

A MATRIX FORMULATION OF FROBENIUS POWER SERIES SOLUTIONS USING PRODUCTS OF 4×4 MATRICES

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ABSTRACT. In Coddington and Levinson [7, p. 119, Thm. 4.1] and Balsler [4, p. 18-19, Thm. 5], matrix formulations of Frobenius theory, near a regular singular point, are given using 2×2 matrix recurrence relations yielding fundamental matrices consisting of two linearly independent solutions together with their quasi-derivatives. In this article we apply a reformulation of these matrix methods to the Bessel equation of nonintegral order. The reformulated approach of this article differs from [7] and [4] by its implementation of a new “vectorization” procedure that yields recurrence relations of an altogether different form: namely, it replaces the implicit 2×2 matrix recurrence relations of both [7] and [4] by explicit 4×4 matrix recurrence relations that are implemented by means only of 4×4 matrix products. This new idea of using a vectorization procedure may further enable the development of symbolic manipulator programs for matrix forms of the Frobenius theory.

1. INTRODUCTION

The first appearance of a Bessel function was in a 1738 memoir by Daniel Bernoulli [5], [31, p. 356] and since then, numerous books and papers have been devoted to their properties ([31, ch. XVII], [30], [9, vol. 1, 2, & 3, ch. VI, VII, & XIX], [10], [27, ch. 1, 4], [1, ch. 9–11], [13], [14], [15], [6], [8], [18], [17], [19], [23], [24], [26], [27]–[29]).

In this article, we give a matrix recursion involving products of 4×4 matrices by which the power series representations of two Bessel functions and their quasi-derivatives emerge. This approach appears to be new: see, for example, the historical observations in Section 8 below.

2. MATRIX FORMS OF FROBENIUS THEORY FOR SECOND ORDER EQUATIONS

Consider the first order system with $A(x)$ being a 2×2 matrix,

$$X' = \frac{1}{x}A(x)X, \text{ where } A(x) = \sum_{n=0}^{\infty} A_n x^n \text{ is convergent for } |x| < a \text{ for some } a > 0. \quad (2.1)$$

This system is said to have a regular singular point (RSP) at $x = 0$ and will therefore have two linearly independent solutions defined for $x \in (-a, 0) \cup (0, a)$.

2010 *Mathematics Subject Classification*. 34B30, 33C10, 68W30, 34-03, 01A55.

Key words and phrases. Matrix power series; Frobenius theory; Bessel equation.

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Submitted January 12, 2015. Published August 17, 2015.

We now give the matrix formulation of Frobenius theory from [7, p. 119, Thm. 4.1, Eq. 4.4] and [4, p. 20, Ex. 2] stated as Theorem 2.1.

Theorem 2.1. *Assume A_0 has two distinct real eigenvalues r_1 and r_2 , $r_1 > r_2$, $r_1 - r_2 \neq n$, $n = 1, 2, 3, \dots$. Let P be the invertible change of basis matrix for which*

$$B_0 = P^{-1} \cdot A_0 \cdot P = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}. \quad (2.2)$$

Under the above assumptions, every system (2.1) has a unique fundamental matrix solution of the form

$$Y(x) = T(x)x^{B_0}, \quad T(x) = \sum_{n=0}^{\infty} T_n x^n, \quad T_0 = P, \quad 0 < |x| < a, \quad (2.3)$$

where the 2×2 matrix coefficients T_n are uniquely determined by the recurrence relation

$$T_n (B_0 + nI) - A_0 \cdot T_n = \sum_{m=0}^{n-1} A_{n-m} \cdot T_m, \quad n \geq 1. \quad (2.4)$$

Remark 2.2. When $P(x)y'' + Q(x)y' + R(x)y = 0$ with a RSP at $x = 0$ is converted to the system form (2.1), the indicial roots, r_1 and r_2 , of this scalar equation, are identically the eigenvalues of A_0 .

Remark 2.3. B_0 in (2.2)–(2.4) is a “monodromy matrix” and may be calculated as in (2.2) by a diagonalization of A_0 while x^{B_0} may then be explicitly computed by means of the formula: $x^{B_0} = \exp(B_0 \cdot \ln x)$.

Remark 2.4. The “uniqueness” of the 2×2 matrix solution in (2.3) results from the initialization $T_0 = P$ and from [4, p. 212, lem. 24] or [16, p. 225, Eq. 32] which will be discussed further in Section 3.

We now state a definition from Balser [4, p. 18].

Definition 2.5. System (2.1) has a “good spectrum” if no two eigenvalues of A_0 differ by a natural number.

Using the definition above, we now give the matrix formulation of Frobenius theory from [7, p. 119, Thm. 4.1, Eq. 4.2] and [4, p. 19, Thm. 5] stated as Theorem 2.6.

Theorem 2.6. *Every system (2.1) with good spectrum has a unique fundamental matrix solution:*

$$X(x) = S(x)x^{A_0}, \quad S(x) = \sum_{n=0}^{\infty} S_n x^n, \quad S_0 = I, \quad 0 < |x| < \rho, \quad (2.5)$$

where the 2×2 matrix coefficients S_n are uniquely determined by the recurrence relation

$$S_n (A_0 + nI) - A_0 S_n = \sum_{m=0}^{n-1} A_{n-m} S_m, \quad n \geq 1. \quad (2.6)$$

Remark 2.7. Any implementation of recursion relations (2.4) or (2.6) poses considerable difficulty as these relations prohibit explicit representations of the unknown matrices S_n or T_n in terms of previous S_n 's or T_n 's owing to their appearance once on the left and once on the right in matrix products.

To overcome this difficulty indicated above, we now introduce the new idea, a vectorization procedure.

3. PRELIMINARIES FOR THE VECTORIZATION PROCEDURE USING THEOREM 2.1: BESSEL EQUATION

For the Bessel Equation,

$$(xy')' + \left(\frac{-\nu^2}{x} + \lambda x\right)y = 0, \quad x > 0, \quad (3.1)$$

we obtain the 1st order system (2.1) as demonstrated by Baker in [3], where $\lambda = 1$. Thus we set

$$X = \begin{pmatrix} y \\ xy' \end{pmatrix} \quad (3.2)$$

to yield

$$X' = \frac{1}{x}A(x)X, \quad (3.3)$$

where

$$A(x) = A_0 + A_1x + A_2x^2 \quad (3.4)$$

with

$$A_0 = \begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix}, \quad A_1 = \begin{matrix} O \\ 2 \times 2 \end{matrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix}.$$

Before we apply Theorem 2.1 and later Theorem 2.6 to system (3.3)–(3.4), we first observe that the assumptions $r_1 - r_2 \neq n$, $n = 1, 2, 3, \dots$, from Theorem 2.1 and that of a good spectrum from Theorem 2.6 prohibit both the nonzero integer and the nonzero half-integer values of the indicial roots of (3.1), $r_{1,2} = \pm\nu$. However, for system (3.3)–(3.4), the assumptions of Theorems 2.1 and 2.6 may be slightly relaxed to admit the nonzero half-integer values of ν . This analysis follows from a theorem of Kaplan [21, p. 369, Thm. 5, Eq. 9–40]. Recall in the scalar Frobenius theory, with the solution ansatz

