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$\mathcal{S}$-ASYMPTOTICALLY ω-PERIODIC SOLUTIONS FOR ABSTRACT NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article we study the existence of $\mathcal{S}$-asymptotically ω-periodic solutions for abstract neutral functional differential equations recently, introduced in the literature. An application involving a partial neutral differential equation is presented.

1. INTRODUCTION

In this work we study the existence of $\mathcal{S}$-asymptotically ω-periodic solutions for a class of abstract neutral differential equations of the form

$$u'(t) = Au(t) + f(t, u_t, u'_t), \quad t \in [0, \infty), \tag{1.1}$$

$$u_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, the history u_t belongs to an abstract Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ defined axiomatically, u'_t denotes the derivative at t of the function $s \to u_s$ and $f(\cdot)$ is a suitable function.

The neutral system (1.1)–(1.2) was introduced recently by Hernández and O' Regan [21]. As pointed out in [21], the study of the existence of solutions for this class of problems via semigroup methods and fixed point techniques is highly nontrivial since the temporal derivative of the solution appears in the integral equation used to define the concept of mild solution of (1.1)–(1.2), see Definition 2.4. As a consequence, it is necessary to work on spaces of differentiable functions which is a complex problem under the semigroup framework.

To the best of our knowledge, the paper [21] is the first and only work treating neutral problems described in the abstract form (1.1)–(1.2). On the current state of the theory of abstract neutral differential equations we cite [1, 7, 10, 11, 15, 16, 17, 18, 19, 20, 22, 23, 33] and the references therein.

The concept of $\mathcal{S}$-asymptotically ω-periodic was introduced recently in the literature, see [13, 14]. We note that a continuous function $u(\cdot)$ defined on $[0, \infty)$ is said to be $\mathcal{S}$-asymptotically periodic if there exists $\omega \in \mathbb{R}$ such that $\lim_{t\to\infty}[f(t + \omega) - f(t)] = 0$.

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For qualitative properties of $\mathcal{S}$-asymptotically ω-periodic functions, we cite [13, 28, 14]. Concerning the problem of the existence of $\mathcal{S}$-asymptotically ω-periodic solutions for differential equations, we cite [8, 12, 31, 32, 34] for ordinary differential equations on finite dimensional spaces and [4, 5, 6, 13, 14] for ordinary differential equations defined on abstract Banach spaces.

The abstract problem (1.1)–(1.2) arises, for example, in the theory of heat conduction in fading memory material. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depends linearly on the temperature $u(\cdot)$ and on its gradient $\nabla u(\cdot)$. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [9, 29], the internal energy and the heat flux are described as functionals of u and u_x. The next system, see [2, 3, 25], has been frequently used to describe these phenomena,

$$\begin{gathered}\frac{d}{dt}(u(t,x)+\int_{-\infty}^{t}k_1(t-s)u(s,x)ds)=c\Delta u(t,x)+\int_{-\infty}^{t}k_2(t-s)\Delta u(s,x)ds,\\ u(t,x)=0,\quad x\in\partial\Omega.\end{gathered}\tag{1.3}$$

In this system, $\Omega\subset\mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x)\in[0,\infty)\times\Omega$, $u(t,x)$ denotes the temperature in x at time t, c is a physical constant and $k_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are the internal energy and the heat flux relaxation respectively. If we assume that $k_1=\gamma_1+\gamma_2$ and the solution $u(\cdot)$ is known on $(-\infty,0]$, we can study the above system via the initial-value problem

$$\begin{gathered}\frac{d}{dt}(u(t,x)+\int_{-\infty}^{t}\gamma_1(t-s)u(s,x)ds)\\ =c\Delta u(t,x)+\int_{-\infty}^{t}k_2(t-s)\Delta u(s,x)ds-\int_{-\infty}^{t}\gamma_2(t-s)u'(s,x)ds,\\ u(s,x)=\varphi(s,x),\quad s\geq 0,x\in\Omega,\end{gathered}\tag{1.4}$$

which can be represented in the abstract form (1.1)–(1.2). For additional applications and examples on neutral differential equations we cite our recent papers [18, 21] and the references therein.

Next, we include some notations, definitions and technicalities. Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. In this paper, $\mathcal{L}(Z,W)$ denotes the space of bounded linear operators from Z into W endowed with the norm of operators denoted $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z=W$. We use the notation $Z\hookrightarrow W$ to indicate that Z is continuously included in W.

Let $I\subset\mathbb{R}$. As usual, $C(I;Z)$ is the space formed by all the bounded continuous functions from I into Z endowed with the sup-norm denoted by $\|\cdot\|_{C(I;Z)}$ and $C^1(I;Z)$ the space formed by all the functions $u\in C(I;Z)$ such that $u'\in C(I;Z)$ endowed with the norm $\|u\|_{C^1(I;Z)}=\|u\|_{C(I;Z)}+\|u'\|_{C(I;Z)}$. In addition, $C^\gamma([0,\infty);Z)$ (with $\gamma\in(0,1)$) is the space formed by all the functions $\xi\in C([0,\infty);Z)$ such that

$$[\xi]_{C^\gamma([0,\infty);Z)}=\sup_{t,s\in[0,\infty),t\neq s}\frac{\|\xi(s)-\xi(t)\|_Z}{|\,t-s\,|^\gamma}$$

is finite, provided with the norm $\|\xi\|_{C^{\gamma}([0,\infty);Z)} = \|\xi\|_{C([0,\infty);Z)} + [\xi]_{C^{\gamma}([0,\infty);Z)}$. The notation $C^{1+\gamma}([0,\infty);Z)$ is used for the space of all the differentiable functions $\xi \in C^{\gamma}([0,\infty);Z)$ such that $\xi' \in C^{\gamma}([0,\infty);Z)$ endowed with the norm $\|\xi\|_{C^{1+\gamma}([0,\infty);Z)} = \|\xi\|_{C^{\gamma}([0,\infty);Z)} + \|\xi'\|_{C^{\gamma}([0,\infty);Z)}$.

From [13] we note the following concept.

Definition 1.1. A function $f \in C([0,\infty), Z)$ is said to be $\mathcal{S}$-asymptotically periodic if there exists $\omega \in \mathbb{R}$ such that $\lim_{t\to\infty}[f(t+\omega) - f(t)] = 0$. In this case, we say that ω is an asymptotic period of $f(\cdot)$ and that $f(\cdot)$ is $\mathcal{S}$-asymptotically ω-periodic.

Here $SAP_{\omega}(Z)$ denotes the space formed by the Z-valued $\mathcal{S}$-asymptotically ω-periodic functions provided with the norm $\|\cdot\|_{C([0,\infty);Z)}$.

In this article, $A : D(A) \subset X \to X$ is the generator of an uniformly stable analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X and γ, C_i, $i \in \mathbb{N}$, are positive constants such that

$$\|A^i T(t)\|_{\mathcal{L}(X)} \leq \frac{C_i e^{-\gamma t}}{t^i}$$

for all $t > 0$ and each $i \in \mathbb{N}$. For $\beta > 0$, we represent by X_{β} the domain of the fractional power $(-A)^{\beta}$ of $-A$ endowed with the norm $\|x\|_{\beta} = \|(-A)^{\beta}x\|$. In addition, for $\beta > 0$ we assume that $C_{\beta} > 0$ is such that

$$\|AT(t)\|_{\mathcal{L}(X_{\beta},X)} \leq \frac{C_{\beta}e^{-\gamma t}}{t^{\beta}}, \quad \forall t > 0.$$

The notation $D_A(\eta,\infty)$, $\eta \in (0,1)$, stands for the space

$$D_A(\eta,\infty) = \{x \in X : [x]_{\eta,\infty} = \sup_{t\in(0,1)} \|t^{1-\eta}AT(t)x\| < \infty\},$$

with the norm $\|x\|_{\eta,\infty} = [x]_{\eta,\infty} + \|x\|$ and we assume that for all $k \in \mathbb{N}\cup\{0\}$ there exits a constant $C_{k,\eta}$ such that

$$\|A^k T(t)\|_{\mathcal{L}(D_A(\eta,\infty),X)} \leq \frac{C_{k,\eta}e^{-\gamma t}}{t^{1-\eta}}$$

for all $t > 0$. For additional details on analytic semigroups and interpolation spaces we cite [26].

In this article, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space formed by functions defined from a connected interval $\{0\} \subset J \subset (-\infty,0]$ into X, satisfying the following conditions.

