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BOUNDARY-VALUE PROBLEMS FOR A THIRD-ORDER LOADED PARABOLIC-HYPERBOLIC EQUATION WITH VARIABLE COEFFICIENTS

BOZOR ISLOMOV, UMIDA I. BALTAEVA

Abstract. We prove the unique solvability of a boundary-value problems for a third-order loaded integro-differential equation with variable coefficients, by reducing the equation to a Volterra integral equation.

1. Introduction

The theory of mixed type equations is one of the principal parts of the general theory of partial differential equations. The interest for these kinds of equations arises intensively due to both theoretical and practical uses of their applications. Many mathematical models of applied problems require investigations of this type of equations. The first fundamental results in this direction were obtained in 1923 by Tricomi. The works of Gellerstedt, Lavrent'ev, Bitsadze, Frankl, Protter and Morawetz, Salakhitdinov, Djuraev, Rassias have had a great impact in this theory, where outstanding theoretical results were obtained and pointed out important practical values.

Currently, the concept of mixed-type equations has expanded to include all possible combinations of two or three classic types of equations.

The necessity of the consideration of the parabolic-hyperbolic type equation was specified for the first time in 1956 by Gel'fand [8]. He gave an example connected to the movement of the gas in a channel surrounded by a porous environment. The movement of the gas inside the channel was described by the equation, outside by the diffusion equation [2, 3, 15, 18].

A systematic study of the third and higher order mixed and mixed-composite type PDEs, containing in the main part parabolic-hyperbolic, hyperbolic-elliptic and elliptic-parabolic operators began in the early seventies and intensively developed by many mathematicians [4, 5, 16, 17].

In the recent years, in connection with intensive research on problems of optimal control of the agro economical system, long-term forecasting and regulating the level of ground waters and soil moisture, it has become necessary to investigate a new class of equations called as "loaded equations". Such equations were investigated

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for the first time by Knezer [10], Lichtenstein [11]. However, they did not use the term "loaded equation". This terminology has been introduced by Nakhushev [12], where the most general definition of a loaded equation is given and various loaded equations are classified in detail, e.g., loaded differential, integral, integro-differential, functional equations etc., and numerous applications are described [20, 13].

Definition 1.1. An equation

$$Au(x) = f(x) \tag{1.1}$$

is called loaded in an n-dimensional Euclidean domain Ω if (part of) the operator A depends on the restriction of the unknown function $u(x)$ to a closed subset of $\overline{\Omega}$, of measure strictly less than n.

Definition 1.2. A loaded equation is called a loaded differential equation in the domain $\Omega \subseteq \mathbb{R}^n$ if it contains at least one derivative of the unknown solution in a subset of $\overline{\Omega}$ of nonzero measure.

Basic questions of the theory of boundary value problems for PDEs are the same for the boundary value problems for the loaded differential equations. However, the existence of the loaded part operator A does not always make it possible to apply directly the known theory of boundary value problems for equations

$$L(x) = f(x).$$

On the other hand, searching for solutions of loaded differential equation pre-assigned classes it might reduce to new problems for non-loaded equations.

Works of Nakhushev, Shkhankov, Borodin, Borok, Kaziev, Pomraning, Larsen, Pul'kina, Eleev, Dzhenaliev, Attaev, Wiener, Islomov, Khubiev et al. are devoted to loaded second-order partial differential equations. However, we would like to note that boundary-value problems for third-order loaded equations of a hyperbolic, parabolic-hyperbolic, elliptic-hyperbolic types are not well studied. We indicate only the works [6, 7, 19] in which study-case, when loaded part contain only track or derivative track from unknown solutions. It can be explained with the absence of the representation of the general solution for such equations; on the other hand, these problems will be reduced to integral equations with stir [1], which are not investigated in detail.