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r},$$

(3.1) gives rise to the two-term recurrence relation

$$c_n = -\frac{\lambda}{(n+r+\nu)(n+r-\nu)}c_{n-2},$$

hence the theorem of Kaplan applies ensuring that two linearly independent non-logarithmic power series solutions exist whenever the indicial roots $r_{1,2}$ of (3.1) satisfy $(r_1 - r_2)/k \neq n$, $n = 1, 2, 3, \dots$

For the Bessel equation (3.1), Kaplan's parameter k is 2, so it follows that the fundamental matrix solutions to (3.3)–(3.4), given by both Theorems 2.1 and 2.6, are valid so long as $\nu \neq 1, 2, 3, \dots$

Summarizing, the assumptions on the indicial roots $r_{1,2}$, in Theorems 2.1 and 2.6, are sufficient (but not necessary) to exclude all the logarithmic cases in the Frobenius theory as required for Theorems 2.1 and 2.6 to apply. Our objective now is to apply the matrix form of Frobenius theory, as stated in Theorem 2.1, to obtain a fundamental matrix solution $Y(x, \lambda)$ to the system form of the Bessel

equation (3.3)–(3.4). To this end, we compute the monodromy matrix B_0 in (2.2) by a diagonalization of A_0 whose eigenvalues are $r_{1,2} = \pm\nu$. We find

$$B_0 = P^{-1} \cdot A_0 \cdot P = \begin{bmatrix} \nu & 0 \\ 0 & -\nu \end{bmatrix}, \quad (3.5)$$

where the matrices

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\nu} \\ \frac{1}{2} & -\frac{1}{2\nu} \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ \nu & -\nu \end{bmatrix}.$$

Next we compute the matrix exponential x^{B_0} as

$$x^{B_0} = \exp\left(\begin{bmatrix} \nu & 0 \\ 0 & -\nu \end{bmatrix} \ln x\right) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \begin{bmatrix} \nu & 0 \\ 0 & -\nu \end{bmatrix}^n \ln^n x\right) \quad (3.6)$$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} \nu^n \ln^n x & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} (-\nu)^n \ln^n x \end{bmatrix} = \begin{bmatrix} x^\nu & 0 \\ 0 & x^{-\nu} \end{bmatrix}. \quad (3.7)$$

Observe now that the recurrence relation (2.4) for the Bessel equation (3.3)–(3.4) becomes for $n = 2, 3, 4, \dots$,

$$T_n \begin{bmatrix} \nu + n & 0 \\ 0 & -\nu + n \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix} T_n = \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix} T_{n-2}, \quad (3.8)$$

with the initialization given by (2.3):

$$T_0 = P = \begin{bmatrix} 1 & 1 \\ \nu & -\nu \end{bmatrix}.$$

It follows from $A_1 = O_{2 \times 2}$, that $T_1 = O_{2 \times 2}$ and then from (3.8) that:

$$T_{2n-1} = O_{2 \times 2}, \quad n = 1, 2, 3, \dots \quad (3.9)$$

Now to get T_2 from T_0 , we observe that for $n = 2$, (3.8) becomes

$$\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \nu + 2 & 0 \\ 0 & -\nu + 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\lambda & -\lambda \end{bmatrix}, \quad (3.10)$$

where we have set

$$T_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

Here we can now observe the main obstacle in solving (3.10) for T_2 : namely, the matrix T_2 appears in a “commutator” form, multiplying a 2×2 matrix once on the left and once on the right, thus prohibiting the explicit solution for T_2 in the matrix form of the recurrence relation (3.10). However, T_2 is uniquely determined by matrix theory from Gantmacher [16, p. 225, Eq. 32] or the restatement as a lemma in Balsler [4, p. 212, Eq. A.1], which asserts that the general commutator matrix equation for square matrices A , B , C , and X :

$$A \cdot X - X \cdot B = C \quad (3.11)$$

has a unique solution X if A and B have no common eigenvalues. This criteria holds in (3.10) with

$$A = -\begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix}, \quad B = -\begin{bmatrix} \nu + 2 & 0 \\ 0 & -\nu + 2 \end{bmatrix}, \quad \nu \neq 1, 2, 3, \dots, \quad (3.12)$$

so the existence of a unique solution of (3.10) for $X = T_2$ follows directly from this matrix theory.

4. A VECTORIZATION PROCEDURE

To overcome the commutator equation (3.10), so that a solution procedure for T_2 and more generally T_n emerges, we now introduce a rather simple approach that seems to have eluded others: namely, we will first write the four scalar equations for the four unknowns a_2 , b_2 , c_2 and d_2 (well-known) and then generate, by means of products of suitable 4×4 matrices (the new idea), an explicit solution formula. This first yields, by computation of the matrix products in (3.10), the equivalent 4×4 linear system

$$\begin{aligned}(\nu + 2)a_2 - c_2 &= 0 \\ (-\nu + 2)b_2 - d_2 &= 0 \\ (\nu + 2)c_2 - \nu^2 a_2 &= -\lambda \\ (-\nu + 2)d_2 - \nu^2 b_2 &= -\lambda\end{aligned}\tag{4.1}$$

Now we introduce a “vectorization” of the matrices T_0, T_2, T_4, \dots as follows:

$$\vec{T}_0 := \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \nu \\ -\nu \end{pmatrix}, \quad \text{where } T_0 = P = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \nu & -\nu \end{bmatrix},\tag{4.2}$$

and for $n = 1, 2, 3, \dots$,

$$\vec{T}_{2n} := \begin{pmatrix} a_{2n} \\ b_{2n} \\ c_{2n} \\ d_{2n} \end{pmatrix}, \quad \text{where } T_{2n} = \begin{bmatrix} a_{2n} & b_{2n} \\ c_{2n} & d_{2n} \end{bmatrix}.\tag{4.3}$$

Thus the rows of the 2×2 matrices T_{2n} are “stacked” on top of one another to form the 4×1 vectors \vec{T}_{2n} . Now in the second stage of our vectorization procedure, the 4×4 linear system (4.1) is written as

$$M_2 \vec{T}_2 = \Lambda \vec{T}_0, \quad M_2 = \begin{bmatrix} (\nu + 2) & 0 & -1 & 0 \\ 0 & (-\nu + 2) & 0 & -1 \\ -\nu^2 & 0 & (\nu + 2) & 0 \\ 0 & -\nu^2 & 0 & (-\nu + 2) \end{bmatrix},\tag{4.4}$$

$$\Lambda := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{bmatrix}.\tag{4.5}$$

By the assumption that $\nu \neq \pm 1$, M_2 is invertible and hence:

$$\begin{aligned}\vec{T}_2 &= M_2^{-1} \cdot \Lambda \cdot \vec{T}_0 \\ &= \begin{bmatrix} \frac{2+\nu}{4(1+\nu)} & 0 & \frac{1}{4(1+\nu)} & 0 \\ 0 & \frac{2-\nu}{4(1-\nu)} & 0 & \frac{1}{4(1-\nu)} \\ \frac{\nu^2}{4(1+\nu)} & 0 & \frac{2+\nu}{4(1+\nu)} & 0 \\ 0 & \frac{\nu^2}{4(1-\nu)} & 0 & \frac{2-\nu}{4(1-\nu)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ \nu \\ -\nu \end{pmatrix}\end{aligned}\tag{4.6}$$

$$= \begin{pmatrix} \frac{-\lambda}{2^2(\nu+1)} \\ \frac{-\lambda}{2^2(1-\nu)} \\ \frac{-\lambda}{2^2(\nu+1)} \\ \frac{-\lambda}{2^2(1-\nu)} \end{pmatrix}.$$

