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MULTIPLE SOLUTIONS FOR P-LAPLACIAN BOUNDARY-VALUE PROBLEMS WITH IMPULSIVE EFFECTS

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Abstract. In this article we study a class of boundary value problems with impulsive effects. First by using Morse theory in combination with local linking arguments, the existence result of at least two nontrivial solutions are obtained. Next we prove that the problems have k distinct pairs of solutions by using the Clark theorem. Recent results from the literature are improved and extended.

1. Introduction and statement of main results

In this article, we consider the impulsive boundary value problem

$$\begin{gathered} -(\rho(x)\Phi_p(u'(x)))' + s(t)\Phi_p(u(x)) = f(x,u(x)), \quad \text{a.e. } x\in(a,b),\\ \Delta(\rho(x_j)\Phi_p(u'(x_j))) = \iota_j(u(x_j)), \quad j=1,2,\dots,m,\\ \alpha_1 u'(a^+) - \alpha_2 u(a) = 0, \quad \beta_1 u'(b^-) + \beta_2 u(b) = 0, \end{gathered} \tag{1.1}$$

where $\Phi_p(u)=|u|^{p-2}u$, $p>1$, $\rho, s\in L^\infty[a,b]$ with $\operatorname{ess\,inf}_{[a,b]}\rho>0$, $\operatorname{ess\,inf}_{[a,b]} s>0$, $0<\rho(a),\rho(b)<\infty$, $\alpha_1,\alpha_2,\beta_1,\beta_2>0$, $a=x_0<x_1<x_2<\dots<x_m<x_{m+1}=b$, $u'(x_j^+))$ and $u'(x_j^-))$ denotes the right and left limit of $u'(x_j)$ at $x=x_j$, respectively, $\iota_j\in C(\mathbb{R},\mathbb{R})$, $j=1,2,\dots,m$, $f\in C([a,b]\times\mathbb{R},\mathbb{R})$.

Since many evolution processes exhibit impulsive effects in the real world, the theory of impulsive differential equations has developed rapidly in recent years. For the significance, it is important to study the solvability of impulsive differential equations. We refer some recent works on the theory of impulsive differential equations that developed by a large number of mathematicians [2, 8, 14, 18, 20, 21, 33, 34]. Classical approaches to such problems include fixed point theory, topological degree theory and comparison method and so on. More recently, variational method is one of the most promising techniques for differential equations, especially for the boundary value problems of impulsive differential equations, and the literature on this technique has grown extensively, see [4, 5, 6, 17, 19, 25, 27, 30, 31, 35, 36] and the references therein.

Morse theory and local linking arguments are powerful tools in modern nonlinear analysis [7, 11, 12, 23, 28], especially for the problems with resonance [13, 24]. However, to the best of our knowledge, there are few papers dealing with the

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existence of solutions for impulsive boundary value problems by using Morse theory. Recently, in [1], the authors considered the following problem

$$
\begin{gathered}
-u'' = f(x,u), \quad x \in (0,1) \setminus \{x_1, x_2, \dots x_m\}, \\
\Delta u'(x_j) = \iota_j(u(x_j)), \quad j = 1,2,\dots,m, \\
u(0) = u(1) = 0.
\end{gathered} \tag{1.2}
$$

They obtained the existence of one nontrivial solution for (1.2) when the impulses are asymptotically linear near zero via computing the critical groups at zero.

Inspired by the above facts, the goal of this paper is to consider the multiplicity of nontrivial solutions for (1.1). Under some suitable assumptions, by using Morse theory in combination with local linking arguments, the existence result of at least two nontrivial solutions are obtained. Next we prove that the problems have k distinct pairs of solutions by using the Clark theorem.

Before stating our main results, we present the following assumptions on ι_j $(j = 1,2,\dots,m)$:

- (I1) $\iota_j(t)t \geq 0$ and there exist $a_j > 0$ and $0 \leq \gamma_j < p-1$ such that
$$|\iota_j(t)| \leq a_j |t|^{\gamma_j}, \quad j = 1,2,\dots,m;$$
- (I2) $\iota_j(-t) = -\iota_j(t)$, $j = 1,2,\dots,m$.

