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CRITICAL EXPONENT FOR A DAMPED WAVE SYSTEM WITH FRACTIONAL INTEGRAL

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Abstract. We shall present the critical exponent

$$F(p,q,\alpha) := \max\left\{\alpha + \frac{(\alpha+1)(p+1)}{pq-1}, \alpha + \frac{(\alpha+1)(q+1)}{pq-1}\right\} - \frac{1}{2}$$

for the Cauchy problem

$$\begin{gathered} u_{tt} - u_{xx} + u_t = J^{\alpha}_{0|t}(|v|^p), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v_{tt} - v_{xx} + v_t = J^{\alpha}_{0|t}(|u|^q), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}, \\ (v(0,x), v_t(0,x)) = (v_0(x), v_1(x)), \quad x \in \mathbb{R}, \end{gathered}$$

where $p, q \geq 1, pq > 1$ and $0 < \alpha < 1/2$; that is, the small data global existence of solutions can be derived to the problem above if $F(p,q,\alpha) < 0$. Furthermore, in the case of $F(p,q,\alpha) \geq 0$ the non-existence of global solution can be obtained with the initial data having positive average value.

1. Introduction and statement of main results

In this article, we consider the Cauchy problem of damped wave system

$$\begin{gathered} u_{tt} - u_{xx} + u_t = J^{\alpha}_{0|t}(|v|^p), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v_{tt} - v_{xx} + v_t = J^{\alpha}_{0|t}(|u|^q), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}, \\ (v(0,x), v_t(0,x)) = (v_0(x), v_1(x)), \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $p, q \geq 1, pq > 1$ and $0 < \alpha < 1/2$. The initial values satisfy

$$\operatorname{supp}\{u_i, v_i\} \subset \{|x| \leq K\}, \quad K > 0,\ i = 0, 1, \tag{1.2}$$

$$(u_0, u_1, v_0, v_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}). \tag{1.3}$$

The notation $J^{\alpha}_{0|t}$ stands for the Riemann-Liouville fractional integral [9]

$$J^{\alpha}_{0|t} f(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases}$$

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for $f \in L^p(0,T)(1 \le p \le \infty)$ and $\Gamma(\cdot)$ is the Euler gamma function.

In recent decades, nonlinear hyperbolic equations and systems have been studied extensively (see, for example, [4, 5, 9, 10, 11, 18, 19] and the rich references therein). Todorova and Yordanov [17] considered the semilinear wave equation

$$\begin{gathered} u_{tt} - \Delta u + u_t = |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.4}$$

and proved that the critical exponent of (1.4) is $p_c(n) = 1 + 2/n$. More precisely, if $p > p_c(n)$ there exists a unique global solution of (1.4) for sufficiently small initial data, while, if $1 < p < p_c(n)$ any solution with positive initial data must blow up in finite time. Later Zhang [20] proved that the exponent $1 + 2/n$ belongs to the blow up region. Fino [2] considered the damped wave equation with nonlinear memory term

$$u_{tt} - \Delta u + u_t = J^{\alpha}_{0|t}(|u|^p)(t). \tag{1.5}$$

The existence of global solutions and the asymptotic behavior of small data solutions to (1.5) as $t \to \infty$ were established when $1 \le n \le 3$. If $p > 1+2(1+\alpha)/(n-2\alpha)$, the blow up result was also derived under some positive data in any dimensional space. Comparing the results of [17] with that of [2], we derive that the fractional integral $J^{\alpha}_{0|t}$ has an influence on the solution. The problems with the fractional integral term are interesting.