- (A1) If $x : (J+\{\sigma\}) \cup [\sigma,\sigma+b) \to X$, $b > 0$, $\sigma \in \mathbb{R}$, is continuous on $[\sigma,\sigma+b)$ and $x_{\sigma} \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+b)$ the following conditions hold:
  - (i) the function $s \to x_s$ belongs to $C([\sigma,\sigma+b),\mathcal{B})$,
  - (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
  - (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_{\sigma}\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M \in C([0,\infty);\mathbb{R}^+)$ and $H, K(\cdot), M(\cdot)$ are independent of $x(\cdot)$.
  - (iv) If $(\psi^n)_{n\in\mathbb{N}}$ is a sequence in $C(J,X) \cup \mathcal{B}$ and $\psi^n \to \psi$ uniformly on compact subsets of J, then $\psi \in \mathcal{B}$ and $\|\psi^n - \psi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Remark 1.2. For $\beta > 0$, we represent by $\mathcal{B}_{\beta}$ the space $\mathcal{B}_{\beta} = \{(-A)^{-\beta}\psi : \psi \in \mathcal{B}\}$ endowed with the norm $\|\psi\|_{\mathcal{B}_{\beta}} = \|(-A)^{\beta}\psi\|_{\mathcal{B}_{\beta}}$. We note that $\mathcal{B}_{\beta}$ verifies the axiom (A1) with X_{β} in place X.

Remark 1.3. In the remainder of this paper, to simplify, we assume that $\mathcal{K} > 0$ is a constant such that $\max\{K(t), M(t)\} \leq \mathcal{K}$ for all $t \geq 0$.

From from [21] we note the following result. In this result, P_u is the function defined by $P_u(t) = u_t$. We will use this notation for the remainder of this article.

Lemma 1.4. *If $u \in C^1(J \cup [0,b]; X)$, then $P_u \in C^1([0,b]; \mathcal{B})$ and $\frac{d}{dt}P_u(t) = P_{\frac{du}{dt}}(t)$ for all $t \in [0,b]$.*

This article has three sections. In the next section we study the existence of $\mathcal{S}$-asymptotically ω-periodic strict solutions for the problem (1.1)–(1.2). In the last section, an application involving a partial neutral differential equation is presented.

2. Existence results

In this section we study the existence of $\mathcal{S}$-asymptotically ω-periodic strict solution for the abstract neutral problem (1.1)–(1.2). To prove our results, we consider the following conditions.

- (H1) There are $\alpha \in (0,1)$, $\beta \in (0,1]$ and functions $f_1 \in C^\alpha([0,\infty); \mathcal{L}(\mathcal{B}_\beta, X))$, $f_2 \in C^\alpha([0,\infty); \mathcal{L}(\mathcal{B}, X))$ such that $f(t,\psi_1,\psi_2) = f_1(t,\psi_1) + f_2(t,\psi_2)$ for all $t \ge 0$, $\psi_1 \in \mathcal{B}_\beta$ and $\psi_2 \in \mathcal{B}$.
- (H2) There is a Banach space $(Y, \|\cdot\|_Y) \hookrightarrow (X, \|\cdot\|)$, a integrable function $H \in L^1([0,\infty); \mathbb{R}^+)$, a continuous function $L_f \in C([0,\infty); \mathbb{R}^+)$ and $\omega > 0$ such that $f \in C([0,\infty) \times \mathcal{B}_\beta \times \mathcal{B}; Y)$, $\|AT(s)\|_{\mathcal{L}(Y,X)} \le H(s)$ for all $s > 0$, and
$$\|f(t,\psi_1,\zeta_1) - f(t,\psi_2,\zeta_2)\|_Y \le L_f(t)(\|\psi_1 - \psi_2\|_{\mathcal{B}_\beta} + \|\zeta_1 - \zeta_2\|_{\mathcal{B}}), \tag{2.1}$$
for all $\psi_i \in \mathcal{B}_\beta$, $\zeta_i \in \mathcal{B}$, $i = 1,2$, and every $t \ge 0$.
- (H3) There is a Banach space $(Y, \|\cdot\|_Y) \hookrightarrow (X, \|\cdot\|)$, such that the function $f(\cdot)$ belongs to $C([0,\infty) \times \mathcal{B}_\beta \times \mathcal{B}; Y)$ and $f(\cdot)$ is uniformly $\mathcal{S}$-asymptotically ω-periodic on bounded sets; that is,
$$\lim_{t\to\infty} \sup_{\|\psi\|_{\mathcal{B}_\beta} \le r, \|\xi\|_{\mathcal{B}} \le r} \|f(t+\omega,\psi,\xi) - f(t,\psi,\xi)\|_Y = 0.$$

To prove our results, is convenient to include some comments on the problem
$$u'(t) = Au(t) + \xi(t), \quad t \in I, \tag{2.2}$$
$$u(0) = x \quad \in X, \tag{2.3}$$
with $I = [0,a]$ or $I = [0,\infty)$ and $\xi \in L^1(I;X)$. We note that the function $u : I \to X$ given by $u(t) = T(t)x + \int_0^t T(t-s)\xi(s)ds$ is called a mild solution of (2.2)-(2.3) on I and that a function $v \in C(I;X)$ is said to be a strict solution (2.2)-(2.3) on I if $v \in C^1(I;X) \cap C(I;X_1)$ and $v(\cdot)$ satisfies (2.2)-(2.3).

The proof of Proposition 2.1 is similar to the proof of [26, Theorem 4.3.1]. We include some details of the proof for completeness.

Proposition 2.1. *Assume $\xi \in C^\alpha([0,\infty); X)$, $x \in X_1$ and $Ax + \xi(0) \in D_A(\alpha,\infty)$. If $u(\cdot)$ is the mild solution of (2.2)-(2.3) on $[0,\infty)$, then $u(\cdot)$ is a strict solution, $u \in C^\alpha([0,\infty); X_1)$ and*
$$u'(t) = AT(t)x + \int_0^t AT(t-s)(\xi(s) - \xi(t))ds + T(t)\xi(t), \quad \forall t \ge 0. \tag{2.4}$$
Moreover,
$$[u]_{C^\alpha([0,\infty);X_1)} \le \frac{C_{1,\alpha}}{\alpha}\|Ax + \xi(0)\|_{\alpha,\infty} + \Lambda\|\xi\|_{C^\alpha([0,\infty);X)}, \tag{2.5}$$

$$[u']_{C^\alpha([0,\infty);X)} \le \frac{C_{1,\alpha}}{\alpha}\|Ax+\xi(0)\|_{\alpha,\infty} + (\Lambda+1)\|\xi\|_{C^\alpha([0,\infty);X)}, \tag{2.6}$$

$$\|u\|_{C([0,\infty);X_1)} \le C_0\|Ax\| + \Lambda_1[\xi]_{C^\alpha([0,\infty);X)} + \Lambda_2\|\xi\|_{C([0,\infty);X)}, \tag{2.7}$$

$$\|u'\|_{C([0,\infty);X)} \le C_0\|Ax\| + \Lambda_1[\xi]_{C^\alpha([0,\infty);X)} + (\Lambda_2+1)\|\xi\|_{C([0,\infty);X)}, \tag{2.8}$$

where $\Lambda = (\frac{2C_1}{\alpha} + 3C_0 + 1 + \frac{C_2}{\alpha(1-\alpha)})$, $\Lambda_1 = C_1(\frac{1}{\gamma}+\frac{1}{\alpha})$ *and* $\Lambda_2 = (C_0+1)$.

Proof. Let $T>0$. From [26, Theorem 4.3.1] we know that $u_{|[0,T]}$ is a strict solution of the problem (2.2)-(2.3) on $[0,T]$, $u \in C^\alpha([0,T]);X_1)$ and that the representation (2.4) is valid on $[0,T]$. Moreover, a review of the proof of [26, Theorem 4.3.1] permit us to assert that

$$[u]_{C^\alpha([0,T];X_1)} \le \frac{C_{1,\alpha}}{\alpha}\|Ax+\xi(0)\|_{\alpha,\infty} + (\frac{2C_1}{\alpha} + 3C_0 + 1 + \frac{C_2}{\alpha(1-\alpha)})[\xi]_{C^\alpha([0,T];X)}.$$

From the above, we infer that $u(\cdot)$ is a strict solution of the problem (2.2)-(2.3) on $[0,\infty)$, the representation (2.4) is satisfied on $[0,\infty)$ and (2.5) is valid with $\Lambda = (\frac{2C_1}{\alpha} + 3C_0 + 1 + \frac{C_2}{\alpha(1-\alpha)})$. In addition, from the representation

$$Au(t) = T(t)Ax + \int_0^t AT(t-s)(\xi(s)-\xi(t))ds + (T(t)-I)\xi(t),$$

it is easy to see that

$$\|u\|_{C([0,\infty);X_1)} \le C_0\|Ax\| + C_1(\frac{1}{\gamma}+\frac{1}{\alpha})[\xi]_{C^\alpha([0,\infty);X)} + (C_0+1)\|\xi\|_{C([0,\infty);X)},$$

which establish (2.7). Finally, from (2.5) and (2.7), and the fact that $u(\cdot)$ is a strict solution we obtain (2.6) and (2.8). This completes the proof. □

For completeness, from [21] we quote the followings result.