Remark 1.1. From condition (I1), we can see that

$$|I_j(t)| \leq a_j |t|^{\gamma_j+1} \quad \text{and} \quad I_j(t) \geq 0 \quad (j = 1,2,\dots,m),$$

here and in the sequel $I_j(t) = \int_0^t \iota_j(s)ds$.

Furthermore, we assume that the nonlinearity $f(x,u)$ satisfies the conditions:

- (F1) there exist $c_1 > 0$ and $0 \leq \alpha < p-1$ such that
$$|f(x,u)| \leq c_1 |u|^\alpha, \quad \forall (x,u) \in [a,b] \times \mathbb{R};$$
- (F2) there exist small constants $0 < r < r_0$, $c_2 > 0$, $0 < c_3 < \frac{1}{pS_p^p}$, $1 < \gamma < \max\{\gamma_j + 1\}$ such that
$$c_3 |u|^p > F(x,u) \geq c_2 |u|^\gamma, \quad r \leq |u| \leq r_0 \quad \text{a.e. } x \in [a,b],$$
here and in the sequel $F(x,u) = \int_0^u f(x,s)ds$, furthermore, S_p is the Sobolev constant from $W^{1,p}([a,b])$ to $L^p([a,b])$;
- (F3) $f(x,-u) = -f(x,u)$.

Now, we are ready to state the main results of this article.

Theorem 1.2. *Assume that* (I1), (F1), (F2) *hold. Then* (1.1) *has at least two nontrivial solutions.*

Theorem 1.3. *Assume that* (I1), (I2), (F1)–(F3) *hold. Then* (1.1) *has at least k distinct pairs of solutions.*

The remainder of this article is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of our main result. Finally, an example is given to demonstrate the applicability of our main results in Section 4. Furthermore, we want to point out that a similar approach can be used to study different elliptic problems, such as in the paper [10].

2. Preliminaries and variational setting

Throughout this article, C, C_i denotes positive constants which may vary; $\to$ denotes the strong and $\rightharpoonup$ the weak convergence; B_r denotes the ball of radius r and E^* denotes the dual space of E.

The Sobolev space $E = W^{1,p}([a,b])$ is equipped with the norm

$$\|u\| = \Big(\int_a^b \rho(x)|u'(x)|^p + s(x)|u(x)|^p\Big)^{1/p},$$

which is equivalent to the usual one.

As usual, for $1 \leq p < +\infty$, we let

$$\|u\|_p = \Big(\int_a^b |u(x)|^p dx\Big)^{1/p}, \quad u \in L^p([a,b]),$$
$$\|u\|_\infty = \max_{x\in[a,b]}|u(x)|, \quad u \in C([a,b]).$$

Lemma 2.1 ([30, Lemma 2.6]). *For $u \in E$, then we have $\|u\|_\infty \leq C_1\|u\|$, where*

$$C_1 = 2^{1/q}\max\Big\{\frac{1}{(b-a)^{1/p}(\operatorname{ess\,inf}_{[a,b]} s)^{1/p}}, \frac{(b-a)^{1/q}}{(\operatorname{ess\,inf}_{[a,b]} \rho)^{1/p}}\Big\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Now we begin describing the variational formulation of problem (1.1). Consider the functional $\varphi : E \to \mathbb{R}$ defined by

$$\begin{aligned} \varphi(u) = \frac{\|u\|^p}{p} &+ \sum_{j=1}^m I_j(u(x_j)) + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p \\ &- \int_a^b F(x,u)dx. \end{aligned} \tag{2.1}$$

Since f and $\iota_j (j = 1, 2, \dots, m)$ are continuous, we deduce that φ is of class C^1 and its derivative is given by

$$\begin{aligned} \varphi'(u)v = \int_a^b &\rho(x)\Phi_p(u'(x))v'(x)dx + \int_a^b s(x)\Phi_p(u(x))v(x)dx \\ &+ \rho(a)\Phi_p(\frac{\alpha_2 u(a)}{\alpha_1})v(a) + \rho(b)\Phi_p(\frac{\beta_2 u(b)}{\beta_1})v(b) + \sum_{j=1}^m \iota_j(u(x_j))v(x_j) \\ &- \int_a^b f(x,u(x))v(x)dx, \end{aligned} \tag{2.2}$$

for all $u, v \in E$. Then we can infer that $u \in E$ is a critical point of φ if and only if it is a solution of (1.1).