The problem (1.1) with $\alpha = 0$ can be considered as the following weakly coupled system

$$\begin{gathered} u_{tt} - \Delta u + u_t = |v|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ v_{tt} - \Delta v + v_t = |u|^q, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^n, \\ (v(0,x), v_t(0,x)) = (v_0(x), v_1(x)), \quad x \in \mathbb{R}^n. \end{gathered} \tag{1.6}$$

Sun and Wang [16] considered (1.6) and obtained the critical exponent

$$F(p,q,n) := \max\Big\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\Big\} - \frac{n}{2}.$$

The authors proved that if $F(p,q,n) < 0$, there exists a unique global solution of (1.6) with suitably small initial data for $n = 1$ or $n = 3$, and if $F(p,q,n) \ge 0$, any solution of (1.6) with initial data having positive integral values does not exist globally for any $n \ge 1$. Based on some conditions on nonlinear term, the asymptotic behavior of solutions of (1.6) was considered in [12]. Recently, Kenji and Yuta [13] showed that the number $F(p,q,n)$ is the critical exponent of (1.6) for any dimensional space.

Motivated by the work of [2] and [16], we aim at determining the critical exponent of (1.1). The global result is proved by the weighted energy method (see [17]). For the non-existence of a global solution, we shall use the test function method (see [2]). Our basic definition of the solution to problem (1.1) is the following.

Definition 1.1. Let $T > 0$. We say that a pair of functions (u,v) in $L^q((0,T), L^q_{\rm loc}(\mathbb{R})) \times L^p((0,T), L^p_{\rm loc}(\mathbb{R}))$ is a weak solution of the Cauchy problem (1.1) with the initial data $(u_i, v_i) \in [L_{\rm loc}(\mathbb{R})]^2$ if (u, v) satisfies

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} u(\varphi_{tt} - \varphi_{xx} + \varphi_t)\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^n} (J^\alpha_{0|t}|v|^p)\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx + \int_{\mathbb{R}^n} u_0(x)(\varphi(0,x) - \varphi_t(0,x))dx, \\
&\int_0^T \int_{\mathbb{R}^n} v(\varphi_{tt} - \varphi_{xx} + \varphi_t)\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^n} (J^\alpha_{0|t}|u|^q)\varphi\,dx\,dt + \int_{\mathbb{R}^n} v_1(x)\varphi(0,x)dx + \int_{\mathbb{R}^n} v_0(x)(\varphi(0,x) - \varphi_t(0,x))dx,
\end{aligned} \tag{1.7}$$

for all compactly supported test functions $\varphi \in C^2([0,T] \times \mathbb{R})$ with $\varphi(T, \cdot) = 0$ and $\varphi_t(T, \cdot) = 0$. If $T = \infty$, we say that (u, v) is a global weak solution of (1.1).

We remark that the above definition of a weak solution is a very weak form which will be used in the proof of non-existence of a global solution. However, to prove the global result we need a much stronger form. We have the following local existence result.

Proposition 1.2. *Let $T > 0$. Under assumptions* (1.2) *and* (1.3)*, there exists a unique solution $(u, v) \in X(T) \times X(T)$ for* (1.1) *satisfying*

$$\operatorname{supp}\{u, v\} \subset B(t + K) = \{(t, x) : |x| \le t + K\},\ K > 0$$

where $X(T) = C([0,T); H^1(\mathbb{R})) \cap C^1([0,T); L^2(\mathbb{R}))$.

Using [6, Proposition 2.3] and [2, Proposition 1], the local solvability and uniqueness of (1.1) can be established by a standard estimation and compactness theory.

Denote $\|\cdot\|_r$ and $\|\cdot\|_{H^m}$ the norms of $L^r(\mathbb{R})$ and $H^m(\mathbb{R})$ respectively. Throughout this article, we use C to stand for a generic positive constant which may be different from line to line. Set

$$F(p, q, \alpha) := \max\big\{\alpha + \frac{(\alpha+1)(p+1)}{pq-1}, \alpha + \frac{(\alpha+1)(q+1)}{pq-1}\big\} - \frac{1}{2}.$$

