Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 219, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NO-FLUX BOUNDARY PROBLEMS INVOLVING $p(x)$-LAPLACIAN-LIKE OPERATORS

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Abstract. In this article we obtain weak solutions for a class nonlinear elliptic problems for the $p(x)$-Laplacian-like operators under no-flux boundary conditions. Our result is obtained using a Fredholm-type result for a couple of nonlinear operators, and the theory of variable exponent Sobolev spaces.

1. Introduction

In this paper we show the existence of weak solutions for the following nonlinear elliptic problem for the $p(x)$-Laplacian-like operators originated from a capillary phenomena,

$$\begin{gathered} \begin{aligned} & -M\big(L(u)\big)\Big[\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}})-|u|^{p(x)-2}u\Big] \\ & = f(x,u)|u|_{s(x)}^{t(x)} \quad \text{in } \Omega, \end{aligned} \\ u=\text{a constant} \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega}\Big(|\nabla u|^{p(x)-2}+\frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\frac{\partial u}{\partial\nu}d\Gamma=0. \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $N \geq 1$, $p,s,t \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$; $M:\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, f is a Caratheodory function and

$$L(u)=\int_\Omega \frac{|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}+|u|^{p(x)}}{p(x)}\,dx$$

is a $p(x)$-Laplacian type operator. The study of differential and partial differential equations with variable exponent has been received considerable attention in recent years. This importance reflects directly into various range of applications. There are applications concerning elastic mechanics [40], thermorheologic and electrorheologic fluids [5, 38], image restoration [14] and mathematical biology [29]. In the context of the study of capillarity phenomena, many results have been obtained, for example [6, 9, 16, 28, 31, 37, 45]. Recently, Avci [6] has considered the existence

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2010 *Mathematics Subject Classification.* 35D05, 35J60, 35J70.
Key words and phrases. $p(x)$-Laplacian; variable exponent Sobolev space; Fredholm alternative.
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Submitted May 18, 2015. Published August 21, 2015.

and multiplicity of solutions for the problem (1.1),without the term $|u|^{p(x)-2}u$ and with boundary condition $u = 0$ on $\partial\Omega$. In this case, we notice that if we choose the functional $L(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$ then we have the problem

$$\begin{gathered}
-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.2}$$

which is called the $p(x)$-Kirchhoff type equation.The problem (1.2) is a generalization of a model introduced by Kirchhoff [13], who studied the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinct feature is that the (1.3) contains a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx$ which depends on the average $\frac{1}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2\,dx$, and hence the equation is no longer a pointwise equation. The parameters in (1.3) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Lions [35] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions [35], various equations of Kirchhoff-type have been studied extensively, see e.g. [3, 13] and [17]-[24]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [17, 18, 22, 24]) and $p(x)$-Laplacian (see [4, 15, 19, 20, 21, 30, 42]).

The nonlocal boundary condition in (1.1) have been studied by Berestycki and Brezis [8], Ortega [36] , Amster et al. [2], Zhao et al. [44], Boureanou et al. [11], Cabanillas et al. [12], Afrouzi et al. [1] and the references therein. They arise from certain models in plasma physics:specifically,a model describing the equilibrium of a plasma confined in a toroidal cavity, called a Tokamak machine. A detailed description of this model can be found in the Appendix of [41].

Motivated by the above papers and the results in Avci [6], we consider (1.1) to study the existence of weak solutions. We note that our problem has no variational structure, so the most usual variational techniques can not used to study it. To attack it we will employ a Fredholm type theorem for a couple of nonlinear operators due to Dinca [23].

This article is organized as follows. In Section 2, we present some preliminaries about variable exponent Sobolev spaces. In Sections 3, we give some existence results of weak solutions of problem (1.1) and their proofs.

2. Preliminaries

To discuss problem (1.1), we need some theory on $W^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space (for details, see [25]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$\begin{gathered}
C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},\\
h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).
\end{gathered}$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| \equiv \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([25]). *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([25]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$. For any $u \in L^{p(x)}(\Omega)$, then*

1. *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
2. *$|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*
3. *if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;*
4. *if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
5. *$\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;*
6. *$\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.*

Proposition 2.3 ([25, 26]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N,\\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.4 ([25, 27]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the Hölder-type inequality*

$$\Big|\int_\Omega uv\,dx\Big| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

Theorem 2.5 ([23]). *Let X and Y be real Banach spaces and two nonlinear operators $T, S : X \to Y$ such that*

1. *T is bijective and T^{-1} is continuous.*
2. *S is compact.*
3. *Let $\lambda \neq 0$ be a real number such that: $\|(\lambda T - S)(x)\| \to +\infty$ as $\|x\| \to +\infty$;*
4. *There is a constant $R > 0$ such that $\|(\lambda T - S)(x)\| > 0$ if $\|x\| \ge R$, $d_{LS}(I - T^{-1}(\frac{S}{\lambda}), B(\theta, R), 0) \neq 0$.*

Then $\lambda I - S$ is surjective from X onto Y.