Finally, we revert from the vector form of the solution to obtain the 2×2 matrix solution to (3.10) as

$$T_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \frac{-\lambda}{2^2(\nu+1)} & \frac{-\lambda}{2^2(1-\nu)} \\ \frac{-\lambda}{2^2(\nu+1)} & \frac{-\lambda}{2^2(1-\nu)} \end{bmatrix}. \quad (4.7)$$

Clearly, the 4×4 matrix equation (4.4) is preferable to the 2×2 commutator matrix equation (3.10). Accordingly, we can now proceed to generate a ‘‘closed form’’ solution of the matrix recurrence relation (3.8) for the Bessel equation by employing the above vectorization procedure in the general case. Thus, putting $n = 2j$ in (3.8) and taking

$$T_{2j} = \begin{bmatrix} a_{2j} & b_{2j} \\ c_{2j} & d_{2j} \end{bmatrix}, \quad \vec{T}_{2j} = \begin{pmatrix} a_{2j} \\ b_{2j} \\ c_{2j} \\ d_{2j} \end{pmatrix},$$

then the recurrence relation (3.8) becomes

$$M_{2j} \vec{T}_{2j} = \Lambda \vec{T}_{2j-2}, \quad \text{for } j = 1, 2, 3, \dots, \quad (4.8)$$

where

$$M_{2j} = \begin{bmatrix} (\nu + 2j) & 0 & -1 & 0 \\ 0 & (-\nu + 2j) & 0 & -1 \\ -\nu^2 & 0 & (\nu + 2j) & 0 \\ 0 & -\nu^2 & 0 & (-\nu + 2j) \end{bmatrix}.$$

By the assumption that $\nu \neq \pm 1, \pm 2, \pm 3, \dots$, M_{2j} is invertible and hence:

$$\vec{T}_{2j} = M_{2j}^{-1} \cdot \Lambda \cdot \vec{T}_{2j-2}, \quad (4.9)$$

where

$$M_{2j}^{-1} = \begin{bmatrix} \frac{2j+\nu}{4j(j+\nu)} & 0 & \frac{1}{4j(j+\nu)} & 0 \\ 0 & \frac{2j-\nu}{4j(j-\nu)} & 0 & \frac{1}{4j(j-\nu)} \\ \frac{\nu^2}{4j(j+\nu)} & 0 & \frac{2j+\nu}{4j(j+\nu)} & 0 \\ 0 & \frac{\nu^2}{4j(j-\nu)} & 0 & \frac{2j-\nu}{4j(j-\nu)} \end{bmatrix}.$$

Now proceeding to implement the recursion for $n = 2j - 2, 2j - 4, \dots, 2$ gives

$$\vec{T}_{2j} = (M_{2j}^{-1} \Lambda) \vec{T}_{2j-2} = (M_{2j}^{-1} \Lambda) (M_{2j-2}^{-1} \Lambda) (M_{2j-4}^{-1} \Lambda) \cdots (M_2^{-1} \Lambda) \vec{T}_0, \quad (4.10)$$

for $j = 1, 2, 3, \dots$. So, by our vectorization procedure, it follows that (3.8) becomes

$$\vec{T}_{2j} = \left(\prod_{N=0}^{j-1} (M_{2j-2N}^{-1} \Lambda) \right) \vec{T}_0, \quad j = 1, 2, 3, \dots \quad (4.11)$$

As all matrices in (4.11) are known, we can compute the product, obtaining

$$\overrightarrow{T}_{2j} = \begin{pmatrix} \frac{(-1)^j \cdot \lambda^j}{2^{2j} \cdot j! \cdot (\nu+1)_j} \\ \frac{(-1)^j \cdot \lambda^j}{2^{2j} \cdot j! \cdot (1-\nu)_j} \\ \frac{(-1)^j \cdot \lambda^j \cdot (2j+\nu)}{2^{2j} \cdot j! \cdot (\nu+1)_j} \\ \frac{(-1)^j \cdot \lambda^j \cdot (2j-\nu)}{2^{2j} \cdot j! \cdot (1-\nu)_j} \end{pmatrix} \tag{4.12}$$

where $\alpha_j = (\alpha)(\alpha + 1) \cdots (\alpha + j - 1)$ is the Pochhammer symbol. Finally, we revert from vector form to yield the closed form 2×2 matrix solution of (3.8),

$$T_{2j} = \begin{bmatrix} \frac{(-1)^j \cdot \lambda^j}{2^{2j} \cdot j! \cdot (\nu+1)_j} & \frac{(-1)^j \cdot \lambda^j}{2^{2j} \cdot j! \cdot (1-\nu)_j} \\ \frac{(-1)^j \cdot \lambda^j \cdot (2j+\nu)}{2^{2j} \cdot j! \cdot (\nu+1)_j} & \frac{(-1)^j \cdot \lambda^j \cdot (2j-\nu)}{2^{2j} \cdot j! \cdot (1-\nu)_j} \end{bmatrix}. \tag{4.13}$$

From (2.3), (3.7), and (4.13), the fundamental solution to (3.3)–(3.4) given by Theorem 2.1 is then

$$\begin{aligned} Y(x, \lambda) &= \begin{bmatrix} y_1(x, \lambda) & y_2(x, \lambda) \\ xy'_1(x, \lambda) & xy'_2(x, \lambda) \end{bmatrix} \\ &= T(x)x^{B_0} = \left(\sum_{j=0}^{\infty} T_{2j} \right) x^{2j} \cdot \begin{bmatrix} x^\nu & 0 \\ 0 & x^{-\nu} \end{bmatrix} \\ &= \left(\sum_{j=0}^{\infty} T_{2j} \cdot \begin{bmatrix} x^\nu & 0 \\ 0 & x^{-\nu} \end{bmatrix} \right) x^{2j} \\ &= \begin{bmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j \cdot \lambda^j \cdot x^{2j+\nu}}{2^{2j} \cdot j! \cdot (\nu+1)_j} & \sum_{j=0}^{\infty} \frac{(-1)^j \cdot \lambda^j \cdot x^{2j-\nu}}{2^{2j} \cdot j! \cdot (1-\nu)_j} \\ \sum_{j=0}^{\infty} \frac{(-1)^j \cdot \lambda^j \cdot (2j+\nu) \cdot x^{2j+\nu}}{2^{2j} \cdot j! \cdot (\nu+1)_j} & \sum_{j=0}^{\infty} \frac{(-1)^j \cdot \lambda^j \cdot (2j-\nu) \cdot x^{2j-\nu}}{2^{2j} \cdot j! \cdot (1-\nu)_j} \end{bmatrix}. \end{aligned} \tag{4.14}$$

As system (3.3)–(3.4) is the matrix formulation of the Bessel equation (3.1), we may expect the fundamental matrix (4.14) to contain Bessel functions. This connection will now be made in the next section.