Lemma 2.2. *Assume the condition* (H2) *is satisfied,* $x \in X_1$ *and* $\xi \in C([0,b];Y)$. *Then, the mild solution* $w(\cdot)$ *of* (2.2)-(2.3) *is a strict solution and*

$$w'(t) = T(t)Ax + \int_0^t AT(t-s)\xi(s)ds + \xi(t), \quad \forall t \in I.$$

Remark 2.3. In the remainder of this paper, for $u \in C(J\cup[0,\infty);X_\beta)$, $v \in C(J\cup[0,\infty);X)$ with $u_0 \in \mathcal{B}_\beta$ and $v_0 \in \mathcal{B}$, we use the notations $\mathcal{P}_v$, $\mathcal{P}_u$ and $f_{u,v}$ for the functions $\mathcal{P}_v : [0,\infty) \to \mathcal{B}$, $\mathcal{P}_u : [0,\infty) \to \mathcal{B}_\beta$ and $f_{u,v} : [0,\infty) \to X$ given by $\mathcal{P}_v(t) = v_t$, $\mathcal{P}_u(t) = u_t$ and $f_{u,v}(t) = f(t,u_t,v_t)$.

By considering the above remarks, we adopt the following concepts of solution.

Definition 2.4. A function $u : J\cup[0,\infty) \to X$ is called a mild solution of the abstract problem (1.1)–(1.2) if $u_0 = \varphi$, $\mathcal{P}_u \in C^1([0,\infty);\mathcal{B}) \cap C([0,\infty);\mathcal{B}_\beta)$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)f_{u,u'}(s)ds, \quad \forall t \in [0,\infty).$$

Definition 2.5. A function $u : J\cup[0,\infty) \to X$ is said to be a strict solution of (1.1)–(1.2) if $\mathcal{P}_u \in C^1([0,\infty);\mathcal{B}) \cap C([0,\infty);\mathcal{B}_\beta)$, $u \in C([0,\infty);X_1)$ and $u(\cdot)$ satisfies (1.1)–(1.2).

Next, we include some results on $\mathcal{S}$-asymptotically ω-periodic functions. In the next results, $(Z,\|\cdot\|_Z)$ is a Banach space.

Lemma 2.6. *Assume* $(Z, \|\cdot\|_Z) \hookrightarrow (X, \|\cdot\|)$, $u \in C(J \cup [0,\infty); Z)$ *and* $u_0 \in \mathcal{B}$. *If* $u\big|_{[0,\infty)} \in SAP_\omega(Z)$ *and* $M(t) \to 0$ *as* $t \to \infty$, *then* $\mathcal{P}_u \in SAP_\omega(\mathcal{B})$. *Similarly, if* $(Z, \|\cdot\|_Z) \hookrightarrow (X_\beta, \|\cdot\|)$, $w \in C(J \cup [0,\infty); Z)$, $w_0 \in \mathcal{B}_\beta$, $w\big|_{[0,\infty)} \in SAP_\omega(Z)$ *and* $M(t) \to 0$ *as* $t \to \infty$, *then* $\mathcal{P}_w \in SAP_\omega(\mathcal{B}_\beta)$.

Proof. We only prove the assertion involving the space X. Let $i_{Z,X} : Z \to X$ be the inclusion map from Z into X. For $1 > \varepsilon > 0$ there exists $L_\varepsilon > 0$ such that $\|u(t+\omega) - u(t)\|_Z \le \varepsilon$ and $M(t) \le \varepsilon$ for all $t \ge L_\varepsilon$. Then, for $t \ge 2L_\varepsilon$ we obtain

$$
\begin{aligned}
&\|\mathcal{P}_u(t+\omega) - \mathcal{P}_u(t)\|_{\mathcal{B}} \\
&= \|u_{t+\omega} - u_t\|_{\mathcal{B}} \\
&\le M(t - L_\varepsilon)\|u_{L_\varepsilon}\|_{\mathcal{B}} + \mathcal{K}\|i_{Z,X}\|_{\mathcal{L}(Z,X)} \sup_{s \ge L_\varepsilon} \|u(s+\omega) - u(s)\|_Z \\
&\le M(t - L_\varepsilon)\big(M(L_\varepsilon)\|u_0\|_{\mathcal{B}} + \mathcal{K}\|i_{Z,X}\|_{\mathcal{L}(Z,X)}\|u\|_{C([0,\infty);Z)}\big) + \mathcal{K}\|i_{Z,X}\|_{\mathcal{L}(Z,X)}\varepsilon \\
&\le \varepsilon\big(\varepsilon\|u_0\|_{\mathcal{B}} + \mathcal{K}\|i_{Z,X}\|_{\mathcal{L}(Z,X)}\|u\|_{C([0,\infty);Z)} + \mathcal{K}\|i_{Z,X}\|_{\mathcal{L}(Z,X)}\big),
\end{aligned}
$$

which implies that $\lim_{t\to\infty} \|\mathcal{P}_u(s+\omega) - \mathcal{P}_u(s)\|_{\mathcal{B}} = 0$ and $\mathcal{P}_u \in SAP_\omega(\mathcal{B})$. □

Lemma 2.7. *Assume* $\mathcal{Q} \in C([0,\infty); \mathcal{L}(Z,X)) \cap L^1([0,\infty); \mathcal{L}(Z,X))$, $v \in SAP_\omega(Z)$ *and let* $u : [0,\infty) \to X$ *be the function given by* $u(t) = \int_0^t \mathcal{Q}(t-s)v(s)ds$. *Then* $u \in SAP_\omega(X)$.

Proof. From Bochner's criteria for integrable functions and the estimate

$$
\|u(t)\| \le \int_0^t \|\mathcal{Q}(t-s)\|_{\mathcal{L}(Z,X)}\|v(s)\|_Z ds \le \|\mathcal{Q}\|_{L^1([0,\infty),\mathcal{L}(Z,X))}\|v\|_{C([0,\infty);Z)},
$$

it follows that $v \in C([0,\infty); X)$. On the other hand, for $\varepsilon > 0$ given there is $L_\varepsilon > 0$ such that $\|v(s+\omega) - v(s)\|_Z \le \varepsilon$ and $\|\mathcal{Q}\|_{L^1([L_\varepsilon,\infty),\mathcal{L}(Z,X))} \le \varepsilon$ for all $s \ge L_\varepsilon$. Then, for $t \ge 2L_\varepsilon$ we obtain

$$
\begin{aligned}
&\|u(t+\omega) - u(t)\| \\
&\le \int_0^\omega \|\mathcal{Q}(t+\omega-s)\|_{\mathcal{L}(Z,X)}\|v(s)\|_Z ds \\
&\quad + \int_0^t \|\mathcal{Q}(t-s)\|_{\mathcal{L}(Z,X)}\|v(s+\omega) - v(s)\|_Z ds \\
&\le \|v\|_{C([0,\infty);Z)} \int_t^{t+\omega} \|\mathcal{Q}(s)\|_{\mathcal{L}(Z,X)} ds \\
&\quad + 2\|v\|_{C([0,\infty);Z)} \int_0^{L_\varepsilon} \|\mathcal{Q}(t-s)\|_{\mathcal{L}(Z,X)} ds + \varepsilon \int_{L_\varepsilon}^t \|\mathcal{Q}(t-s)\|_{\mathcal{L}(Z,X)} ds \\
&\le \|v\|_{C([0,\infty);Z)}\|\mathcal{Q}\|_{L^1([L_\varepsilon,\infty);\mathcal{L}(Z,X))} + 2\|v\|_{C([0,\infty);Z)}\|\mathcal{Q}\|_{L^1([L_\varepsilon,\infty);\mathcal{L}(Z,X))} \\
&\quad + \varepsilon\|\mathcal{Q}\|_{L^1([0,\infty);\mathcal{L}(Z,X))} \\
&\le \varepsilon(3\|v\|_{C([0,\infty);Z)} + \|\mathcal{Q}\|_{L^1([0,\infty);\mathcal{L}(Z,X))}),
\end{aligned}
$$

which implies that $\lim_{t\to\infty}[u(s+\omega) - u(s)] = 0$ and $u \in SAP_\omega(X)$. □

In addition to the above, from [13] we note (without proof) the following corollary.

Corollary 2.8 ([13, Corollary 3.2]). *Let $w : [0,\infty) \to Z$ be a S-asymptotically ω-periodic function and assume that w' is bounded and uniformly continuous. Then w' is S-asymptotically ω-periodic.*

We include now some Lemmas on α-Hölder functions. We omit the proofs.