Here $d_{LS}(G, B, 0)$ denotes the Leray-Schauder degree. Throughout this paper, let

$$V = \{u \in W^{1,p(x)}(\Omega) : u|_{\partial\Omega} = \text{constant}\}.$$

The space V is a closed subspace of the separable and reflexive Banach space $W^{1,p(x)}(\Omega)$ (See [10]), so V is also separable and reflexive Banach space with the usual norm of $W^{1,p(x)}(\Omega)$. The space V is the space where we will try to find weak solutions for problem (1.1).

Definition 2.6. A function $u \in V$ is said to be a weak solution of (1.1) if

$$\begin{aligned}&M\Big(L(u)\Big)\Big[\int_\Omega\Big(|\nabla u|^{p(x)-2}+\frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla u\nabla v\,dx+\int_\Omega |u|^{p(x)-2}uv\,dx\Big]\\&=\int_\Omega f(x,u)|u|_{s(x)}^{t(x)}v\,dx\,,\end{aligned}$$

for all $v \in V$.

We assume that M and f satisfy the following hypotheses:

(H0) $M:[0,+\infty[\to[m_0,+\infty[$ is a continuous and nondecreasing function with $m_0>0$.

(H1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and there exist positive constants c_1 and c_2 such that

$$|f(x,s)|\le c_1+c_2|s|^{\alpha(x)-1}),\quad \forall x\in\Omega, s\in\mathbb{R},$$

for some $\alpha\in C_+(\Omega)$ such that $1<\alpha(x)<p^*(x)$ for $x\in\overline{\Omega}$.

3. Existence of solutions

In this section we discuss the existence of weak solutions of (1.1). Our main result is as follows.

Theorem 3.1. *Assume that* (H0) *and* (H1) *hold. Then* (1.1) *has a weak solution in* V.

Proof. To apply theorem (2.5), we take $Y=V'$ and the operators $T,S:V\to V'$ in as follows:

$$\begin{aligned}&\langle Tu,v\rangle\\&=M\Big(L(u)\Big)\Big[\int_\Omega\Big(|\nabla u|^{p(x)-2}+\frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla u\nabla v\,dx+\int_\Omega |u|^{p(x)-2}uv\,dx\Big]\end{aligned}$$

$$\langle Su,v\rangle=\int_\Omega f(x,u)|u|_{s(x)}^{t(x)}v\,dx$$

for all $u,v\in V$. Then $u\in V$ is a solution of (1.1) if and only if

$$Tu=Su\quad\text{in } V'.$$

Next, we split the proof in several steps.

Step 1. We prove that T is an injection. First we observe that

$$\Phi(u)=\widehat{M}\Big(L(u)\Big),\quad\text{where } \widehat{M}(s)=\int_0^s M(t)\,dt,$$

is a continuously Gâteaux differentiable function whose Gâteaux derivative at the point $u\in V$ is the functional $\Phi'(u)\in V'$ given by

$$\langle\Phi'(u),v\rangle=\langle T(u),v\rangle\quad\text{for all } v\in V.$$

On the other hand, $L\in C^1(V,\mathbb{R})$ and

$$\langle L'(u),v\rangle=\int_\Omega\Big(|\nabla u|^{p(x)-2}+\frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla u\nabla v\,dx+\int_\Omega|u|^{p(x)-2}uv\,dx$$

for all $u, v \in V$. From [37, Prop. 3.1] and taking into account the inequality [39, (2.2)],