5. CONNECTION TO BESSEL FUNCTIONS

From (4.14), the Bessel functions and the modified Bessel functions are not yet transparent. This is because these special functions receive their definitions from the standard Frobenius theory, which uses a scalar recurrence relation for the equations:

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0 \quad \text{Bessel equation} \tag{5.1}$$

$$x^2u' + xu' - (x^2 + \nu^2)u = 0 \quad \text{modified Bessel equation} \tag{5.2}$$

where no eigenparameter λ appears. The Bessel functions then receive their definitions by making special choices of constants multiplying the Frobenius solutions $u_i(x)$, $i = 1, 2, 3, 4$ to (5.1) and (5.2) where $u_1(x) = y_1(x, 1)$, $u_2(x) = y_2(x, 1)$, $u_3(x) = y_1(x, -1)$, and $u_4 = y_2(x, -1)$, so that

$$J_\nu(x) = \frac{1}{2^\nu \Gamma(\nu + 1)} u_1(x) = \sum_{j=0}^{\infty} \left(\frac{(-1)^j}{j! \Gamma(\nu + 1 + j)} \right) \left(\frac{x}{2} \right)^{2j+\nu}, \tag{5.3}$$

$$J_{-\nu}(x) = \frac{1}{2^{-\nu}\Gamma(\nu+1)} u_2(x) = \sum_{j=0}^{\infty} \left(\frac{(-1)^j}{j!\Gamma(1-\nu+j)} \right) \left(\frac{x}{2} \right)^{2j-\nu}, \quad (5.4)$$

$$I_{\nu}(x) = \frac{1}{2^{\nu}\Gamma(\nu+1)} u_3(x) = \sum_{j=0}^{\infty} \left(\frac{1}{j!\Gamma(\nu+1+j)} \right) \left(\frac{x}{2} \right)^{2j+\nu}, \quad (5.5)$$

$$I_{-\nu}(x) = \frac{1}{2^{-\nu}\Gamma(\nu+1)} u_4(x) = \sum_{j=0}^{\infty} \left(\frac{1}{j!\Gamma(1-\nu+j)} \right) \left(\frac{x}{2} \right)^{2j+\nu}. \quad (5.6)$$

We now wish to identify which Bessel functions are generated in the fundamental matrix solution $Y(x, \lambda)$ in (4.14). Comparison between (4.14), (5.3)–(5.6) reveals that the entries of $Y(x, \lambda)$ contain weighted Bessel functions of the arguments $\sqrt{\lambda}x$ and $\sqrt{|\lambda|x}$, or more specifically,

$$Y(x, \lambda) = \begin{bmatrix} 2^{\nu}\Gamma(\nu+1)\lambda^{-\nu/2}J_{\nu}(\sqrt{\lambda}x) & 2^{-\nu}\Gamma(1-\nu)\lambda^{\nu/2}J_{-\nu}(\sqrt{\lambda}x) \\ 2^{\nu}\Gamma(\nu+1)\lambda^{-\nu/2}xJ'_{\nu}(\sqrt{\lambda}x) & 2^{-\nu}\Gamma(1-\nu)\lambda^{\nu/2}xJ'_{-\nu}(\sqrt{\lambda}x) \end{bmatrix} \quad (5.7)$$

for $\lambda \geq 0$, and

$$Y(x, \lambda) = \begin{bmatrix} 2^{\nu}\Gamma(\nu+1)|\lambda|^{-\nu/2}I_{\nu}(\sqrt{|\lambda|x}) & 2^{-\nu}\Gamma(1-\nu)|\lambda|^{\nu/2}I_{-\nu}(\sqrt{|\lambda|x}) \\ 2^{\nu}\Gamma(\nu+1)|\lambda|^{-\nu/2}xI'_{\nu}(\sqrt{|\lambda|x}) & 2^{-\nu}\Gamma(1-\nu)|\lambda|^{\nu/2}xI'_{-\nu}(\sqrt{|\lambda|x}) \end{bmatrix} \quad (5.8)$$

for $\lambda < 0$.

Remark 5.1. (1) One consequence of Theorem 2.1, due to the monodromy matrix B_0 and the initialization $T_0 = P$ in the recurrence relation (2.4), is that the fundamental matrix solution (2.3) to (3.3)–(3.4) captures the Frobenius solutions to the scalar Bessel equation (3.1) in the first row of (4.14) and their quasi-derivatives in the second row.

(2) Evidently the 4×4 matrix recurrence relation (4.11) has enabled two Bessel functions and their quasi-derivatives to emerge by a novel means: namely, as products of 4×4 matrices.

We now demonstrate that our vectorization procedure applies equally well in implementing Theorem 2.6.

6. AN EXAMPLE OF THE VECTORIZATION PROCEDURE USING THEOREM 2.6: BESSEL EQUATION

In this section we take note of the fact that the vectorization procedure, used to solve the matrix recurrence relation (2.4) in Theorem 2.1, applies equally well to solve the recurrence relation (2.6) in Theorem 2.6. As it turns out, Theorem 2.6 yields in the first row of its fundamental matrix $X(x, \lambda)$, linear combinations of the two Frobenius solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$, obtained earlier by application of Theorem 2.1.

We find $S_{2n-1} = O_{2 \times 2}$, $n = 1, 2, 3, \dots$, while for $n = 2$, the recurrence relation (2.6) yields the system

$$\begin{aligned} 2a_2 + \nu^2 b_2 - c_2 &= 0 \\ a_2 + 2b_2 - d_2 &= 0 \\ -\nu^2 a_2 + 2c_2 + \nu^2 d_2 &= -\lambda \\ -\nu^2 b_2 + c_2 + 2d_2 &= 0 \end{aligned} \quad (6.1)$$