Lemma 2.9. *Assume that $u \in C^{\alpha}([0,b];X)$, $\psi \in \mathcal{B} \cap C^{\alpha}(J;X)$ and $u(0) = \psi(0)$. Let v be the function $v : J \cup [0,b] \to X$ given by $v = \psi$ on J and $v = u$ on $[0,b]$. Then $v \in C^{\alpha}(J \cup [0,b];X)$ and $[v]_{C^{\alpha}(J\cup[0,b];X)} \le [\psi]_{C^{\alpha}(J;X)} + [u]_{C^{\alpha}([0,b];X)}$. Moreover, the assertion is valid replacing X by X_β.*

Lemma 2.10. *Assume that the condition* (H1) *is satisfied, $z \in C^{\alpha}([0,\infty);\mathcal{B}_\beta)$ and $w \in C^{\alpha}([0,\infty);\mathcal{B})$. Then $f_1(\cdot,w(\cdot)) \in C^{\alpha}([0,\infty);X)$, $f_2(\cdot,w(\cdot)) \in C^{\alpha}([0,\infty);X)$ and*

$$\begin{aligned}
[f_1(\cdot,z)]_{C^{\alpha}([0,\infty);X)} &\le [f_1]_{C^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B}_\beta,X))}\|z\|_{C([0,\infty);\mathcal{B}_\beta)} \\
&\quad + \|f_1\|_{C([0,\infty);\mathcal{L}(\mathcal{B}_\beta,X))}[z]_{C^{\alpha}([0,\infty);\mathcal{B}_\beta)}, \\
[f_2(\cdot,w)]_{C^{\alpha}([0,\infty);X)} &\le [f_2]_{C^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B},X))}\|w\|_{C([0,\infty);\mathcal{B})} \\
&\quad + \|f_2\|_{C([0,\infty);\mathcal{L}(\mathcal{B},X))}[w]_{C^{\alpha}([0,\infty);\mathcal{B})}.
\end{aligned}$$

We can establish now our first result on the existence of an $\mathcal{S}$-asymptotically ω-periodic strict solution for (1.1)–(1.2). Next, Λ, Λ_1 and Λ_2 are the constant in Proposition 2.1 and $\tilde{\varphi} : J \cup [0,\infty) \to X$ is the function given by $\tilde{\varphi}(t) = \varphi(t)$ for $t \in J$ and $\tilde{\varphi}(t) = T(t)\varphi(0)$ for $t \ge 0$.

Theorem 2.11. *Assume* (H1) *is satisfied, $C(J;X) \hookrightarrow \mathcal{B}$, $M(t) \to 0$ as $t \to \infty$, $f_1 \in SAP_\omega(\mathcal{L}(\mathcal{B}_\beta,X))$ and $f_2 \in SAP_\omega(\mathcal{L}(\mathcal{B},X))$. Suppose that the function $\varphi(\cdot)$ belongs to $C^{1+\alpha}(J;X) \cap C^{\alpha}(J;X_\beta)$, $\varphi(0) \in X_1$, $\frac{d^-\varphi}{dt}(0) = A\varphi(0) \in D_A(\alpha,\infty)$, $f(0,\varphi,\varphi') = 0$, $\mathcal{P}_{\tilde{\varphi}} \in C^{1+\alpha}([0,\infty);\mathcal{B}) \cap C^{\alpha}([0,\infty);\mathcal{B}_\beta)$ and*

$$\Xi = \mathcal{K}\big[(\mathcal{K}+1)(\mathcal{A}\Lambda+1) + \mathcal{A}(\Lambda_1+\Lambda_2) + 1\big]\Theta(f_1,f_2) < 1,$$

where

$$\Theta(f_1,f_2) = \|f_1\|_{C^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B}_\beta,X))} + \|f_2\|_{C^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B},X))}$$

and $\mathcal{A} = \|A^{-1}\| + \|(-A)^{\beta-1}\| + 1$. Then there exists a unique $\mathcal{S}$-asymptotically ω-periodic strict solution $u \in C^{1+\alpha}(J \cup \mathbb{R}^+;X) \cap C(J \cup \mathbb{R}^+;X_\beta)$ of (1.1)–(1.2).

Proof. Let $\mathcal{Y} = C^{1+\alpha}(J \cup \mathbb{R}^+;X) \cap C^{\alpha}(J \cup \mathbb{R}^+;X_\beta)$ endowed with the norm $\|\cdot\|_{\mathcal{Y}}$ given by $\|\cdot\|_{\mathcal{Y}} = \|\cdot\|_{C^{1+\alpha}(J\cup\mathbb{R}^+;X)} + \|\cdot\|_{C(J\cup\mathbb{R}^+;X_\beta)}$ and $\mathfrak{S}$ be the space

$$\mathfrak{S} = \{u \in \mathcal{Y} : u\big|_{[0,\infty)} \in SAP_\omega(X),\ \mathcal{P}_u \in C^{1+\alpha}([0,\infty);\mathcal{B}) \cap C^{\alpha}([0,\infty);\mathcal{B}_\beta)\}, \quad (2.9)$$

endowed with the metric

$$\Phi(u,v) = \|P_u - P_v\|_{C^{1+\alpha}([0,\infty);\mathcal{B})} + \|P_u - P_v\|_{C^{\alpha}([0,\infty);\mathcal{B}_\beta)}.$$

Let Γ be the map $\Gamma : \mathfrak{S} \to \mathfrak{S}$ given by $(\Gamma u)_0 = \varphi$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s)f_{u,u'}(s)ds, \quad t \ge 0, \quad (2.10)$$

where $f_{u,u'} : [0,\infty) \to X$ is the function defined by $f_{u,u'}(t) = f(t,u_t,u'_t)$.

In the remainder of this proof we show that Γ is a contraction on $\mathfrak{S}$. To this end, next we assume that $u, v \in \mathfrak{S}$.

Step 1. $\Gamma u_{|_{[0,\infty)}} \in SAP_\omega(X) \cap C^\alpha(\mathbb{R}^+; X_1)$ and $\Gamma u \in C^{1+\alpha}(J \cup \mathbb{R}^+; X) \cap C^\alpha(J \cup \mathbb{R}^+; X_\beta)$.

From the definition of $\mathcal{Y}$ we have that $u'(\cdot)$ is uniformly continuous on $[0,\infty)$, which implies via Corollary 2.8 that $u' \in SAP_\omega(X)$. Moreover, from Lemma 2.6 and Lemma 2.10 we infer that $f_1 \circ P_u$ and $f_2 \circ P'_u$ belong to $SAP_\omega(X)$ and from Lemma 2.7 (with $\mathcal{Q}(s) = T(s)$ and $Z = X$) we obtain that $\Gamma u_{|_{[0,\infty)}} \in SAP_\omega(X)$.

On the other hand, from Lemma 2.10 we have that $f_{u,u'} \in C^\alpha([0,\infty); X)$ which implies via Proposition 2.1 that $\Gamma u_{|_{[0,\infty)}} \in C^\alpha([0,\infty); X_1) \cap C^{1+\alpha}([0,\infty); X) \cap C^\alpha([0,\infty); X_\beta)$ and

$$[\Gamma u]_{C^\alpha([0,\infty);X_1)} \le \frac{C_{1,\alpha}}{\alpha} \|A\varphi(0)\|_{\alpha,\infty} + \Lambda [f_{u,u'}]_{C^\alpha([0,\infty);X)}, \tag{2.11}$$

$$\|\Gamma u\|_{C([0,\infty);X_1)} \le C_0 \|A\varphi(0)\| + (\Lambda_1 + \Lambda_2) \|f_{u,u'}\|_{C^\alpha([0,\infty);X)}, \tag{2.12}$$

$$[(\Gamma u)']_{C^\alpha([0,\infty);X)} \le \frac{C_{1,\alpha}}{\alpha} \|Ax\|_{\alpha,\infty} + (\Lambda + 1)[f_{u,u'}]_{C^\alpha([0,\infty);X)}, \tag{2.13}$$

$$\|(\Gamma u)'\|_{C([0,\infty);X)} \le C_0 \|A\varphi(0)\| + (\Lambda_1 + \Lambda_2 + 1) \|f_{u,u'}\|_{C^\alpha([0,\infty);X)}, \tag{2.14}$$

$$\|\Gamma u\|_{C([0,\infty);X_\beta)} \le \|(-A)^{\beta-1}\| \|\Gamma u\|_{C([0,\infty);X_1)}, \tag{2.15}$$

$$[\Gamma u]_{C^\alpha([0,\infty);X_\beta)} \le \|(-A)^{\beta-1}\| [\Gamma u]_{C^\alpha([0,\infty);X_1)}. \tag{2.16}$$

Moreover, by noting that $(\Gamma u)'(0) = A\varphi(0) + f_{u,u'}(0) = A\varphi(0) = \varphi'(0)$ and $\Gamma u(0) = \varphi(0)$, from the properties of the function $\tilde{\varphi}$ and Lemma 2.9 we infer that $\Gamma u \in C^{1+\alpha}(J \cup [0,b]; X) \cap C^\alpha(J \cup [0,b]; X_\beta)$, which completes the proof of this step.