Based on Proposition 1.2, our main results read as follows

Theorem 1.3. *Assume that* (1.2) *and* (1.3) *hold. If $F(p, q, \alpha) < 0$, then there is a small constant ε such that under the conditions*

$$\begin{aligned}
I_{0,u} &= \|u_0\|_{H^1} + \|u_0\|_1 + \|u_1\|_2 + \|u_1\|_1 < \varepsilon, \\
I_{0,v} &= \|v_0\|_{H^1} + \|v_0\|_1 + \|v_1\|_2 + \|v_1\|_1 < \varepsilon,
\end{aligned} \tag{1.8}$$

problem (1.1) *admits a unique global solution*

$$(u, v) \in [C((0,\infty); H^1(\mathbb{R})) \cap C^1((0,\infty); L^2(\mathbb{R}))]^2.$$

Moreover,

$$\begin{aligned}
\|Du(t)\|_2 &\le (1+t)^{-\frac{(\alpha+1)(p+1)}{(pq-1)} - \frac{1}{4}}, \quad t \to \infty, \\
\|Dv(t)\|_2 &\le (1+t)^{-\frac{(\alpha+1)(q+1)}{(pq-1)} - \frac{1}{4}}, \quad t \to \infty,
\end{aligned} \tag{1.9}$$

where $Du = (u_t, u_x)$.

Theorem 1.4. *Assumed that* (1.2), (1.3) *hold and*

$$\int_{\mathbb{R}} u_i dx > 0, \quad \int_{\mathbb{R}} v_i dx > 0, \quad i = 0, 1. \tag{1.10}$$

If $F(p, q, \alpha) \geq 0$, *then the weak solution* (u, v) *of* (1.1) *does not exist globally.*

Remark 1.5. $F(p, q, \alpha)$ is the critical exponent of (1.1).

Remark 1.6. If $\alpha = 0$, $F(p, q, \alpha)$ is consistent with the critical exponent of (1.6) for $n = 1$.

The remainder of this paper is organized as follows. In Section 2, some preliminaries are collected. We will prove our global result (Theorem 1.3) in Section 3. Section 4 is devoted to proving the blow up result (Theorem 1.4).

2. Preliminaries

We shall start this section with some basic definitions and properties on the Riemann-Liouville fractional calculus. We refer to [7]-[3] for more details.

Let $AC[0, T]$ denote the space of all absolutely continuous functions on $[0, T]$. Then, if $f \in AC[0, T]$, the left-sided and the right-sided Riemann-Liouville fractional derivatives of the function f of order $\alpha \in (0, 1)$ are defined by

$$D^{\alpha}_{0|t} f(t) := \partial_t J^{1-\alpha}_{0|t} f(t), \quad D^{\alpha}_{t|T} f(t) := -\frac{1}{\Gamma(1-\alpha)} \partial_t \int_t^T (s-t)^{-\alpha} f(s) ds.$$

Set

$$AC^{n+1}[0, T] := \{f : [0, T] \to \mathbb{R} text and \partial_t^n f \in AC[0, T]\}.$$

Then for all $f \in AC^{n+1}[0, T]$, the following propositions are obtained in [7, 14, 15], respectively.

Proposition 2.1 ([7])**.** *Let* $0 < \alpha < 1$ *and* $p \geq 1$. *If* $f \in L^p(0, T)$,

$$(D^{\alpha}_{0|t} J^{\alpha}_{0|t} f)(t) = f(t), \quad (-1)^n \partial_t^n D^{\alpha}_{t|T} f = D^{n+\alpha}_{t|T} f,$$

for almost everywhere on $[0, T]$.

Proposition 2.2 ([15])**.** *Let* $0 < \alpha < 1$. *For every* $f, g \in C([0, T])$ *such that* $(D^{\alpha}_{0|t} f)(t), (D^{\alpha}_{t|T} g)(t)$ *exist and are continuous, the formula of integration by parts is*

$$\int_0^T (D^{\alpha}_{0|t} f)(t) g(t) dt = \int_0^T f(t) (D^{\alpha}_{t|T} g)(t) dt, \quad t \in [0, T].$$