We will use Morse theory in combination with local linking arguments to obtain the critical points of φ. Now, it is necessary to recall the following definitions and results.

Definition 2.2. Let E be a real reflexive Banach space. We say that φ satisfies the (PS)-condition, i.e. every sequence $\{u_n\} \subset E$ satisfying $\varphi(u_n)$ bounded and $\lim_{n\to\infty} \varphi'(u_n) = 0$ contains a convergent subsequence.

Let E be a real Banach space and $\varphi \in C^1(E,\mathbb{R})$. $K = \{u \in E : \varphi'(u) = 0\}$, then the q-th critical group of φ at an isolated critical point $u \in K$ with $\varphi(u) = c$

is defined by

$$C_q(\varphi,u):=H_q(\varphi^c\cap U,\varphi^c\cap U\setminus\{u\}),\quad q\in\mathbb{N}:=\{0,1,2,\dots\},$$

where $\varphi^c=\{u\in E:\varphi(u)\le c\}$, U is a neighborhood of u, containing the unique critical point, H_* is the singular relative homology with coefficient in an Abelian group G.

We say that $u\in E$ is a homological nontrivial critical point of φ if at least one of its critical groups is nontrivial. Now, we present the following propositions which will be used later.

Proposition 2.3 ([15, Proposition 2.1]). *Assume that φ has a critical point $u=0$ with $\varphi(0)=0$. Suppose that φ has a local linking at 0 with respect to $E=V\oplus W$, $k=\dim V<\infty$; that is, there exists $\rho>0$ small such that*

$$\begin{gathered}\varphi(u)\le 0,\quad u\in V,\quad \|u\|\le\rho;\\ \varphi(u)>0,\quad u\in W,\quad 0<\|u\|\le\rho.\end{gathered}$$

Then $C_k(\varphi,0)\ncong 0$, hence 0 is a homological nontrivial critical point of φ.

Proposition 2.4 ([15, Theorem 2.1]). *Let E be a real Banach space and let $\varphi\in C^1(E,\mathbb{R})$ satisfy the (PS)-condition and is bounded from below. If φ has a critical point that is homological nontrivial and is not a minimizer of φ, then φ has at least three critical points.*

Proposition 2.5 ([22, Theorem 9.1]). *Let E be a real Banach space, $\varphi\in C^1(E,\mathbb{R})$ with φ even, bounded from below, and satisfying (PS)-condition. Suppose $\varphi(0)=0$, there is a set $K\subset E$ such that K is homeomorphic to S^{j-1} by an odd map, and $\sup_K\varphi<0$. Then φ possesses at least j distinct pairs of critical points.*

3. Proof of main results

In this section, we prove Theorems 1.2 and 1.3. To complete the proof, we need the following lemmas.

Lemma 3.1. *Suppose that φ satisfies* (I1), (F1), *then φ satisfies the (PS)-condition.*

Proof. We first prove that φ is coercive. It follows from $(I1)$ and (F1) that

$$\begin{aligned}\varphi(u)&=\frac{\|u\|^p}{p}+\sum_{j=1}^m I_j(u(x_j))+\frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p+\frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p-\int_a^b F(x,u)dx\\ &\ge\frac{\|u\|^p}{p}+\frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p+\frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p-\int_a^b c_1|u|^{\alpha+1}dx\\ &\ge\frac{\|u\|^p}{p}+\frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p+\frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p-C_2\|u\|^{\alpha+1}\end{aligned}$$

Since $\alpha+1<p$, it follows that $\varphi(u)\to+\infty$ as $\|u\|\to\infty$.