Step 2. $\mathcal{P}_{\Gamma u} \in C^{1+\alpha}([0,\infty); \mathcal{B})$ and

$$[P_{\Gamma u} - P_{\Gamma v}]_{C^\alpha([0,b];\mathcal{B})} \le \mathcal{K}(\mathcal{K}+1) \|A^{-1}\| \Lambda [f_{u,u'} - f_{v,v'}]_{C^\alpha([0,\infty);X)}, \tag{2.17}$$

$$\|P_{\Gamma u} - P_{\Gamma v}\|_{C([0,b];\mathcal{B})} \le \mathcal{K} \|A^{-1}\| (\Lambda_1 + \Lambda_2) \|f_{u,u'} - f_{v,v'}\|_{C^\alpha([0,\infty);X)}, \tag{2.18}$$

$$[(P_{\Gamma u})' - (P_{\Gamma v})']_{C^\alpha([0,b];\mathcal{B})} \le \mathcal{K}(\mathcal{K}+1)(\Lambda+1)[f_{u,u'} - f_{v,v'}]_{C^\alpha([0,\infty);X)}, \tag{2.19}$$

$$\|(P_{\Gamma u})' - (P_{\Gamma v})'\|_{C([0,b];\mathcal{B})} \le \mathcal{K}(\Lambda_1 + \Lambda_2 + 1) \|f_{u,u'} - f_{v,v'}\|_{C^\alpha([0,\infty);X)}). \tag{2.20}$$

From the properties of the space $\mathcal{B}$ it is easy to see that $\mathcal{P}_{\Gamma u} \in C([0,\infty); \mathcal{B})$. To show that $\mathcal{P}_{\Gamma u} \in C^\alpha([0,\infty); \mathcal{B})$ is convenient to estimate $\|P_{\Gamma u}(t) - \varphi\|_{\mathcal{B}}$. By using the properties of $P_{\tilde{\varphi}}$, for $t \ge 0$ we obtain

$$\begin{aligned}
&\|P_{\Gamma u}(t) - \varphi\|_{\mathcal{B}} \\
&\le \|P_{\Gamma u}(t) - P_{\widetilde{\varphi}}(t)\|_{\mathcal{B}} + \|P_{\widetilde{\varphi}}(t) - \varphi\|_{\mathcal{B}} \\
&\le \mathcal{K} \sup_{s\in[0,t]} \|\Gamma u(s) - \widetilde{\varphi}(0)\| + \mathcal{K} \sup_{s\in[0,t]} \|\widetilde{\varphi}(0) - \widetilde{\varphi}(s)\| + t^\alpha [P_{\widetilde{\varphi}}]_{C^\alpha([0,\infty);\mathcal{B})} \\
&\le t^\alpha \mathcal{K} [\Gamma u]_{C^\alpha([0,\infty);X)} + t^\alpha (\mathcal{K}H + 1)[P_{\widetilde{\varphi}}]_{C^\alpha([0,\infty);\mathcal{B})}.
\end{aligned}$$

From this estimate, for $t > 0$ and $h > 0$, we see that

$$\begin{aligned}
&\|P_{\Gamma u}(t+h) - P_{\Gamma u}(t)\|_{\mathcal{B}} \\
&\le \mathcal{K} \|P_{\Gamma u}(h) - \varphi\|_{\mathcal{B}} + \mathcal{K} \sup_{s\in[0,t]} \|\Gamma u(s+h) - \Gamma u(s)\| \\
&\le \mathcal{K}^2 h^\alpha [\Gamma u]_{C^\alpha([0,\infty);X)} + h^\alpha \mathcal{K}(\mathcal{K}H + 1)[P_{\widetilde{\varphi}}]_{C^\alpha([0,\infty);\mathcal{B})} + \mathcal{K} h^\alpha [\Gamma u]_{C^\alpha([0,\infty);X)},
\end{aligned}$$

which implies that

$$[P_{\Gamma u}]_{C^\alpha([0,\infty);\mathcal{B})} \le \mathcal{K}(\mathcal{K}+1)[\Gamma u]_{C^\alpha([0,\infty);X)} + \mathcal{K}(\mathcal{K}H + 1)[P_{\widetilde{\varphi}}]_{C^\alpha([0,\infty);\mathcal{B})}, \tag{2.21}$$

and $\mathcal{P}_{\Gamma u} \in C^\alpha([0,\infty);\mathcal{B})$ since $[\Gamma u]_{C^\alpha([0,\infty);X)} \le \|A^{-1}\|[\Gamma u]_{C^\alpha([0,\infty);X_1)} < \infty$. Moreover, by noting that $\Gamma u - \Gamma v$ is the mild solution of (2.2)-(2.3) with $x = 0$ and $\xi = f_{u,u'} - f_{v,v'}$, from the above remarks, the estimative (2.21) and Proposition 2.1 we infer

$$\begin{aligned}[P_{\Gamma u} - P_{\Gamma v}]_{C^\alpha([0,\infty);\mathcal{B})} &\le \mathcal{K}(\mathcal{K}+1)[\Gamma u - \Gamma v]_{C^\alpha([0,\infty);X)}\\ &\le \mathcal{K}(\mathcal{K}+1)\|A^{-1}\|[\Gamma u - \Gamma v]_{C^\alpha([0,\infty);X_1)}\\ &\le \mathcal{K}(\mathcal{K}+1)\|A^{-1}\|\Lambda[f_{u,u'} - f_{v,v'}]_{C^\alpha([0,\infty);X)},\end{aligned}$$

which establish the estimate (2.17).

Proceeding as above, we also can prove that $(\mathcal{P}_{\Gamma u})' \in C^\alpha([0,\infty);\mathcal{B})$. Since $\varphi' \in \mathcal{B}$ and $(\Gamma u)'_0 = \varphi'$ (note that $(\Gamma u)'(0) = A\varphi(0) + f_{u,u'}(0) = A\varphi(0) = \varphi'(0)$), from the properties of $\mathcal{B}$ and the estimate (2.14) we obtain that $(\mathcal{P}_{\Gamma u})' \in C([0,\infty);\mathcal{B})$. In addition, for $t \ge 0$ it is easy to see that

$$\|P_{(\Gamma u)'}(t) - \varphi'\|_{\mathcal{B}} \le t^\alpha \mathcal{K}[(\Gamma u)']_{C^\alpha([0,\infty);X)} + t^\alpha(\mathcal{K}H+1)[P_{\widetilde{\varphi}'}]_{C^\alpha([0,\infty);\mathcal{B})}. \tag{2.22}$$

Using this estimate, we have that

$$\begin{aligned}&\|P_{(\Gamma u)'}(t+h) - P_{(\Gamma u)'}(t)\|_{\mathcal{B}}\\ &\le \mathcal{K}\|P_{(\Gamma u)'}(h) - \varphi'\|_{\mathcal{B}} + \mathcal{K}\sup_{s\in[0,t]}\|(\Gamma u)'(s+h) - (\Gamma u)'(s)\|\\ &\le \mathcal{K}(h^\alpha\mathcal{K}[(\Gamma u)']_{C^\alpha([0,\infty);X)} + h^\alpha(\mathcal{K}H+1)[P_{\widetilde{\varphi}'}]_{C^\alpha([0,\infty);\mathcal{B})})\\ &\quad + \mathcal{K}h^\alpha[(\Gamma u)']_{C^\alpha([0,\infty);X)}\\ &\le h^\alpha\mathcal{K}(\mathcal{K}+1)[(\Gamma u)']_{C^\alpha([0,\infty);X)} + h^\alpha\mathcal{K}(\mathcal{K}H+1)[P_{\widetilde{\varphi}'}]_{C^\alpha([0,\infty);\mathcal{B})}),\end{aligned}$$

which implies

$$[P_{(\Gamma u)'}]_{C^\alpha([0,\infty);\mathcal{B})} \le \mathcal{K}(\mathcal{K}+1)[(\Gamma u)']_{C^\alpha([0,\infty);X)} + \mathcal{K}(\mathcal{K}H+1)[P_{\widetilde{\varphi}'}]_{C^\alpha([0,\infty);\mathcal{B})}, \tag{2.23}$$

and $P_{(\Gamma u)'} \in C^\alpha([0,\infty);\mathcal{B})$ since $[(\Gamma u)']_{C^\alpha([0,\infty);X)}$ is finite. This completes the proof that $P_{\Gamma u} \in C^{1+\alpha}([0,\infty);\mathcal{B})$. Proceeding as above, we also note that

$$\begin{aligned}[P_{(\Gamma u)'} - P_{(\Gamma v)'}]_{C^\alpha([0,\infty);\mathcal{B})} &\le \mathcal{K}(\mathcal{K}+1)[(\Gamma u)' - (\Gamma v)']_{C^\alpha([0,\infty);X)}\\ &\le \mathcal{K}(\mathcal{K}+1)(\Lambda+1)[f_{u,u'} - f_{v,v'}]_{C^\alpha([0,\infty);X)},\end{aligned}$$

which establish (2.19).