Proposition 2.3 ([14])**.** *Set* $\varphi_2(t) := (1 - t/T)^{\eta}_+$. *Then* $\varphi_2(t)$ *satisfies*

$$D^{\alpha}_{t|T} \varphi_2(t) = CT^{-\eta}(T-t)^{\eta-\alpha}_+, \quad D^{\alpha+1}_{t|T} \varphi_2(t) = CT^{-\eta}(T-t)^{\eta-\alpha-1}_+,$$
$$D^{\alpha+2}_{t|T} \varphi_2(t) = CT^{-\eta}(T-t)^{\eta-\alpha-2}_+,$$

and

$$D^{\alpha}_{t|T} \varphi_2(T) = 0, \quad D^{\alpha}_{t|T} \varphi_2(0) = CT^{-\alpha},$$
$$D^{\alpha+1}_{t|T} \varphi_2(T) = 0, \quad D^{\alpha+1}_{t|T} \varphi_2(0) = CT^{-\alpha-1}.$$

Consider the linear damped wave equation

$$\begin{gathered} U_{tt} - U_{xx} + U_t = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ (U(0,x), U_t(0,x)) = (U_0(x), U_1(x)), \quad x \in \mathbb{R}. \end{gathered} \tag{2.1}$$

When $U_0 = 0$, the unique solution $U(t,x)$ to (2.1) can be denoted by $S(t)U_1$. Then the Duhamel's principle implies the solution to (1.4) solves the integral system

$$\begin{aligned} u(t,x) &= S(t)(u_0+u_1) + \partial_t(S(t)u_0) + \int_0^t S(t-\tau) J^{\alpha}_{0|\tau}(|v|^p) d\tau \\ &= u_L + \int_0^t S(t-\tau) J^{\alpha}_{0|\tau}(|v|^p) d\tau, \\ v(t,x) &= S(t)(v_0+v_1) + \partial_t(S(t)v_0) + \int_0^t S(t-\tau) J^{\alpha}_{0|\tau}(|u|^q) d\tau \\ &= v_L + \int_0^t S(t-\tau) J^{\alpha}_{0|\tau}(|u|^q) d\tau. \end{aligned} \tag{2.2}$$

The following lemmas will be used in the proof of Theorem 1.3.

Lemma 2.4 ([17, Proposition2.5])**.** *Let $m \in [1,2]$. Then*

$$\|\partial_t^k \nabla_x^\nu S(t) f\|_2 \le C(1+t)^{n/4 - n/(2m) - |\nu|/2 - k} (\|f\|_m + \|f\|_{H^{k+|\nu|-1}}), \tag{2.3}$$

for each $f \in L^m(\mathbb{R}^n) \bigcap H^{k+|\nu|-1}(R^n)$.

Lemma 2.5 ([17, Proposition2.4])**.** *Let $\theta(r) = n(1/2 - 1/r)$ and $0 \le \theta(r) \le 1, 0 < \delta \le 1$. If $u \in H^1(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B(t+K)$, then*

$$\|e^{\delta\psi(t,\cdot)} u\|_r \le C(1+t)^{(1-\theta(r))/2} \|e^{\psi(t,\cdot)} \nabla u\|_2^\delta \|\nabla u\|_2^{1-\delta}, \tag{2.4}$$

where $\psi(t,x) = (t + K - \sqrt{(t+K)^2 - |x|^2})/2$.

Lemma 2.6 ([1])**.** *Suppose that $0 \le \theta < 1, a \ge 0$ and $b \ge 0$. Then there exists a constant $C > 0$ depending only on a, b and θ such that for all $t > 0$,*

$$\begin{aligned} &\int_0^t (t-\tau)^{-\theta} (1+t-\tau)^{-a} (1+\tau)^{-b} d\tau \\ &\le \begin{cases} C(1+t)^{-\min\{a+\theta,b\}}, & \max\{a+\theta,b\} > 1, \\ C(1+t)^{-\min\{a+\theta,b\}} \ln(2+t), & \max\{a+\theta,b\} = 1, \\ C(1+t)^{1-\theta-a-b}, & \max\{a+\theta,b\} < 1. \end{cases} \end{aligned}$$