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2 \\ C_p\frac{|x-y|^2}{(|x|+|y|)^{p-2}}, \ (x,y) \neq (0,0) & \text{if } 1 < p < 2, \end{cases} \tag{3.1}$$

for all $x, y \in \mathbb{R}^N$. Then we obtain

$$\langle L'(u) - L'(v), u - v\rangle > 0 \quad \text{for all } u, v \in V \text{ with } u \neq v$$

which means that L' is strictly monotone. So, by [43, Prop. 25.10], L is strictly convex. Moreover, since M is nondecreasing, $\widehat{M}$ is convex in $[0, +\infty[$. Thus, for every $u, v \in X$ with $u \neq v$, and every $s, t \in (0,1)$ with $s + t = 1$, one has

$$\widehat{M}(L(su + tv)) < \widehat{M}(sL(u) + tL(v)) \leq s\widehat{M}(L(u)) + t\widehat{M}(L(v)).$$

This shows that Φ is strictly convex, and as $\Phi'(u) = T(u)$ in V' we infer that T is strictly monotone in V, then T is an injection.

Step 2. We prove that the inverse $T^{-1} : V' \to V$ of T is continuous. For any $u \in V$ with $\|u\| > 1$, one has

$$\begin{aligned} \frac{\langle T(u), u\rangle}{\|u\|} &= M\big(L(u)\big[\int_\Omega \Big(|\nabla u|^{p(x)} + \frac{|\nabla u|^{2p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)\,dx + \int_\Omega |u|^{p(x)}\,dx\big]/\|u\| \\ &\geq m_0\Big(c\int_\Omega \sqrt{1 + |\nabla u|^{2p(x)}} + \int_\Omega |u|^{p(x)}\,dx\Big) \geq c_0\|u\|^{p^- - 1}, \end{aligned}$$

from which we have the coercivity of T. Since T is the Fréchet derivative of Φ, T is continuous. Thus in view of the well known Minty-Browder theorem T is a surjection and so $T^{-1} : V' \to V$ and it is bounded.

Now we prove the continuity of T^{-1}. First, we verify that T is of type (S_+).In fact, if $u_\nu \rightharpoonup u$ in V (so there exists $R > 0$ such that $\|u_\nu\| \leq R$) and the strict monotonicity of T we have

$$0 = \limsup_{\nu\to\infty}\langle Tu_\nu - Tu, u_\nu - u\rangle = \lim_{\nu\to\infty}\langle Tu_\nu - Tu, u_\nu - u\rangle$$

Then

$$\lim_{\nu\to\infty}\langle Tu_\nu, u_\nu - u\rangle = 0$$

That is,

$$\begin{aligned} &\lim_{\nu\to\infty} M\big(L(u_\nu)\big)\Big[\int_\Omega \Big(|\nabla u_\nu|^{p(x)-2}\nabla u_\nu \\ &+ \frac{|\nabla u_\nu|^{2p(x)-2}\nabla u_\nu}{\sqrt{1 + |\nabla u_\nu|^{2p(x)}}}\Big)(\nabla u_\nu - \nabla u)\,dx + \int_\Omega |u_\nu|^{p(x)-2}u_\nu(u_\nu - u)\,dx\Big] = 0 \end{aligned} \tag{3.2}$$

Now, we have

$$\begin{aligned} |L(u_\nu)| &\leq \frac{1}{p^-}\int_\Omega (2|\nabla u_\nu|^{p(x)} + 1 + |u_\nu|^{p(x)})\,dx \\ &\leq \frac{1}{p^-}(|\Omega| + C^{p_0} \end{aligned} \tag{3.3}$$

where $p_0 = p^+$ if $\|u_\nu\| \leq 1$, $p_0 = p^-$ if $\|u_\nu\| > 1$. So, $\big(L(u_\nu)\big)_{\nu\geq 1}$ is bounded. Then, since M is continuous, up to a subsequence, there is $t_0 \geq 0$ such that

$$M\big(L(u_\nu)\big) \to M(t_0) \geq m_0 \quad \text{as } \nu \to \infty$$

This and (3.2) imply

$$\begin{aligned}&\lim_{\nu\to\infty}\Big[\int_\Omega\Big(|\nabla u_\nu|^{p(x)-2}\nabla u_\nu+\frac{|\nabla u_\nu|^{2p(x)-2}\nabla u_\nu}{\sqrt{1+|\nabla u_\nu|^{2p(x)}}}\Big)(\nabla u_\nu-\nabla u)\,dx\\&+\int_\Omega|u_\nu|^{p(x)-2}u_\nu(u_\nu-u)\,dx\Big]=0\end{aligned}$$

Using a similar method as in [37], we have

$$\lim_{\nu\to\infty}\int_\Omega\Big(|\nabla u_\nu-\nabla u|^{p(x)}+|u_\nu-u|^{p(x)}\Big)\,dx=0.$$

Therefore,

$$u_\nu\to u\quad\text{strongly in } W^{1,p(x)}(\Omega)\text{as } \nu\to\infty$$

Since $(u_\nu)\subseteq V$ and V is a closed subspace of $W^{1,p(x)}(\Omega)$, we have $u\in V$, so $u_\nu\to u$ in V.