Suppose that $\{u_n\}$ is a (PS) sequence, then $\{u_n\}$ is bounded, there exists a constant $M>0$ such that

$$\|u_n\|\le M,\quad\forall n\in\mathbb{N}.\tag{3.1}$$

Going to a subsequence, if necessary, we can assume that $u_n \rightharpoonup u_0$ in E. Hence, by compact embedding theorem of Sobolev space, we have

$$u_n \to u_0 \text{ in } L^p([a,b]), \quad u_n \to u_0 \text{ a.e. } x \in [a,b].$$

By (2.2), we have

$$\begin{aligned}
&(\varphi'(u_n) - \varphi'(u_0), u_n - u_0) \\
&= \int_a^b \rho(x)(\Phi_p(u_n'(x)) - \Phi_p(u_0'(x)))(u_n'(x) - u_0'(x))dx \\
&\quad + \int_a^b s(x)(\Phi_p(u_n(x)) - \Phi_p(u_0(x)))(u_n(x) - u_0(x))dx \\
&\quad + \rho(a)(\Phi_p(\frac{\alpha_2 u_n(a)}{\alpha_1}) - \Phi_p(\frac{\alpha_2 u_0(a)}{\alpha_1}))(u_n(a) - u_0(a)) \\
&\quad + \rho(b)(\Phi_p(\frac{\beta_2 u_n(b)}{\beta_1}) - \Phi_p(\frac{\beta_2 u_0(b)}{\beta_1}))(u_n(b) - u_0(b)) \\
&\quad + \sum_{j=1}^m (\iota_j(u_n(x_j)) - \iota_j(u_0(x_j)))(u_n(x_j) - u_0(x_j)) \\
&\quad - \int_a^b (f(t, u_n(x)) - f(t, u_0(x)))(u_n(x) - u_0(x))dx.
\end{aligned} \tag{3.2}$$

If $p \geq 2$, it is easy to show that for any $x, y \in \mathbb{R}$, there exists $c_p > 0$ such that

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq c_p|x - y|^p, \quad p \geq 2.$$

Combining this inequality with (3.2), we have

$$\begin{aligned}
c_p\|u_n - u_0\|^p &\leq \|\varphi'(u_n) - \varphi'(u_0)\|\|u_n - u_0\| \\
&\quad - \rho(a)(\Phi_p(\frac{\alpha_2 u_n(a)}{\alpha_1}) - \Phi_p(\frac{\alpha_2 u_0(a)}{\alpha_1}))(u_n(a) - u_0(a)) \\
&\quad - \rho(b)(\Phi_p(\frac{\beta_2 u_n(b)}{\beta_1}) - \Phi_p(\frac{\beta_2 u_0(b)}{\beta_1}))(u_n(b) - u_0(b)) \\
&\quad - \sum_{j=1}^l (\iota_j(u_n(x_j)) - \iota_j(u_0(x_j)))(u_n(x_j) - u_0(x_j)) \\
&\quad + \int_a^b (f(x, u_n(x)) - f(x, u_0(x)))(u_n(x) - u_0(x))dx.
\end{aligned}$$

It follows directly that $u_n \to u_0$ in E.

If $1 < p < 2$, by the results of [4], there exists $d_p > 0$ such that

$$\begin{aligned}
&\int_a^b \rho(x)(\Phi_p(u_n'(x)) - \Phi_p(u_0'(x)))(u_n'(x) - u_0'(x))dx \\
&+ \int_a^b s(x)(\Phi_p(u_n(x)) - \Phi_p(u_0(x))) \\
&\geq \frac{d_p 2^{p-2}\|u_n - u_0\|^2}{(\|u_n\| + \|u_0\|)^{2-p}}
\end{aligned}$$

Similarly, we can obtain that $u_n \to u_0$ in E, i.e. φ satisfies the (PS)-condition. □

We choose an orthogonal basis $\{e_j\}$ of E and define $X_j := \operatorname{span}\{e_j\}$, $j = 1, 2, \dots$, $Y_k := \oplus_{j=1}^{k} X_j$, $Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j}$, then $E = Y_k \oplus Z_k$.

Lemma 3.2. *Suppose that* Φ *satisfies* (I1), (F2)*, then there exists* $k_0 \in \mathbb{N}$ *such that* $C_{k_0}(\varphi, 0) \not\cong 0$.

Proof. Since $F(x, 0) = 0$ and $I_j(0) = 0 (j = 1, 2, \dots, m)$, then the zero function is a critical point of φ. So we only need to prove that φ has a local linking at 0 with respect to $E = Y_k \oplus Z_k$.