3. Proof of Theorem 1.3

Let $T_{\max}$ be the maximal existence time of the local solution of (u,v) to the problem (1.1). Denote

$$M(t) = \sup_{0 \le \tau < t} ((1+\tau)^k \|Du(\tau)\|_2 + (1+\tau)^j \|Dv(\tau)\|_2), \quad \forall t \in [0, T_{\max}), \tag{3.1}$$

where k, j will be determined later. We will prove the estimate

$$M(t) \le C(\varepsilon + M(t)^p + M(t)^q), \quad \forall t \in [0, T_{\max}), \tag{3.2}$$

with C is independent of ε. Taking ε and C_1 sufficiently small such that

$$C\varepsilon < C_1, \quad 2^{p-1} C M(t)^{p-1} + 2^{q-1} C M(t)^{q-1} < 1,$$

then as the argument in [4, Proposition 2.1], we find from (3.2) that

$$M(t) \le 2C_1, \quad \forall t \in [0, T_{\max}).$$

We have that

$$\|Du(t)\|_2 \le C(1+t)^{-k}, \quad \|Dv(t)\|_2 \le C(1+t)^{-j}, \quad \forall t \in [0, T_{\max}),$$

which imply $T_{\max} = \infty$, the solution of (1.1) exists globally in time.

Now, we prove (3.2). From (2.2), we deduce that

$$\|Du(t)\|_2 \le \|Du_L(t)\|_2 + \int_0^t \|DS(t-\tau) J^\alpha_{0|\tau}(|v|^p)(\tau)\|_2 d\tau. \tag{3.3}$$

Applying Lemma 2.4 with $m = 1$ and $n = 1$, we see that

$$\|Du_L(t)\|_2 \le C I_{0,u}(1+t)^{-3/4}, \tag{3.4}$$

and

$$\begin{aligned}
&\int_0^t \|DS(t-\tau) J^\alpha_{0|\tau}(|v|^p)(\tau)\|_2 d\tau \\
&\le C \int_0^t (1+t-\tau)^{-3/4} (\|J^\alpha_{0|\tau}(|v|^p)(\tau)\|_1 + \|J^\alpha_{0|\tau}(|v|^p)(\tau)\|_2) d\tau \\
&\le C \int_0^t (1+t-\tau)^{-3/4} \int_0^\tau (\tau - s)^{-(1-\alpha)} (\|v(s)\|_p^p + \|v(s)\|_{2p}^p) ds d\tau.
\end{aligned} \tag{3.5}$$

Next, we transform the L^p norm into a weighted L^{2p} norm. Making use of the Cauchy inequality and the fact $\psi(t,x) \ge |x|^2/4(t+K)$ for $x \in B(\tau + K)$, we have

$$\begin{aligned}
\|v(\tau,\cdot)\|_p^p &= \int_{B(\tau+K)} |v(\tau,x)|^p dx \\
&\le \Big(\int_{B(\tau+K)} e^{-2p\delta\psi(\tau,x)} dx \Big)^{1/2} \Big(\int_{B(\tau+K)} e^{2p\delta\psi(\tau,x)} |v(\tau,x)|^{2p} dx \Big)^{1/2} \\
&\le \Big(\int_{B(\tau+K)} e^{-p\delta|x|^2/2(\tau+K)} dx \Big)^{1/2} \|e^{\delta\psi(\tau,\cdot)} v(\tau)\|_{2p}^p \\
&\le C_{K,\delta} (\tau+K)^{1/4} \|e^{\delta\psi(\tau,\cdot)} v(\tau)\|_{2p}^p,
\end{aligned} \tag{3.6}$$

where $\delta > 0$. Obviously,

$$\|v(\tau,\cdot)\|_{2p}^p \le (\tau+K)^{1/4} \|e^{\delta\psi(\tau,\cdot)} v(\tau)\|_{2p}^p. \tag{3.7}$$