Let $(g_\nu)_{\nu\ge1}$ be a sequence of V' such that $g_\nu\to g$ in V'.Let $u_\nu=T^{-1}g_\nu$, $u=T^{-1}g$, then $Tu_\nu=g_\nu$, $Tu=g$. By the coercivity of T, we deduce that $(u_\nu)_{\nu\ge1}$ is bounded in V ;up to a subsequence , we can assume that $u_\nu\rightharpoonup u$ in V. Since $g_n\to g$,

$$\lim_{n\to+\infty}\langle Tu_n-Tu,u_n-u\rangle=\lim_{n\to+\infty}\langle g_n-g,u_n-u\rangle=0.$$

Since T is of type (S_+), $u_n\to u$, so T^{-1} is continuous.

Step 3. We prove that S is a compact operator.

1. S is well defined. Indeed, using (H1) and $t\in C(\overline{\Omega})$, for all u,v in V we have

$$\begin{aligned}|\langle Su,v\rangle|&\le\int_\Omega|f(x,u)||u|_{s(x)}^{t(x)}|v|\,dx\\&\le C|f(x,u)|_{\frac{\alpha(x)}{\alpha(x)-1}}|v|_{\alpha(x)}\le C|f(x,u)|_{\frac{\alpha(x)}{\alpha(x)-1}}\|v\|<\infty\end{aligned}\tag{3.4}$$

2. S is continuous on V. Let $u_\nu\to u$ in V. Then proposition (2.3) implies that $u_\nu\to u$ in $L^{s(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$ So, up to a subsequence we deduce

$$u_\nu\to u\quad\text{a.e. in }\Omega\tag{3.5}$$

$$|u_\nu(x)|^{\alpha(x)}\le k(x)\quad\text{a.e. } x\in\Omega\text{ for some } k\in L^1(\Omega)\tag{3.6}$$

Since $t\in C(\overline{\Omega})$,

$$|u_\nu|_{s(x)}^{t(x)}\to|u|_{s(x)}^{t(x)}\quad\text{a.e. } x\in\Omega.$$

Furthermore,

$$f(x,u_\nu)\to f(x,u)\quad\text{a.e. } x\in\Omega,$$

Thus, we have

$$f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}\to f(x,u)|u|_{s(x)}^{t(x)}\quad\text{a.e. } x\in\Omega.$$

But, it follows from (F1) and (3.6) that

$$\begin{aligned}\Big|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\Big|^{\alpha'(x)}&\le C2^{(\alpha')^+}\big[|f(x,u_\nu)|^{(\alpha')^+}+|f(x,u)|^{(\alpha')^+}\big]\\&\le C(1+k(x)+|u|^{\alpha(x)})\end{aligned}$$

Note that $C(1+k(x)+|u|^{\alpha(\cdot)})\in L^1(\Omega)$. Applying the Dominated Convergence Theorem with (3.5), we obtain

$$\lim_{\nu\to\infty}\int_\Omega\Big|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\Big|^{\alpha'(x)}\,dx=0$$

This implies that

$$\lim_{\nu\to\infty}\Big|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\Big|_{\alpha'(x)}=0 \tag{3.7}$$

By direct computations we obtain

$$\begin{aligned}|\langle Su_\nu,v\rangle-\langle Su,v\rangle| &\le \int_\Omega \Big|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\Big||v|\,dx\\ &\le C\big|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\big|_{\alpha'(x)}\|v\|;\end{aligned}$$

therefore, from (3.7)

$$|Su_\nu-Su|\le C\big|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\big|_{\alpha'(x)}\to 0 \tag{3.8}$$

So, $Su_\nu\to Su$ in V'.