where $S_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

Similarly to equation (4.6), we find by employing a vectorization procedure that

$$\begin{aligned} \vec{S}_2 &:= \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = (M_2^{-1}\Lambda)\vec{S}_0 \\ &= \frac{1}{4(\nu^2 - 1)} \underbrace{\begin{bmatrix} \nu^2 - 2 & \nu^2 & -1 & \nu^2 \\ 1 & \nu^2 - 2 & 1 & -1 \\ -\nu^2 & \nu^4 & \nu^2 - 2 & \nu^2 \\ \nu^2 & -\nu^2 & 1 & \nu^2 - 2 \end{bmatrix}}_{M_2^{-1}} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{bmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{S}_0} \\ &= \begin{pmatrix} \frac{-\lambda}{2^5} \left\{ \frac{1}{\nu+1} + \frac{1}{1-\nu} \right\} \\ \frac{-\lambda}{2^5} \left\{ \frac{1}{\nu} \left[\frac{1}{\nu+1} + \frac{1}{1-\nu} \right] \right\} \\ \frac{-\lambda}{2^5} \left\{ \frac{2+\nu}{\nu+1} + \frac{2-\nu}{1-\nu} \right\} \\ \frac{-\lambda}{2^5} \left\{ \frac{1}{\nu} \left[\frac{1}{\nu+1} + \frac{1}{1-\nu} \right] \right\} \end{pmatrix}. \\ \vec{S}_{2j} &= (M_{2j}^{-1}\Lambda)\vec{S}_{2j-2} = (M_{2j}^{-1}\Lambda)(M_{2j-2}^{-1}\Lambda)(M_{2j-4}^{-1}\Lambda)\cdots(M_2^{-1}\Lambda)\vec{S}_0 \\ &= \left(\prod_{N=0}^{j-1} (M_{2j-2N}^{-1}\Lambda) \right) \vec{S}_0 \\ &= \begin{pmatrix} \frac{(-1)^j \cdot \lambda^j}{2^{2j+1} \cdot j!} \left\{ \frac{1}{(\nu+1)_j} + \frac{1}{(1-\nu)_j} \right\} \\ \frac{(-1)^j \cdot \lambda^j}{2^{2j+1} \cdot j!} \left\{ \frac{1}{\nu} \left[\frac{1}{(\nu+1)_j} - \frac{1}{(1-\nu)_j} \right] \right\} \\ \frac{(-1)^j \cdot \lambda^j}{2^{2j+1} \cdot j!} \left\{ \frac{2j+\nu}{(\nu+1)_j} + \frac{2j-\nu}{(1-\nu)_j} \right\} \\ \frac{(-1)^j \cdot \lambda^j}{2^{2j+1} \cdot j!} \left\{ \frac{1}{\nu} \left[\frac{1}{(\nu+1)_j} + \frac{1}{(1-\nu)_j} \right] \right\} \end{pmatrix}, \quad j = 1, 2, 3, \dots, \end{pmatrix} \tag{6.2} \end{aligned}$$

and

$$M_{2j}^{-1} = \frac{1}{4(\nu^2 - j^2)} \begin{bmatrix} \frac{\nu^2 - 2j^2}{j} & \nu^2 & -1 & \frac{\nu^2}{j} \\ 1 & \frac{\nu^2 - 2j^2}{j} & \frac{1}{j} & -1 \\ -\nu^2 & \frac{\nu^4}{j} & \frac{\nu^2 - 2j^2}{j} & \nu^2 \\ \frac{\nu^2}{j} & -\nu^2 & 1 & \frac{\nu^2 - 2j^2}{j} \end{bmatrix}. \tag{6.3}$$

Accordingly, the fundamental solution matrix (2.5) is readily calculated as

$$\begin{aligned} X(x, \lambda) &= \begin{bmatrix} y_3(x, \lambda) & y_4(x, \lambda) \\ xy_3'(x, \lambda) & xy_4'(x, \lambda) \end{bmatrix} = S(x)x^{A_0} \\ &= \sum_{j=0}^{\infty} \left\{ S_{2j} \cdot \begin{bmatrix} \frac{x^\nu + x^{-\nu}}{2} & \frac{x^\nu - x^{-\nu}}{2\nu} \\ \nu \frac{x^\nu - x^{-\nu}}{2} & \frac{x^\nu + x^{-\nu}}{2} \end{bmatrix} \right\} x^{2j} \end{aligned} \tag{6.4}$$

The above expression is a matrix whose first row is

$$\frac{1}{2} \sum_{j=0}^{\infty} A_{2j} x^{2j+\nu} + \frac{1}{2} \sum_{j=0}^{\infty} B_{2j} x^{2j-\nu}, \quad \frac{1}{2\nu} \sum_{j=0}^{\infty} A_{2j} x^{2j+\nu} - \frac{1}{2\nu} \sum_{j=0}^{\infty} B_{2j} x^{2j-\nu},$$

and its second row is

$$\begin{aligned} & \frac{1}{2} \sum_{j=0}^{\infty} (2j + \nu) A_{2j} x^{2j+\nu} + \frac{1}{2} \sum_{j=0}^{\infty} (2j - \nu) B_{2j} x^{2j-\nu}, \\ & \frac{1}{2\nu} \sum_{j=0}^{\infty} (2j + \nu) A_{2j} x^{2j+\nu} - \frac{1}{2\nu} \sum_{j=0}^{\infty} (2j - \nu) B_{2j} x^{2j-\nu}. \end{aligned}$$

Here A_{2j} and B_{2j} are given by

$$A_{2j} = \frac{(-1)^j \cdot \lambda^j}{2^{2j} \cdot j! \cdot (\nu + 1)_j}, \quad B_{2j} = \frac{(-1)^j \cdot \lambda^j}{2^{2j} \cdot j! \cdot (1 - \nu)_j}, \quad (6.5)$$

and we have used

$$\begin{aligned} x^{A_0} &= \exp \left(\begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix} \cdot \ln x \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\left(P \cdot \begin{bmatrix} \nu & 0 \\ 0 & -\nu \end{bmatrix} \cdot P^{-1} \right)^n}_{A_0} \cdot \ln^n x \\ &= \begin{bmatrix} \frac{x^\nu + x^{-\nu}}{2} & \frac{x^\nu - x^{-\nu}}{2\nu} \\ \frac{\nu(x^\nu - x^{-\nu})}{2} & \frac{x^\nu + x^{-\nu}}{2} \end{bmatrix}. \end{aligned} \quad (6.6)$$

Comparison between (6.4)–(6.5) and (4.14) now reveals:

$$\begin{aligned} X(x, \lambda) &= \begin{bmatrix} y_3(x, \lambda) & y_4(x, \lambda) \\ xy_3'(x, \lambda) & xy_4'(x, \lambda) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}y_1(x, \lambda) + \frac{1}{2}y_2(x, \lambda) & \frac{1}{2\nu}y_1(x, \lambda) - \frac{1}{2\nu}y_2(x, \lambda) \\ x \left(\frac{1}{2}y_1'(x, \lambda) + \frac{1}{2}y_2'(x, \lambda) \right) & x \left(\frac{1}{2\nu}y_1'(x, \lambda) - \frac{1}{2\nu}y_2'(x, \lambda) \right) \end{bmatrix}, \end{aligned} \quad (6.7)$$

where the first row of $X(x, \lambda)$ may be represented in terms of Bessel functions as

$$\begin{aligned} & y_3(x, \lambda) \\ &= \begin{cases} \frac{1}{2}\lambda^{-\nu/2} \cdot 2^\nu \Gamma(\nu + 1) J_\nu(\sqrt{\lambda}x) + \frac{1}{2}\lambda^{\nu/2} \cdot 2^{-\nu} \Gamma(1 - \nu) J_{-\nu}(\sqrt{\lambda}x), & \lambda \geq 0 \\ \frac{1}{2}|\lambda|^{-\nu/2} \cdot 2^\nu \Gamma(\nu + 1) I_\nu(\sqrt{|\lambda}|x) + \frac{1}{2}|\lambda|^{\nu/2} \cdot 2^{-\nu} \Gamma(1 - \nu) I_{-\nu}(\sqrt{|\lambda}|x), & \lambda < 0 \end{cases} \end{aligned} \quad (6.8)$$