From (3.5)-(3.7), we obtain

$$\begin{aligned}
&\int_0^t \|DS(t-\tau) J^\alpha_{0|\tau}(|v|^p)(\tau)\|_2 d\tau \\
&\le C \sup_{[0,t)} \Big[(1+\tau)^{\beta_1} \|e^{\delta\psi(\tau,\cdot)} v(\tau)\|_{2p} \Big]^p \\
&\quad \times \int_0^t (1+t-\tau)^{-3/4} \int_0^\tau (\tau-s)^{-(1-\alpha)} (1+s)^{-(p\beta_1 - 1/4)} ds d\tau.
\end{aligned} \tag{3.8}$$

Taking $\beta_1 = (\alpha+1)(q+1)/(pq-1) - 1/4p$ such that $1/4 < p\beta_1 < 5/4$ and applying Lemma 2.6, we have

$$\begin{aligned} &\int_0^t \|DS(t-\tau)J^{\alpha}_{0|\tau}(|v|^p)(\tau)\|_2 d\tau \\ &\leq C(1+t)^{-k} \sup_{[0,t)} \big[(1+\tau)^{\beta_1} \|e^{\delta\psi(\tau,\cdot)} v(\tau)\|_{2p}\big]^p, \end{aligned} \tag{3.9}$$

where $k = (\alpha+1)(p+1)/(pq-1) + 1/4$. To estimate the weighted L^{2p} norm, we use Lemma 2.5 with $r = 2p$ and $n = 1$,

$$\begin{aligned} \|e^{\delta\psi(\tau,\cdot)} v(\tau)\|_{2p} &\leq C(1+\tau)^{(1-\theta(2p))/2} \|v_x\|_2^{1-\delta} \|e^{\psi(\tau,\cdot)} v_x\|_2^{\delta} \\ &\leq C(1+\tau)^{(1-\theta(2p))/2} \|Dv\|_2. \end{aligned} \tag{3.10}$$

From (3.9), (3.10) and (3.1), we derive that

$$\int_0^t \|DS(t-\tau)J^{\alpha}_{0|\tau}(|v|^p)(\tau)\|_2 d\tau \leq C(1+t)^{-k} \sup_{[0,t)} \big[(1+\tau)^{\beta_1 + (1-\theta(2p))/2 - j} M(t)\big]^p. \tag{3.11}$$

Multiplying (3.3) by $(1+t)^k$ and from (3.4), (3.11), we obtain that

$$(1+t)^k \|Du(t)\|_2 \leq CI_{0,u}(1+t)^{k-3/4} + C \sup_{[0,t)} \Big[(1+\tau)^{\beta_1 + (1-\theta(2p))/2 - j} M(\tau)\Big]^p. \tag{3.12}$$

Similarly, we can deduce that

$$(1+t)^j \|Dv(t)\|_2 \leq CI_{0,v}(1+t)^{j-3/4} + C \sup_{[0,t)} \big[(1+\tau)^{\beta_2 + (1-\theta(2q))/2 - k} M(\tau)\big]^q, \tag{3.13}$$

where we choose $\beta_2 = (\alpha+1)(p+1)/(pq-1) - 1/4q$ such that $1/4 < q\beta_2 < 5/4$ and get $j = (\alpha+1)(q+1)/(pq-1) + 1/4$. It can be easily checked that

$$\begin{aligned} &\beta_1 + (1-\theta(2p))/2 - j = 0, \\ &\beta_2 + (1-\theta(2q))/2 - k = 0,. \end{aligned} \tag{3.14}$$

and

$$k - 3/4 < F(p,q,\alpha) - \alpha, \quad j - 3/4 < F(p,q,\alpha) - \alpha.$$

As $q\beta_2 < 5/4$ and $p\beta_1 < 5/4$ imply $F(p,q,\alpha) < 0$, we have

$$k - 3/4 < -\alpha, \quad j - 3/4 < -\alpha. \tag{3.15}$$

Combining (3.12)-(3.15), we have (3.2). Theorem 1.3 is proved.