Concerning the estimate (2.18), from Proposition 2.1 we have

$$\begin{aligned}&\|P_{\Gamma u}(t) - P_{\Gamma v}(t)\|_{\mathcal{B}}\\ &\le \mathcal{K}\sup_{s\in[0,t]}\|\Gamma u(s) - \Gamma v(s)\|\\ &\le \mathcal{K}\|A^{-1}\|\sup_{s\in[0,t]}\|\Gamma u(s) - \Gamma v(s)\|_{X_1}\\ &\le \mathcal{K}\|A^{-1}\|(\Lambda_1[f_{u,u'} - f_{v,v'}]_{C^\alpha([0,\infty);X)} + \Lambda_2\|f_{u,u'} - f_{v,v'}\|_{C([0,\infty);X)}),\end{aligned}$$

which proves (2.18). Finally, from the estimate

$$\begin{aligned}&\|(P_{\Gamma u})' - (P_{\Gamma v})'\|_{C([0,b];\mathcal{B})}\\ &\le \mathcal{K}\|(\Gamma u)' - (\Gamma v)'\|_{C([0,b];X)}\\ &\le \mathcal{K}(\|\Gamma u - \Gamma v\|_{C([0,b];X_1)} + \|f_{u,u'} - f_{v,v'}\|_{C([0,\infty);X)})\\ &\le \mathcal{K}(\Lambda_1[f_{u,u'} - f_{v,v'}]_{C^\alpha([0,\infty);X)} + \Lambda_2\|f_{u,u'} - f_{v,v'}\|_{C([0,\infty);X)})\\ &\quad + \mathcal{K}\|f_{u,u'} - f_{v,v'}\|_{C([0,\infty);X)},\end{aligned}$$

we obtain (2.20).

Step 3. The function $\mathcal{P}_{\Gamma u}$ belongs to $C^\alpha([0,\infty);\mathcal{B}_\beta)$ and

$$\|P_{\Gamma u}-P_{\Gamma v}\|_{C([0,b];\mathcal{B}_\beta)} \le \mathcal{K}\|(-A)^{\beta-1}\|(\Lambda_1+\Lambda_2)\|f_{u,u'}-f_{v,v'}\|_{C^\alpha([0,\infty);X)}), \tag{2.24}$$

$$[P_{\Gamma u}-P_{\Gamma v}]_{C^\alpha([0,b];\mathcal{B}_\beta)} \le \mathcal{K}(\mathcal{K}+1)\|(-A)^{\beta-1}\|\Lambda[f_{u,u'}-f_{v,v'}]_{C^\alpha([0,\infty);X)}. \tag{2.25}$$

From Proposition 2.1 it follows that

$$\begin{aligned}\|P_{\Gamma u}-P_{\Gamma v}\|_{C([0,b];\mathcal{B}_\beta)} &\le \mathcal{K}\|\Gamma u-\Gamma v\|_{C([0,b];X_\beta)}\\ &\le \mathcal{K}\|(-A)^{\beta-1}\|\|\Gamma u-\Gamma v\|_{C([0,b];X_1)}\\ &\le \mathcal{K}\|(-A)^{\beta-1}\|(\Lambda_1+\Lambda_2)\|f_{u,u'}-f_{v,v'}\|_{C^\alpha([0,\infty);X)},\end{aligned}$$

which establish (2.24). Concerning (2.25) we note that

$$\begin{aligned}&\|P_{\Gamma u}(t+h)-P_{\Gamma v}(t+h)-(P_{\Gamma u}(t)-P_{\Gamma v}(t))\|_{\mathcal{B}_\beta}\\ &\le \mathcal{K}\|P_{\Gamma u}(h)-P_{\Gamma v}(h)\|_{\mathcal{B}_\beta}+\mathcal{K}\sup_{s\in[0,h]}\|\Gamma u(s+h)-\Gamma u(s+h)-(\Gamma u(s)-\Gamma u(s))\|_\beta\\ &\le \mathcal{K}^2\sup_{s\in[0,h]}\|\Gamma u(s)-\Gamma u(h)-(\Gamma u(0)-\Gamma u(0))\|_\beta\\ &\quad+\mathcal{K}\sup_{s\in[0,t]}\|\Gamma u(s+h)-\Gamma u(s+h)-(\Gamma u(s)-\Gamma u(s))\|_\beta\\ &\le \mathcal{K}^2\|(-A)^{\beta-1}\|\sup_{s\in[0,t]}\|\Gamma u(s+h)-\Gamma u(s+h)-(\Gamma u(s)-\Gamma u(s))\|_{X_1}\\ &\quad+\mathcal{K}\|(-A)^{\beta-1}\|\sup_{s\in[0,t]}\|\Gamma u(s+h)-\Gamma u(s+h)-(\Gamma u(s)-\Gamma u(s))\|_{X_1}\\ &\le \mathcal{K}(\mathcal{K}+1)\|(-A)^{\beta-1}\|[\Gamma u-\Gamma u]_{C^\alpha([0,\infty);X_1)},\end{aligned}$$

which, via Proposition 2.1, implies

$$[P_{\Gamma u}-P_{\Gamma v}]_{C^\alpha([0,b];\mathcal{B}_\beta)} \le \mathcal{K}(\mathcal{K}+1)\|(-A)^{\beta-1}\|\Lambda[f_{u,u'}-f_{v,v'}]_{C^\alpha([0,\infty);X)}.$$

From Steps 2 and 3 we obtain that $\Phi(\Gamma u,\Gamma v)\le \Xi\Phi(u,v)$ which proves that Γ is a contraction on $\mathfrak{S}$. Finally, from the contraction mapping principle and Proposition 2.1 we infer that there exists a unique $\mathcal{S}$-asymptotically ω-periodic strict solution $u\in C^{1+\alpha}(J\cup\mathbb{R}^+;X)\cap C(J\cup\mathbb{R}^+;X_\beta)$ of the problem (1.1)–(1.2). □

In the next theorem we prove the existence of a strict solution via the conditions (H2) and (H3). In this result, Y is the space in condition (H2) and $i_{Y,X}$ denotes the inclusion map from Y into X.

Theorem 2.12. *Assume the conditions* (H2) *and* (H3) *are satisfied,* $C(J;X)\hookrightarrow\mathcal{B}$, $M(t)\to 0$ *as* $t\to\infty$, $\varphi\in C^1(J;X)\cap\mathcal{B}_\beta$, $\varphi(0)\in X_1$, $\frac{d^-\varphi}{dt}(0)=A\varphi(0)$ *and* $f(0,\varphi,\varphi')=0$. *Suppose, in addition, there are* $\psi_1\in\mathcal{B}_\mathcal{B}$ *and* $\psi_2\in\mathcal{B}$ *such that* $f(\cdot,\psi_1,\psi_2)\in C([0,\infty);Y)$ *and*

$$\begin{aligned}\Theta&=\mathcal{K}(\|(-A)^{\beta-1}\|+1)\|H*L_f\|_{C([0,\infty);\mathbb{R}^+)}\\ &\quad+\mathcal{K}\|L_f\|_{C([0,\infty);\mathbb{R}^+)}\|i_{Y,X}\|_{\mathcal{L}(Y,X)}(\frac{C_0}{\gamma}+1)<1,\end{aligned} \tag{2.26}$$

where $H*L_f(t)=\int_0^t H(t-s)L_f(s)ds$. *Then there exists a unique* $\mathcal{S}$*-asymptotically* ω*-periodic strict solution* $u\in C^1(J\cup\mathbb{R}^+;X)\cap C(J\cup\mathbb{R}^+;X_\beta)$ *of* (1.1)–(1.2).

Proof. Let $\mathcal{S} = C^1(J \cup [0,\infty); X) \cap C(J \cup [0,\infty); X_\beta)$ and

$$\mathfrak{F} = \{u \in \mathcal{S} : u\big|_{[0,\infty)}, u'\big|_{[0,\infty)} \in SAP_\omega(X), \mathcal{P}_u \in C([0,\infty); \mathcal{B}_\beta) \cap C^1([0,\infty); \mathcal{B})\},$$

endowed with the metric $\Phi(u,v) = \|\mathcal{P}_u - \mathcal{P}_v\|_{C^1([0,\infty);\mathcal{B})} + \|\mathcal{P}_u - \mathcal{P}_v\|_{C([0,\infty);\mathcal{B}_\beta)}$. Let $\Gamma : \mathfrak{F} \to \mathfrak{F}$ the map defined in the proof of Theorem 2.11. Next, we show that Γ is a contraction on $\mathfrak{F}$. To begin, we show that Γ is a well defined function from $\mathfrak{F}$ into $\mathfrak{F}$. In the remainder of this proof we assume that $u, v \in \mathfrak{F}$.