$y_4(x, \lambda)$

$$= \begin{cases} \frac{1}{2\nu}\lambda^{-\nu/2} \cdot 2^\nu \Gamma(\nu + 1) J_\nu(\sqrt{\lambda}x) - \frac{1}{2\nu}\lambda^{\nu/2} \cdot 2^{-\nu} \Gamma(1 - \nu) J_{-\nu}(\sqrt{\lambda}x), & \lambda \geq 0 \\ \frac{1}{2\nu}|\lambda|^{-\nu/2} \cdot 2^\nu \Gamma(\nu + 1) I_\nu(\sqrt{|\lambda}|x) - \frac{1}{2\nu}|\lambda|^{\nu/2} \cdot 2^{-\nu} \Gamma(1 - \nu) I_{-\nu}(\sqrt{|\lambda}|x), & \lambda < 0 \end{cases} \quad (6.9)$$

Remark 6.1. One consequence of Theorem 2.6, due to the monodromy matrix A_0 and the initialization $S_0 = I$ in recurrence relation (2.6), is that the fundamental matrix solution (2.5) to (3.3)–(3.4) captures linear combinations of Bessel Functions in the first row of (6.4) with their corresponding quasi-derivatives in the second row.

Lastly we sketch the vectorization procedure as it may be applied to a general Frobenius-type problem of the non-logarithmic cases.

7. A VECTORIZATION PROCEDURE FOR THE GENERAL PROBLEM

For the general problem with $A(x)$ being a 2×2 matrix,

$$X' = \frac{1}{x}A(x)X = \frac{1}{x}[A_0 + A_1x + A_2x + \cdots]X, \quad (7.1)$$

recurrence relation (2.4) of Theorem 2.1 is

$$T_n(B_0 + nI) - A_0 \cdot T_n = \sum_{m=0}^{n-1} A_{n-m} \cdot T_m = A_n T_0 + A_{n-1} T_1 + \cdots + A_1 T_{n-1}, \quad n \geq 1, \quad (7.2)$$

where, as in Theorem 2.1,

$$B_0 = P^{-1} \cdot A_0 \cdot P = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \quad T_0 = P. \quad (7.3)$$

If r_1 and r_2 are real with $r_1 > r_2$, then the assumption that (7.1) has a good spectrum is

$$r_1 - r_2 \neq n, \quad n = 0, 1, 2, 3, \dots, \quad (7.4)$$

which rules out the logarithmic cases of Frobenius theory as required and hence the vectorization procedure introduced in Section 4 is applicable to solve (7.2). This fact follows immediately from the lemma of Gantmacher and Balser in (3.11) as applied to (7.2) since for $n = 1, 2, 3, \dots$, the eigenvalues $r_1 + n$ and $r_2 + n$ of $B_0 + nI$ differ from the eigenvalues r_1 and r_2 of A_0 because of assumption (7.4). Thus a unique solution of (7.2) for the 2×2 matrix T_n must exist and since the right hand side of (3.11) is immaterial for the application of this lemma, successive solution for unique T_1, T_2, \dots, T_n is possible. To illustrate the vectorization procedure in this general case, consider the first step for $n = 1$:

$$T_1(B_0 + I) - A_0 T_1 = A_1 T_0. \quad (7.5)$$

Letting

$$T_0 = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_n = \begin{bmatrix} D_n & E_n \\ F_n & G_n \end{bmatrix}, \quad n = 0, 1, 2, 3, \dots,$$

then the matrix equation (7.5) corresponds to the four scalar equations:

$$\begin{aligned} a_1(r_1 + 1) - D_0 a_1 - E_0 c_1 &= D_1 a_0 + E_1 c_0 \\ b_1(r_2 + 1) - D_0 b_1 - E_0 d_1 &= D_1 b_0 + E_1 d_0 \\ c_1(r_1 + 1) - F_0 a_1 - G_0 c_1 &= F_1 a_0 + G_1 c_0 \\ d_1(r_2 + 1) - F_0 b_1 - G_0 d_1 &= F_1 b_0 + G_1 d_0 \end{aligned}$$

Writing these in the form of a 4×4 matrix equation we thus see that the analog of equation (4.4) is

$$M_1 \vec{T}_1 = \Lambda_1 \vec{T}_0, \quad \text{where again } \vec{T}_0 = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}, \quad \vec{T}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}, \quad (7.6)$$

and in this general case we find:

$$M_1 = \begin{bmatrix} r_1 + 1 - D_0 & 0 & -E_0 & 0 \\ 0 & r_2 + 1 - D_0 & 0 & -E_0 \\ -F_0 & 0 & r_1 + 1 - G_0 & 0 \\ 0 & -F_0 & 0 & r_1 + 1 - G_0 \end{bmatrix}, \quad (7.7)$$

$$\Lambda_1 = \begin{bmatrix} D_1 & 0 & E_1 & 0 \\ 0 & D_1 & 0 & E_1 \\ F_1 & 0 & G_1 & 0 \\ 0 & F_1 & 0 & G_1 \end{bmatrix}. \quad (7.8)$$

Hence,

$$\vec{T}_1 = (M_1^{-1} \cdot \Lambda_1) \cdot \vec{T}_0, \quad (7.9)$$

The fact that M_1 is invertible follows (under the assumption of good spectrum) directly from the existence of a unique solution for T_1 of equation (7.5). Likewise in the n^{th} step, recurrence relation (7.2) converts by our vectorization procedure to

$$M_n \vec{T}_n = \sum_{m=0}^{n-1} \left(\Lambda_{n-m} \cdot \vec{T}_m \right), \quad n = 1, 2, 3, \dots, \quad (7.10)$$

where

$$\vec{T}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}, \quad M_n = \begin{bmatrix} r_1 + n - D_0 & 0 & -E_0 & 0 \\ 0 & r_2 + n - D_0 & 0 & -E_0 \\ -F_0 & 0 & r_1 + n - G_0 & 0 \\ 0 & -F_0 & 0 & r_1 + n - G_0 \end{bmatrix},$$

and where

$$\Lambda_{n-m} = \begin{bmatrix} D_{n-m} & 0 & E_{n-m} & 0 \\ 0 & D_{n-m} & 0 & E_{n-m} \\ F_{n-m} & 0 & G_{n-m} & 0 \\ 0 & F_{n-m} & 0 & G_{n-m} \end{bmatrix}.$$

A closed form representation for \vec{T}_n is then

$$\vec{T}_n = M_n^{-1} \cdot \left(\sum_{m=0}^{n-1} (\Lambda_{n-m} \cdot \vec{T}_m) \right) = \sum_{m=0}^{n-1} \left((M_n^{-1} \cdot \Lambda_{n-m}) \cdot \vec{T}_m \right), \quad (7.11)$$

for $n = 1, 2, 3, \dots$, which formulates the 4×1 vector \vec{T}_n explicitly as a sum of products of 4×4 matrices with 4×1 vectors whose representations are all readily obtained from this section. Therefore, under the assumption of good spectrum of system (7.1), our vectorization procedure indeed applies in the general case.

In the last section, we give a brief history of the Bessel equation and the power series method. We also summarize the work of prior authors towards the Bessel equation in chronological order.