4. Proof of Theorem 1.4

In this section we prove the theorem by contraction. In the following, we assume that (u,v) is a global weak solution of (1.1).

Set $\varphi_1(x) := \phi^l(|x|/R), l \gg 1$ with the cut-off function $\phi(r)$ satisfying

$$\phi(r) = \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & r \geq 2, \end{cases} \tag{4.1}$$

$$0 \leq \phi(r) \leq 1, \quad |\phi'(r)| \leq C/r, \quad |\phi''(r)| \leq C/r, \tag{4.2}$$

and $\varphi_2(t) := (1-t/T)_+^{\eta}$, with $\eta \gg 1$. The supports of φ_1 and $(\varphi_1)_{xx}$ are denoted as B_{2R} and $B_{2R} \setminus B_R$ respectively, where

$$B_{2R} = \{x \in \mathbb{R} : |x| \leq 2R\}, \quad B_{2R} \setminus B_R = \{x \in \mathbb{R} : R \leq |x| \leq 2R\}.$$

Denote

$$\varphi(t,x) := \varphi_1(x)(D^{\alpha}_{t|T}\varphi_2)(t). \tag{4.3}$$

From (4.1)-(4.3) and the Proposition 2.1-2.3, we obtain

$$\begin{aligned}
&\int_0^T \int_{B_{2R}} |v(t,x)|^p \varphi(x,t)\,dx\,dt \\
&+ T^{-\alpha}\int_{B_{2R}} (u_1(x)+u_0(x))\varphi_1(x)dx + T^{-\alpha-1}\int_{B_{2R}} u_0(x)\varphi_1(x)dx \\
&= \int_0^T \int_{B_{2R}} u\varphi_1(D^{\alpha+2}_{t|T}\varphi_2(t) + D^{\alpha+1}_{t|T}\varphi_2(t))\,dx\,dt \\
&\quad - \int_0^T \int_{B_{2R}\setminus B_R} u(\varphi_1)_{xx}(D^{\alpha}_{t|T}\varphi_2)(t)\,dx\,dt, \\
&\int_0^T \int_{B_{2R}} |u(t,x)|^q \varphi(x,t)\,dx\,dt + T^{-\alpha}\int_{B_{2R}} (v_1(x)+v_0(x))\varphi_1(x)dx \\
&\quad + T^{-\alpha-1}\int_{B_{2R}} v_0(x)\varphi_1(x)dx \\
&= \int_0^T \int_{B_{2R}} v\varphi_1(x)(D^{\alpha+2}_{t|T}\varphi_2(t) + D^{\alpha+1}_{t|T}\varphi_2(t))\,dx\,dt \\
&\quad - \int_0^T \int_{B_{2R}\setminus B_R} v(\varphi_1)_{xx}(D^{\alpha}_{t|T}\varphi_2)(t)\,dx\,dt.
\end{aligned} \tag{4.4}$$

Set

$$J_p = \int_0^t \int_{B_{2R}} |v(t,x)|^p \varphi(t,x)\,dx\,dt, \tag{4.5}$$

$$J_q = \int_0^t \int_{B_{2R}} |u(t,x)|^q \varphi(t,x)\,dx\,dt. \tag{4.6}$$

From (1.10) and (4.4), we have

$$\begin{aligned}
J_p &\le C\int_0^T \int_{B_{2R}} |u|\varphi_1(D^{2+\alpha}_{t|T}\varphi_2(t) + D^{1+\alpha}_{t|T}\varphi_2(t))\,dx\,dt \\
&\quad + C\int_0^T \int_{B_{2R}\setminus B_R} |u(\varphi_1)_{xx}|(D^{\alpha}_{t|T}\varphi_2)(t)\,dx\,dt = I_1 + I_2.
\end{aligned} \tag{4.7}$$