Step 1. Take $u \in Y_k$, since Y_k is finite dimensional, we have that for given r_0, there exists $0 < \rho < 1$ small such that

$$u \in Y_k, \quad \|u\| \le \rho \Rightarrow |u| < r_0, \quad x \in [a, b]$$

For $0 < r < r_0$, let $\Omega_1 = \{x \in [a, b] : |u(x)| < r\}$, $\Omega_2 = \{x \in [a, b] : r \le |u(x)| \le r_0\}$, $\Omega_3 = \{x \in [a, b] : |u(x)| > r_0\}$, then $[a, b] = \bigcup_{i=1}^{3} \Omega_i$. For the sake of simplicity, let $G(x, u) = F(x, u) - c_2|u|^{\gamma}$. Therefore, it follows form (I1) and (F2) that

$$\begin{aligned}
\varphi(u) &\le \frac{1}{p}\|u\|^p + \sum_{j=1}^{m} a_j |u|^{\gamma_j+1} + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p \\
&\quad - \int_a^b c_2|u|^{\gamma} dx - \Big(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3}\Big) G(x, u) dx \\
&\le \frac{1}{p}\|u\|^p + \sum_{j=1}^{m} a_j \|u\|^{\gamma_j+1} + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p \\
&\quad - \int_a^b c_2|u|^{\gamma} dx - \int_{\Omega_1} G(x, u) dx.
\end{aligned}$$

Note that the norms on Y_k are equivalent to each other, $\|u\|_p$ is equivalent to $\|u\|$ and $\int_{\Omega_1} G(x, u)dx \to 0$ as $r \to 0$. Since $0 < \gamma < max\{\gamma_j + 1\} < p$, then $\Phi(u) \le 0$, for all $u \in Y_k$ with $\|u\| \le \rho$.

Step 2. Take $u \in Z_k$, Since the embedding $E \hookrightarrow L^p([a, b])$ is compact. We have that for given $\varepsilon > 0$, there exists $0 < \rho < 1$ small such that

$$u \in Z_k,\ \|u\| \le \rho \Rightarrow |u| < \varepsilon, \quad x \in [a, b].$$

Therefore, it follows from (I1) and (F2) that

$$\varphi(u) \ge \frac{1}{p}\|u\|^p - \int_a^b c_3|u|^p dx \ge \frac{1}{p}\|u\|^p - \frac{1}{p}\|u\|^p > 0.$$

The proof is complete. □

Proof of Theorem 1.2. By Lemma 3.1, φ satisfies the (PS)-condition and is bounded from below. By Lemma 3.2 and Proposition 2.3, the trivial solution $u = 0$ is homological nontrivial and is not a minimizer. Then it follows immediately from Proposition 2.4 that (1.1) has at least two nontrivial solutions. □

Proof of Theorem 1.3. By (I2) and (F3), we can easily check the functional φ is even. Lemma 3.1 shows that φ satisfies the (PS)-condition and is bounded from below. For $\rho > 0$, let $K = S_\rho = \{u \in Y_k : \|u\| = \rho\}$. Thus, just as shown in the proof of Lemma 3.2, if $\rho > 0$ is small enough, we have that

$$\sup_K \varphi(u) \le 0.$$

By the definition of Y_k, we have $\dim Y_k = k$, then by Proposition 2.5, we have that φ has at least k distinct pairs of critical points. Therefore, (1.1) has at least k distinct pairs of solutions. □

4. An example

In this section, we illustrate our main results with an example. In problem (1.1), let $p = 2$, $\rho(x) = s(x) = 1$,

$$f(x,u) = \frac{1+\sin^2 x}{1+e^{|x|}} \cdot \frac{2n-2}{n}|u|^{-\frac{2}{n}}u,$$
$$\iota_j(u) = \frac{2n-1}{n}|u|^{-\frac{1}{n}}u(j=1,2,\dots,m),$$

then

$$F(x,u) = \frac{1+\sin^2 x}{1+e^{|x|}}|u|^{\frac{2n-2}{n}}, \quad I_j(u) = |u|^{\frac{2n-1}{n}}.$$

When n is an integer (large enough), we know that f satisfies the conditions (F1) and (F2) and impulses ι_j $(j = 1, 2, \dots, m)$ fulfill (I1). By Theorem 1.2, the problem has at least two nontrivial solutions. Furthermore, we can show that the nonlinearity f and the impulses ι_j $(j = 1, 2, \dots, m)$ are all even. Thus by Theorem 1.3, the problem has k distinct pairs of solutions.

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