8. SOME HISTORY ON THE BESSEL EQUATION AND THE POWER SERIES METHOD

The Bessel equation $x^2 w'' + x w' + (x^2 - a^2) w = 0$ takes its name from the German astronomer F. W. Bessel who in 1824, while studying the perturbations of the planets, began the development of the now extensive theory of its solutions (see [6]). In Bessel's treatment, he assumed the parameter a to be an integer and managed to obtain integral definitions of the Bessel functions (Schlömlich and Lipschitz later

named these functions after Bessel [19, p. 319]). Interestingly, the now standard power series representations for the Bessel functions (5.3-5.4) do not appear in Bessel's 1824 seminal paper and they are also missing from the original papers of Frobenius and Fuchs (see [14]–[13]). Given that the power series solution ansatz was originated by L. Euler [2, p. 204] and thus it predates the work of Bessel, Frobenius, and Fuchs, these omissions are notable. In fact, the power series formulas for the Bessel functions arose first not from the Euler/Frobenius/Fuchsian theory but from Bessel's integral representations and thus it is these integral representations that appear in the early books (compare E.C.J. von Lommel [23], C.G. Neumann [24], and G. N. Watson [30, ch. 2]).

Some special instances of series expansions of Bessel functions were however generated before Bessel's 1824 paper through investigations to such varied topics as the oscillations of heavy chains and circular membranes, elliptic motion, and heat conduction in cylinders (see [5] for instance where D. Bernoulli, in 1738, obtained an open form series expansion of a Bessel function of order 0 or [10] where L. Euler, in 1764, generated a Bessel series while investigating vibrations of a stretched circular membrane. Below is a summary of the extent of prior author's treatments of the Bessel equation using the Euler/Frobenius/Fuchsian power series method and its matrix forms (references and pages indicated).

Key for the following table:

- I Solutions of a scalar Bessel equation are obtained as a power series near $x = 0$ using the power series method.
- II A 2×2 matrix formulation of the power series method for 2nd order equations is given.
- III A 2×2 matrix form of the Bessel equation is given.

Author(s); Year; Reference	I	II	III
F. W. Bessel; 1824; [6]	No	No	No
L. Fuchs; 1866, 1868; [14], [15]	No	No	No
C. G. Neumann; 1867; [24]	No	No	No
E. C. J. von Lommel; 1868; [23]	No	No	No
G. Frobenius; 1873; [13]	No	No	No
A. Gray, G. B. Matthews; 1895; [18]	Yes, p. 7–12	No	No
L. Schlesinger; 1897; [28]	No	No	No
L. Schlesinger; 1900, 1904, 1922; [29]	Yes, p. 274-279	Yes, p.132-142	No
A. R. Forsythe; 1902; [12]	Yes, p.100–102	No	No
H. F. Baker; 1902; [3]	No	Yes, p. 335	Yes, p.348
Whittaker, Watson; 1902, 1915, 1920, 1944; [31]	Yes, p.197-202	No	No
A. R. Forsythe; 1903, 3 rd ed.; [11]	Yes, p.176-186, 6 th ed.	No	No
N. Nielsen; 1904; [25]	Yes, p. 4–5	No	No
G. N. Watson; 1922, 1944; [30]	Yes, p.38–42, 57-63	No	No
R. Courant, D. Hilbert; 1924, 1931; [8]	Yes, p.260, 418	No	No
E. L. Ince; 1926; [20]	Yes, p.403-404	Yes, p.408-415	Yes, p.415
A. Erdelyi; 1953; [9]	Yes, p.4–5	No	No
Lappo-Danilevsky; 1936; 1953; [22]	No	Yes, p.143	No
Coddington, Levison; 1955; [7]	No	Yes, ch.4, p.119	No
Abramowitz, Stegun; 1964; [1]	Yes, p.360; 437	No	No
E. Hille; 1969; [19]	Yes, p.319-320	Yes, p.188-191,239-250	No

W. Balsler; 2000; [4]	Yes, p.23	Yes, ch.2, p.19	No
Olver et al; 2010; [26]	Yes, p.217	No	No

So it appears that the power series method was not applied to the Bessel Equation until some years after the original papers of Frobenius and Fuchs, (see [13]–[15]). As indicated in the table, the earliest application of the Euler/Frobenius/Fuchsian power series method to the Bessel Equation that I could find was in the 1895 book of Gray and Matthews [18]. Also, as revealed by the above table, only two authors appear to have written the Bessel equation in a matrix form, first Baker in [3] and then later Ince in [20]. Note though that neither of these authors attempted to apply their 2×2 matrix forms of the power series method to the matrix form of the Bessel equation so as to obtain a 2×2 fundamental matrix as given in this paper in (4.14). This fact appears to stem from the computational difficulty of generating the solution matrix X of (3.11) for use in implementing the matrix recurrence relations (2.4) or (2.6), a matter not addressed by Schlesinger [29], Baker [3], Ince [20], Lappo-Danilevsky [22], Coddington and Levison [7, ch. 4], Hille [19], or Balsler [4, ch. 2], all who formulated a 2×2 matrix power series formulation of the Euler/Frobenius/Fuchsian theory. At least it appears that none of these authors provide a nontrivial example of a matrix equation for which the recurrence relations (2.4) or (2.6), in Theorems 2.1 and 2.6, were successfully solved to yield an explicit representation of the fundamental matrix solution in (2.3) or (2.5).

A search of the literature leads this author to believe that the development of a general method to properly resolve the computational difficulty in (2.4) and (2.6), associated with the 2×2 matrix power series method, has indeed been left undone. This is further supported by a remark of Ince [20, p. 415], where in specific reference to the matrix form of the Bessel Equation (3.3)-(3.4), with $\nu = 1$, he states, “The scope of the matrix method is very wide, but its successful application demands a knowledge of theorems in the calculus of matrices which cannot be given here.” Likewise, after Hille furnishes his solution ansatz to the matrix equation $w'(z) = F(z) \cdot w(z)$ involving a regular singular point at the origin, and defines his “resolvent of the commutator”, he states, “Thus the formal solution is well defined, if not explicitly known”, [19, p. 233].

It should be noted though that two authors did however come close to obtaining an explicit solution of the 2×2 matrix recurrence relation. Both Lappo-Danilevsky in [22] and then later Hille in [19] proposed highly theoretical solutions to the 2×2 matrix recurrence relation using a 2×2 matrix theory. Lappo-Danilevsky’s theory appears to be invoked in Hille’s resolvent of the commutator [19, p. 233], which, in principle, is a matrix inverse operator that if one could explicitly obtain it, then it would theoretically yield the general solution to the 2×2 matrix recurrence relation in (2.4) or (2.6) or (7.2); and, in particular, the solution (4.14) obtained here would be expressible as a product of suitable 2×2 matrices. The fact though that there is no implementation to any specific examples of the Lappo-Danilevsky theory or the Hille theory in their works or in any of the works by the later authors who also gave a 2×2 matrix formulation of the Frobenius theory does indicate that an explicit solution to the 2×2 matrix recurrence relation that could be implemented in the general case as shown in Section 7 has heretofore eluded previous authors.