Applying Holder's inequality with exponents q and $q/(q-1)$, we can achieve that

$$\begin{aligned}
I_1 &\le C\Big(\int_0^T \int_{B_{2R}} |u(t,x)|^q \varphi(t,x)\,dx\,dt\Big)^{1/q} \\
&\quad \times \Big(\int_0^T \int_{B_{2R}} \varphi_1\varphi_2^{-\frac{1}{q-1}}(D^{2+\alpha}_{t|T}\varphi_2 + D^{1+\alpha}_{t|T}\varphi_2)^{q'}\,dx\,dt\Big)^{1/q'} \\
&\le C(T^{-(2+\alpha)+1/q'} + T^{-(1+\alpha)+1/q'})R^{1/q'}J_q^{1/q},
\end{aligned} \tag{4.8}$$

with $q' = q/(q-1)$. In the same way, I_2 can be estimated by

$$\begin{aligned} I_2 &\le C\Big(\int_0^T\int_{B_{2R}} |u(t,x)|^q\varphi(t,x)\,dx\,dt\Big)^{1/q} \\ &\quad\times\Big(\int_0^T\int_{B_{2R}\setminus B_R}\varphi_1^{1-2q'/l}\varphi_2^{-\frac{1}{q-1}}(|\Delta\varphi_1|^{q'}+|\nabla\varphi_1|^{2q'})(D_{t|T}^{2+\alpha}\varphi_2)^{q'}\,dx\,dt\Big)^{1/q'} \\ &\le CT^{-\alpha+1/q'}R^{-2+1/q'}J_q^{1/q}. \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), we deduce that

$$J_p \le CS(q',T,R)J_q^{1/q}, \tag{4.10}$$

where

$$S(q',T,R) = T^{-(2+\alpha)+1/q'}R^{1/q'} + T^{-(1+\alpha)+1/q'}R^{1/q'} + T^{-\alpha+1/q'}R^{-2+1/q'}. \tag{4.11}$$

Similarly, we can prove that

$$J_q \le CS(p',T,R)J_p^{\frac{1}{p}}, \tag{4.12}$$

where

$$S(p',T,R) = T^{-(2+\alpha)+1/p'}R^{1/p'} + T^{-(1+\alpha)+1/p'}R^{1/p'} + T^{-\alpha+1/p'}R^{-2+1/p'}. \tag{4.13}$$

This yields

$$J_p \le CS(q',T,R)S(p',T,R)^{1/q}J_p^{\frac{1}{pq}}. \tag{4.14}$$

Taking $R = \sqrt{T}$ in (4.10)-(4.14), and by Young's inequality, we have

$$J_p \le \frac{1}{2}J_p + CT^{1/2-\alpha-(1+p)(1+\alpha)/(pq-1)}. \tag{4.15}$$

Next, we divide into two cases to discuss the estimate of (4.15).

Case i. $F(p,q,\alpha) > 0$. this implies the exponent of T in (4.15) is negative. Letting $T \to \infty$ in (4.15), we derive that

$$\int_0^\infty\int_{-\infty}^{+\infty} |u(t,x)|^p\,dx\,dt = 0, \tag{4.16}$$

which implies $u(t,x) = 0$ for all t and $x \in \mathbb{R}$ a.e.. This is a contradiction to (1.10).

Case ii. $F(p,q,\alpha) = 0$, we have

$$\lim_{T\to\infty} J_p = \int_0^\infty\int_{\mathbb{R}} |u(x,t)|^p\,dx\,dt \le D. \tag{4.17}$$

It follows from (4.12) that for any $\epsilon > 0$ there exists T_1, such that

$$J_q \le C\epsilon^{1/p}T^{-(1+\alpha)+3(p-1)/2p}, \quad T > T_1, \tag{4.18}$$

where C is independent of ϵ. Combining (4.10) and (4.18), we get that

$$J_p \le C\epsilon^{1/(pq)}, \tag{4.19}$$

and the constant C is also independent of ϵ. The arbitrary of ϵ yields a contradiction with (1.10). This completes the proof of Theorem 1.4.

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