,T] \times \mathbb{R}^n)$*.*

Remark 2.7. On a bounded domain $\Omega \subset \mathbb{R}^n$, it is enough to impose $f \in C^\infty(\Omega)$ and $D \in C^\infty(\Omega; \mathbb{R}^{n\times n})$ to arrive at the same result.

The above result follows immediately by applying Theorem 2.4 for each $m \in \mathbb{N}$, noticing that

$$\partial_t u = \nabla \cdot (D\nabla u) + f$$

is continuous, and bootstrapping this property for higher-order time derivatives.

Remark 2.8. Though we have chosen to demonstrate the method for equations with time-independent coefficients, the same results can be obtained for time-dependent diffusion coefficients D and sources f using the same proof, as long as these functions are sufficiently smooth in t. Additionally, similar arguments can be used to gain regularity of the solution in t, as well.

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