9. CONCLUSION

In the literature, prior matrix treatments of Frobenius theory, like Theorem 2.1 and 2.6, make use of coupled matrix recurrence relations that pose considerable difficulties in their implementation. The new idea outlined in this paper, our vectorization procedure, appears to provide a rather elegant means to reformulate the matrix recurrence relations of both Theorem 2.1 and 2.6 into equivalent recurrence relations that require only computation of well-defined matrix products and sums. It can also be noted that our vectorization procedure seems to generalize and apply to the third order, fourth order, \dots , n^{th} order Frobenius-type problems of the non-logarithmic cases and near regular singular points by generating equivalent recurrence relations that require only matrix multiplications and sums involving 9×9 matrices, 16×16 matrices, \dots , $n^2 \times n^2$ matrices respectively.

Acknowledgements. Special thanks to Dr. Charles Fulton of the Florida Institute of Technology for his many comments and suggestions that have greatly improved this paper's exposition. In particular, in Section 5: Connection to Bessel Functions and Section 8: Some History on the Bessel Equation and the Power Series Method, his expertise and depth of knowledge was very insightful. Also a deep gratitude is reserved for Jonathan Goldfarb for his judicious editing of a rather raw L^AT_EX manuscript to work it into its present form.

REFERENCES

- [1] Abramowitz and Stegun; *Handbook of Mathematical Functions*, National Bureau of Standards, 1964.
- [2] Ravi P. Agarwal, Syamal K. Sen; *Creators of mathematical and computational sciences*, Springer, 2014.
- [3] H. Baker; *On the integration of Linear Differential Equations*, Proc. London Math Society **s1-35** (1902), no. 1, 333–384.
- [4] W. Balseer; *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*, Springer, 2000.
- [5] D. Bernoulli; *Theoremata De Oscillationibus corporum filo flexile connexorum et catenae verticaliter suspensae*, Comm. Acad. Sci. Imp. Petrop. VI. (1732–33), 108–122, [Published 1738].
- [6] F. W. Bessel; *Untersuchung des Theils der planetarischen Störungen welcher aus der Bewegung der Sonne entsteht*, Berliner Abhandlungen (1824), 1–52, [Published 1826].
- [7] Coddington, Levinson; *Theory of Ordinary Differential Equations*, McGraw-Hill, N.Y., 1955.
- [8] R. Courant, D. Hilbert; *Methoden der Mathematischen Physik, Erster Band*, Springer Verlag, Berlin, 1924 (first edition) 1931 (second edition).
- [9] Erdelyi et. al.; *Higher Transcendental Functions*, vol. II, McGraw-Hill, 1955.
- [10] Leonhard Euler; *De motu vibratorio tympanorum*, Novi Commentarii academiae scientiarum Petropolitanae (1764).
- [11] A. R. Forsythe; *A Treatise on Differential Equations*, first ed., Macmillan and Co. limited, London, 1885, 1888 (second edition), 1903 (third edition), 1914 (fourth edition), 1921 (fifth edition), 1929 (sixth edition), Reprinted in 1933, 1943, and 1948.
- [12] A. R. Forsythe; *Theory of Differential Equations, Part III*, vol. IV, Cambridge University Press, 1902, (Reprinted by Dover, 1959).
- [13] G. Frobenius; *Über die Integration der linearen Differentialgleichungen durch Reihen*, Jour. für die reine und angewandte Mathematik **76** (1873), 214–235.
- [14] L. Fuchs; *Zur Theorie der linearen Differentialgleichungen mit veränderlichen Koeffizienten*, Jour. für die reine und angewandte Mathematik **66** (1866), 121–160.
- [15] L. Fuchs; *Zur Theorie der linearen Differentialgleichungen mit veränderlichen Koeffizienten*, Jour. für die reine und angewandte Mathematik **68** (1868), 354–385.
- [16] Gantmacher; *Matrix Theory*, Chelsea Publishing Company, N.Y., 1959.

- [17] A. Gray, G. B. Matthews; *A Treatise on Bessel Functions*, first ed., MacMillan and Co., London, 1895.
- [18] A. Gray, T. M. McRobert; *A Treatise on Bessel Functions and their Applications to Physics*, second ed., MacMillan and Co., London, 1922.
- [19] E. Hille; *Lectures on Ordinary Differential Equations*, Addison-Wesley, Reading, Mass, 1969.
- [20] E. L. Ince; *Ordinary Differential Equations*, Longmans, Green, and Co., London, 1926, (Reprinted by Dover 1956).
- [21] W. Kaplan; *Ordinary Differential Equations*, Addison-Wesley Pub. Co., Reading, Mass, 1958.
- [22] J. A. Lappo-Danilevsky; *Theorie des systemes des equations differentielles lineares*, Chelsea Publishing Company, N.Y., 1936, (Reprinted 1953).
- [23] E. C. J. Lommel; *Studien über die Bessel'schen Funktionen*, Druck und Verlag von B.G. Teubner, Leipzig, 1868.
- [24] C. G. Neumann; *Theorie der Besselschen Funktionen: Eine Analogie zur Theorie der Kugelfunktionen*, B. G. Teubner, Leipzig, 1867.
- [25] N. Nielsen; *Handbuch der Theorie der Zylinderfunktionen*, B. G. Teubner, Leipzig, 1904.
- [26] F. W. J. Olver et al.; *NIST Handbook of Mathematical Functions*, National Institute of Standards and Technology (US Department of Commerce), and Cambridge University Press, 2010.
- [27] L. Schlesinger; *Handbuch der linearen Differentialgleichungen*, Leipzig, 1895–1898.
- [28] L. Schlesinger; *Theorie der linearen Differentialgleichungen II*, vol. 1, 1897.
- [29] L. Schlesinger; *Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf Funktionentheoretischer Grundlage*, Dritte Auflage, Berlin and Leipzig, G. J. Göschen'sche Verlagshandlung,, 1922, (first ed. 1900; second ed. 1904).
- [30] G. N. Watson; *A Treatise on the Theory of Bessel Functions*, second ed., Cambridge University Press, 1922, (second ed. 1944).
- [31] E. T. Whittaker, G. N. Watson; *A Course of Modern Analysis*, Cambridge University Press, 1902, (second ed. 1915, third ed. 1920, fourth ed. 1927).

10. ADDENDUM POSTED ON MAY 11, 2017

The author wants to correct the two 4×4 matrices M_1 and M_n appearing in equations (7.7) and (7.10), respectively. They were printed with typos in the 4th row, 4th column. These matrices should read:

$$M_1 = \begin{bmatrix} r_1 + 1 - D_0 & 0 & -E_0 & 0 \\ 0 & r_2 + 1 - D_0 & 0 & -E_0 \\ -F_0 & 0 & r_1 + 1 - G_0 & 0 \\ 0 & -F_0 & 0 & r_2 + 1 - G_0 \end{bmatrix}, \quad (10.1)$$

$$M_n = \begin{bmatrix} r_1 + n - D_0 & 0 & -E_0 & 0 \\ 0 & r_2 + n - D_0 & 0 & -E_0 \\ -F_0 & 0 & r_1 + n - G_0 & 0 \\ 0 & -F_0 & 0 & r_2 + n - G_0 \end{bmatrix}. \quad (10.2)$$

End of addendum

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