By noting that $f_{u,u'} \in C([0,\infty); Y)$, from condition (H2) it is easy to see that

$$\begin{aligned}
&\|(-A)^\beta \Gamma u(t)\| \\
&\le C_0 e^{-\gamma t}\|(-A)^\beta \varphi(0)\| + \|(-A)^{\beta-1}\|_{\mathcal{L}(X)} \int_0^t \|AT(t-s)\|_{\mathcal{L}(Y,X)} \|f_{u,u'}(s)\|_Y \,ds \\
&\le C_0 e^{-\gamma t}\|(-A)^\beta \varphi(0)\| + \|(-A)^{\beta-1}\|_{\mathcal{L}(X)} \|f_{u,u'}\|_{C([0,\infty);Y)} \|H\|_{L^1([0,\infty),\mathbb{R}^+)},
\end{aligned}$$

which implies that $\Gamma u \in C(J \cup [0,\infty); X_\beta)$ and $\mathcal{P}_{\Gamma u} \in C([0,\infty); \mathcal{B}_\beta)$ since $\varphi \in \mathcal{B}_\beta$. Moreover, from Lemma 2.2 we have that Γu is continuously differentiable and

$$\|(\Gamma u)'(t)\| \le e^{-\gamma t}\|A\varphi(0)\| + \|f_{u,u'}\|_{C([0,\infty);Y)}(\|H\|_{L^1([0,\infty),\mathbb{R}^+)} + 1),$$

which shows that $(\Gamma u)' \in C([0,\infty); X)$. In addition to the above, from Lemma 1.4, the compatibility condition $\frac{d^-\varphi}{dt}(0) = A\varphi(0) + f(0,\varphi,\varphi') = A\varphi(0)$ and the fact that

$$(\Gamma u)'(t) = AT(t)\varphi(0) + \int_0^t AT(t-s) f_{u,u'}(s)ds + f(t, u_t, u_t'), \quad \forall t \ge 0,$$

we infer that $\mathcal{P}_{\Gamma u} \in C^1([0,\infty); \mathcal{B})$ and $(\mathcal{P}_{\Gamma u})' = \mathcal{P}_{(\Gamma u)'}$.

To complete the proof that Γ has values in $\mathfrak{F}$, it remain to prove that the functions $\Gamma u\big|_{[0,\infty)}$ and $(\Gamma u)'\big|_{[0,\infty)}$ belong to $SAP_\omega(X)$. From Lemma 2.6 we have that $\mathcal{P}_u \in SAP_\omega(\mathcal{B}_\beta)$, $\mathcal{P}_{u'} \in SAP_\omega(\mathcal{B})$ and from the conditions (H2) and (H3) is easy to show that $f(\cdot, \mathcal{P}_u(\cdot), \mathcal{P}_{u'}(\cdot)) \in SAP_\omega(Y)$. In addition, by using Lemma 2.7 with $Z = X$ and $\mathcal{Q}(t) = T(t)$ we obtain that $\Gamma u \in SAP_\omega(X)$. Moreover, arguing as above, but using Lemma 2.7 with $Z = Y$ and $\mathcal{Q}(t) = AT(t)$ it follows that $(\Gamma u)'\big|_{[0,\infty)} \in SAP_\omega(X)$. This completes the proof that Γ is a $\mathfrak{F}$-valued function.

On the other hand, for $u, v \in \mathfrak{F}$ and $t \ge 0$ we obtain

$$\|\Gamma u(t) - \Gamma v(t)\|_\beta \le \|(-A)^{\beta-1}\| \int_0^t \|AT(t-s)\|_{\mathcal{L}(Y,X)} L_f(s)\Phi(u,v)ds,$$

and hence,

$$\|\Gamma u - \Gamma v\|_{C([0,\infty);X_\beta)} \le \|(-A)^{\beta-1}\| \|H * L_f\|_{C([0,\infty);\mathbb{R}^+)}\Phi(u,v). \tag{2.27}$$

Similarly, from the inequality

$$\|\Gamma u(t) - \Gamma v(t)\| \le C_0 \int_0^t e^{-\gamma(t-s)} \|i_{Y,X}\|_{\mathcal{L}(Y,X)} \|f_{u,u'}(s) - f_{v,v'}(s)\|_Y \,ds$$

we obtain

$$\|\Gamma u - \Gamma v\|_{C([0,\infty);X)} \le \frac{C_0}{\gamma} \|i_{Y,X}\|_{\mathcal{L}(Y,X)} \|L_f\|_{C([0,\infty);\mathbb{R}^+)}\Phi(u,v). \tag{2.28}$$

In addition, from Lemma 2.2, it follows that

$$\begin{aligned}
\|(\Gamma u)'(t) - (\Gamma v)'(t)\| \le &\int_0^t H(t-s)L_f(s)(\|u_s - v_s\|_{\mathcal{B}_\beta} + \|u_s' - v_s'\|_{\mathcal{B}})ds \\
&+ L_f(t)\|i_{Y,X}\|_{\mathcal{L}(Y,X)}(\|u_t - v_t\|_{\mathcal{B}_\beta} + \|u_t' - v_t'\|_{\mathcal{B}}),
\end{aligned}$$

from which we infer that

$$\begin{aligned}&\|(\Gamma u)' - (\Gamma v)'\|_{C([0,\infty);X)} \\ &\leq \big(\|H * L_f\|_{C([0,\infty);\mathbb{R}^+)} + \|L_f\|_{C([0,\infty);\mathbb{R}^+)}\|i_{Y,X}\|_{\mathcal{L}(Y,X)}\big)\Phi(u,v).\end{aligned} \tag{2.29}$$

From inequalities (2.27)-(2.29) we obtain that $\Phi(\Gamma u, \Gamma v) \leq \Theta\Phi(u,v)$ which proves that Γ is a contraction on $\mathfrak{F}$ and there exits a unique mild solution $u \in \mathfrak{F}$ of (1.1)–(1.2). Finally, from Lemma 2.2 it follows that $u(\cdot)$ is a strict solution of the problem (1.1)–(1.2). This completes the proof. □

3. Applications

Motivated by the examples presented in the introduction, in this section we discuss the existence of a $\mathcal{S}$-asymptotically ω-periodic strict solution for the partial neutral differential problem

$$u'(t,\xi) = \Delta u(t,\xi) - \int_{-\infty}^{t} a(t)k(t-s)u'(s,\xi)ds + g(t), \tag{3.1}$$

$$u(t,0) = u(t,\pi) = 0, \tag{3.2}$$

$$u(s,\xi) = \varphi(s,\xi), \quad s \leq 0, \tag{3.3}$$

for $(t,\xi) \in [0,\infty) \times [0,\pi]$ where $g \in SAP_\omega(\mathbb{R})$ and $\varphi(\cdot)$ is a function defined from $(-\infty,0] \times [0,\pi]$ into $\mathbb{R}$.

To treat the problem (3.1)-(3.3) under the abstract framework in Section 1, we take $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ be the operator $Ax = x''$ on $D(A) = \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and associated normalized eigenvectors $z_n(\xi) = (\frac{2}{\pi})^{1/2}\sin(n\xi)$. We note that $\|T(t)\| \leq e^{-\frac{t}{2}}$, $\|AT(t)\| \leq e^{-\frac{t}{2}}t^{-1}$ and $\|A^2T(t)\| \leq 4e^{-\frac{t}{2}}t^{-2}$ for $t > 0$.

As a phase space we consider the space $\mathcal{B} = C_r \times L^p(\rho, X)$. Let $r \geq 0, 1 \leq p < \infty$ and $\rho : (-\infty,-r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5)-(g-7) in the terminology of [24]. The space $C_r \times L^p(\rho,X)$ is formed by all classes of functions $\psi : (-\infty,0] \to X$ such that $\psi_{|_{[-r,0]}} \in C([-r,0],X)$, $\psi(\cdot)$ is Lebesgue-measurable and $\rho^{\frac{1}{p}}\psi \in L^p((-\infty,-r],X)$. The norm in $C_r \times L^p(\rho,\mathcal{D})$ is given by $\|\psi\|_{\mathcal{B}} = \|\psi\|_{C([-r,0];X)} + \|\rho^{\frac{1}{p}}\psi\|_{L^p((-\infty,-r],X)}$. From [24], we know that $\mathcal{B}$ satisfy the conditions in Section 2 and $\mathcal{B}$ is a uniform fading memory space, which implies that $M(t) \to 0$ as $t \to \infty$ and $K(\cdot)$ is bounded, see [24, pp.190] for details.