3. Every bounded sequence $(u_\nu)_\nu$ has a subsequence (still denoted by $(u_\nu)_\nu$) for which $(Su_\nu)_\nu$ converges. Let $(u_\nu)_\nu$ be a bounded sequence of V, there exists a subsequence again denoted by $(u_\nu)_\nu$ and u in V , such that

$$u_\nu \rightharpoonup u \quad \text{weakly in } W^{1,p(x)}(\Omega)$$

and by the compact embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$, we have

$$u_\nu\to u \quad \text{in } L^{\alpha(x)}(\Omega).$$

Hence, similarly to the proof of (3.8) we obtain

$$|Su_\nu-Su|\le C\big|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\big|_{\alpha'(x)}\to 0$$

So $Su_\nu\to Su$.

Step 4.

$$\|(T-S)(u)\|\to\infty \quad \text{as } \|u\|\to\infty \quad \text{for } u\in V.$$

In fact, after some computations we obtain

$$\|Tu\|\ge C_0\|u\|^{p^--1} \quad \text{for all } u\in V \text{ with } \|u\|>1$$

and

$$\|Su\|\le C_1\|u\|^\theta+C_2 \quad \text{for all } u\in V, \text{ and some } \theta\in[\alpha^--1,\alpha^+-1]$$

Combining the above inequalities, we obtain

$$\|(T-S)(u)\|\ge\|Tu\|-\|Su\|\ge C_0\|u\|^{p^--1}-C_1'\|u\|^{\alpha^+-1}-C_2 \tag{3.9}$$

Since

$$\lim_{t\to\infty}(C_0t^{p^--1}-C_1't^{\alpha^+-1}-C_2)=\infty,$$

from (3.9) we conclude that $\|(T-S)(u)\|\to\infty$ as $\|u\|\to\infty$. Moreover, there exists $r_0>1$ such that $\|(T-S)(u)\|>1$ for all $u\in V$, with $\|u\|>r_0$.

Step 5. Set

$$W=\{u\in V:\exists t\in[0,1] \text{ such that } u=tT^{-1}(Su)\}$$

Next,we prove that W is bounded in V. For $u\in W\setminus 0$, i.e. $u=tT^{-1}(Su)$ for some $t\in[0,1]$ we have

$$\|T(\frac{u}{t})\|=\|Su\|\le C_1\|u\|^\theta+C_2\text{with } \ t>0 \tag{3.10}$$

Then there exist two constants $a,b>0$ such that

$$m_0\|u\|^{p^+-1}\le a\|u\|^{\alpha^--1}+b \quad \text{if } \ 0<\|u\|<t,$$

$$m_0\|u\|^{p^- -1} \le a\|u\|^{\alpha^- -1} + b \quad \text{if} \;\; t \le \|u\| \le 1,$$
$$m_0\|u\|^{p^- -1} \le a\|u\|^{\alpha^+ -1} + b \quad \text{if} \;\; 1 < \|u\|$$

Let $g_1, g_2 : [0,1] \to \mathbb{R}$ and $g_3 :]1,\infty[\to \mathbb{R}$ be defined by

$$g_1(t) = m_0 t^{p^+ -1} - a t^{\alpha^- -1} - b, \quad g_2(t) = m_0 t^{p^- -1} - a t^{\alpha^- -1} - b,$$
$$g_3(t) = m_0 t^{p^- -1} - a t^{\alpha^+ -1} - b.$$

The sets $\{t \in [0,1] : g_1(t) \le 0\}$, $\{t \in [0,1] : g_2(t) \le 0\}$ and $\{t \in]1,\infty[: g_3(t) \le 0\}$ are bounded in $\mathbb{R}$.

From the above inequalities and (3.10) we infer that W is bounded in V, so

$$W \subseteq B(0, r_1) \quad \text{for some} \;\; r_1 > 0$$

Now, taking $r = \max\{r_0, r_1\}$, it follows from [32, theorem 1.8] that

$$d_{LS}(I - tT^{-1}(S), B(0,r), 0) = 1 \quad \text{for all } t \in [0,1].$$

In particular

$$d_{LS}(I - T^{-1}(S), B(0,r), 0) = 1, .$$

Thus, the couple of nonlinear operators (T, S) satisfies the hypotheses of theorem (2.5) for $\lambda = 1$. Then $T - S : V \to V'$ is surjective. Therefore, there exists $u \in V$ such that

$$(T - S)u = 0 \quad \text{in } V'$$

This completes the proof. □

Acknowledgments. This research was supported by grants 11061030 from the NSFC, and from NWNU-LKQN-10-21. The authors want to thank the anonymous referees for their suggestions.

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