Next we assume $k \in C([0,\infty);\mathbb{R})$, $a \in C^\alpha([0,\infty),\mathbb{R}) \cap SAP_\omega(\mathbb{R})$ for some $\alpha \in (0,1)$ and $\omega > 0$, and

$$\Theta = 2\|a\|_{C([0,\infty);\mathbb{R})}\Big(\int_{-\infty}^{0} \frac{k^2(-\tau)}{\rho(\tau)}d\tau\Big)^{1/2} < \infty.$$

Under these conditions, the function $f : [0,\infty) \times \mathcal{B} \to X$ given by $f(t,\psi)(\xi) = \int_{-\infty}^{0} a(t)k(-\tau)\psi(\tau,\xi)d\tau$ is well defined, $f \in C^\alpha([0,\infty);\mathcal{L}(\mathcal{B},X)) \cap SAP_\omega(\mathcal{L}(\mathcal{B}))$ and $\|f\|_{C^\alpha([0,\infty);\mathcal{L}(\mathcal{B},X))} \leq \Theta$.

Next, we say that a function $u \in C^1(J \cup \mathbb{R}^+;X) \cap C(J \cup \mathbb{R}^+;X_1)$ is a strict solution of (3.1)-(3.3) if $u(\cdot)$ is a strict solution of the associated problem (1.1)–(1.2). In the next result, which follows directly from Theorem 2.11, Λ is the number in Proposition 2.1 and $D_A(\alpha,\infty)$, $\mathcal{K}$ are as in Section 1. We also note that Λ appears explicitly in the end of the proof of Proposition 2.1 and that in the current case $\Lambda = (\frac{2}{\alpha} + 4 + \frac{4}{\alpha(1-\alpha)})$, $\Lambda_1 = (2 + \frac{1}{\alpha})$ and $\Lambda_2 = 2$.

Proposition 3.1. *Assume that* $\varphi(0,\cdot) \in X_1$, $\frac{d^-\varphi}{dt}(0,\cdot) = A\varphi(0,\cdot) \in D_A(\alpha,\infty)$, $f(0,\varphi) + f_2(0,\varphi') = 0$, $\varphi \in C^{1+\alpha}(J;X) \cap C^{\alpha}(J;X_\beta)$ *and* $\mathcal{P}_{\tilde{\varphi}} \in C^{1+\alpha}([0,\infty);\mathcal{B})$. *If*

$$\mathcal{K}\big[(\mathcal{K}+1)(\mathcal{A}\Lambda+1) + \mathcal{A}(4+\frac{1}{\alpha}) + 1\big]\Theta < 1, \tag{3.4}$$

where $\mathcal{A} = \|(-A)^{\beta-1}\| + \|A^{-1}\| + 1$, *then there exists a unique* $\mathcal{S}$*-asymptotically* ω*-periodic strict solution of the problem* (3.1)–(3.3).

References

[1] Mostafa Adimy, K. Ezzinbi; A class of linear partial neutral functional-differential equations with nondense domain, *J. Differential Equations.,* 147 (1998), no. 2, 285-332.

[2] P. Cannarsa., D. Sforza; Global solutions of abstract semilinear parabolic equations with memory terms, *NoDEA Nonlinear Differential Equations. Appl.,* 10 (2003), no. 4, 399-430.

[3] Ph. Clément, J. A. Nohel; Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels. *SIAM J. Math. Anal.* 12 (1981), no. 4, 514-535.

[4] Claudio Cuevas, Carlos Lizama; S-asymptotically ω-periodic solutions for semilinear Volterra equations. *Math. Methods Appl. Sci.* 33 (2010), no. 13, 1628-1636.

[5] C. Cuevas., J. C. de Souza; $\mathcal{S}$-asymptotically ω-periodic solutions of semilinear fractional integro-differential equations, *Appl. Math. Lett.,* 22 (2009) 865-870.

[6] C. Cuevas, J. C. de Souza; Existence of $\mathcal{S}$-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay, *Nonlinear Analysis* (2009), doi:10.1016/j.na.2009.09.007.

[7] R. Datko; Linear autonomous neutral differential equations in a Banach space, *J. Differential Equations* 25 (1977), no. 2, 258-274.

[8] R. C. Grimmer; Asymptotically almost periodic solutions of differential equations, *SIAM J. Appl. Math.* 17 (1969) 109-115.

[9] M. E. Gurtin, A. C. Pipkin; A general theory of heat conduction with finite wave speed, *Arch. Rat. Mech. Anal.*, 31 (1968), 113-126. 113-126.

[10] S. Hadd; Singular functional differential equations of neutral type in Banach spaces, *J. Funct. Anal.* 254 (2008), no. 8, 2069-2091.

[11] J. Hale; Partial neutral functional-differential equations, *Rev. Roumaine Math. Pures Appl.* 39 (1994), no. 4, 339-344.

[12] G. Haiyin, W. Ke, W. Fengying, D. Xiaohua; Massera-type theorem and asymptotically periodic logistic equations, *Nonlinear Anal. Real World Appl.*, 7 (2006), no. 5, 1268-1283.

[13] Hernán Henríquez, M. Pierri, P. Táboas; On $\mathcal{S}$-asymptotically ω-periodic functions on Banach spaces and applications, *J. Math. Anal. Appl.*, 343 (2008), no. 2, 1119-1130.

[14] Hernán Henríquez, M. Pierri, P. Táboas; Existence of $\mathcal{S}$-asymptotically ω-periodic solutions for abstract neutral equations, *Bull. Aust. Math. Soc.* 78 (2008), no. 3, 365-382.

[15] Eduardo Hernández, Donal O'Regan; Existence of solutions for abstract non-autonomous neutral differential equations, *Canad. Math. Bull.*, 55 (2012), no. 4, 736-751.

[16] E. Hernández, D. O'Regan; C^{α}-Hölder classical solutions for non-autonomous neutral differential equations, *Discrete Contin. Dyn. Syst.*, 29 (2011), no. 1, 241-260.

[17] E. Hernández, J. P. Dos Santos, H. Henríquez; Existence results for neutral integro-differential equations with unbounded delay, *Journal of Integral Equations and Applications*, 23 (2011) 2, 289-330.

[18] E. Hernández, D. O'Regan; Existence results for abstract partial neutral differential equations, *Proc. Amer. Math. Soc.*, 137 (2009), 10, 3309-3318.

[19] E. Hernández, H. Henríquez; Existence results for partial neutral functional differential equation with unbounded delay, *J. Math. Anal. Appl.*, 221 (1998) 2, 452-475.

[20] E. Hernández; Existence results for partial neutral integro-differential equations with unbounded delay, *J. Math. Anal. Appl.*, 292, (2004) 1, 194-210.

[21] Eduardo Hernández, Donal O'Regan; On a New Class of Abstract Neutral Differential Equations, *J. Functional Analysis.*, 261 (2011), 12, 3457-3481.

[22] Eduardo Hernández, Donal O'Regan; Existence of solutions for abstract neutral integro-differential equations with unbounded delay, *Czechoslovak Mathematical Journal*, vol. 61, no. 3 (2011), pp. 691-706.

[23] E. Hernández, H. Henríquez; Existence of periodic solution of partial neutral functional differential equation with unbounded delay, *J. Math. Anal. Appl.* 221 (1998) 2, 499-522.

[24] Y. Hino, S. Murakami, T. Naito; *Functional differential equations with infinite delay*, Lecture Notes in Mathematics, 1473. Springer-Verlag, Berlin, 1991.

[25] A. Lunardi; Laplace transform methods in integro-differential equations. Integro-differential evolution equations and application, *J. Integral Equations*, 10 (1985), 1-3, 185-211.

[26] A. Lunardi; Analytic semigroups and optimal regularity in parabolic problems, PNLDE Vol. *16*, Birkhäauser Verlag, Basel, 1995.

[27] A. Lunardi; On the linear heat equation with fading memory, *SIAM J. Math. Anal.* 21 (1990), no. 5, 1213-1224.

[28] S. Nicola, M. Pierri; A note on $\mathcal{S}$-asymptotically periodic functions, *Nonlinear Analysis: Real World Applications,* 10, (2009), 2, 2937-2938.

[29] J. W. Nunziato; On heat conduction in materials with memory, *Quart. Appl. Math.* 29 (1971), 187-204.

[30] Michelle Pierri; On $\mathcal{S}$-asymptotically ω-periodic functions and applications, *Nonlinear Anal.*,75 (2012), no. 2, 651-661.

[31] W. R. Utz, P. Waltman; Asymptotic almost periodicity of solutions of a system of differential equations, *Proc. Amer. Math. Soc.* 18 (1967) 597-601.

[32] J. S. W. Wong, T. A. Burton; Some properties of solutions of $u''(t)+ a(t)f(u)g(u') = 0$ II, *Monatsh. Math.* 69 (1965) 368-374.

[33] Wu, Jianhong; *Theory and applications of partial functional-differential equations*, Applied Mathematical Sciences, 119, Springer-Verlag, New York, 1996.

[34] C. L. Zhong; Asymptotically periodic solutions of a class of second order nonlinear differential equations, *Proc. Amer. Math. Soc.*, 99 (1987), no. 4, 693-699.

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