2. Formulating of the problem

Let Ω be a simple connected domain located in the plane of independent variables x and y, in the case $y > 0$, is bounded by the segments AA_0, BB_0, and $A_0B_0(A(0,0),\ B(1,0),\ A_0(0,h),\ B_0(1,h))$, of the straight lines $x = 0$, $x = 1$, and $y = h$, respectively, and in the case $y < 0$, with segments $AC : x + y = 0$, $BC : \eta = x - y = 1$ originating at the point $C(1/2, -1/2)$.

We use the following designation:

$$I = \big\{(x, y) : 0 < x < 1,\ y = 0\big\}, \quad \Omega_1 = \Omega \cap \{y > 0\}, \Omega_2 = \Omega \cup \{y < 0\}.$$

We consider a linear loaded integro-differential equation

$$\big(a\frac{\partial}{\partial x} + c\big)Lu = 0, \tag{2.1}$$

where

$$Lu \equiv \begin{cases} L_1 u \equiv u_{xx} + a_1(x,y)u_x + b_1(x,y)u_y + c_1(x,y)u \\ -\sum_{i=1}^{n} d_i D_{0x}^{\alpha_i} u(x,0), & \text{if } y \geqslant 0, \\ L_2 u \equiv u_{xx} - u_{yy} + a_2(x,y)u_x + b_2(x,y)u_y + c_2(x,y)u \\ -\sum_{i=1}^{n} e_i D_{0\xi}^{\beta_i} u(\xi,0), & \text{if } y \leqslant 0, \end{cases}$$

where a, c are given real parameters, a_i, b_i, c_i, d_i, e_i are given functions on Ω_i $(i = 1, 2)$, and $b_1(x,y) < 0$, $c_1(x,y) \leqslant 0$ on $\bar{\Omega}_1$; moreover the functions a_1, b_1, c_1, d_i, a_{1x}, a_{1y}, b_{1x}, b_{1y}, d_{ix}, d_{iy} on Ω_1 satisfy a Hölder condition, and $a_2, b_2 \in C^2(\bar{\Omega}_2)$, $c_2 \in C^1(\Omega_2)$, $e_i \in C^1(\bar{\Omega}_i)$. $D_{0x}^{\alpha_i}$ is integro-differential operator (in the sense of Riemann-Liouville), $\alpha_i, \beta_i < 1$, $i = 1, \dots, n$.

For equation (2.1) we investigate the following problems ($a \neq 0$).

Problem 2.1. Find a function $u(x,y)$ possessing the following properties:

(1) $u(x,y) \in C(\bar{\Omega}) \cap C^1(\Omega)$;
(2) $u_x(u_y)$ is continuous up to $AA_0 \cup AC$, (AC);
(3) $u(x,y)$ is a regular solution of equation (2.1) in the domains Ω_1 and Ω_2;
(4) $u(x,y)$ satisfies the boundary value conditions

$$\begin{gathered} u(x,y)\big|_{AA_0} = \varphi_1(y), \quad u(x,y)\big|_{BB_0} = \varphi_2(y), \\ u_x(x,y)\big|_{AA_0} = \varphi_3(y), \quad 0 \leqslant y \leqslant h, \end{gathered} \tag{2.2}$$

$$u(x,y)\big|_{AC} = \psi_1(x), \quad 0 \leqslant x \leqslant \frac{1}{2}, \tag{2.3}$$

$$\frac{\partial u(x,y)}{\partial n}\big|_{AC} = \psi_2(x), \quad 0 \leqslant x \leqslant \frac{1}{2}, \tag{2.4}$$

where n is an inner normal, $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\psi_1(x)$ and $\psi_2(x)$ are given real-valued functions, moreover $\varphi_1(0) = \psi_1(0)$, $\psi_1'(0) = \sqrt{2}\psi_2(0) - 2\varphi_1'(0)$.

Problem 2.2. Find a function $u(x,y)$, satisfying the following conditions:

(1) $u(x,y) \in C(\bar{\Omega}) \cap C^1(\Omega)$;
(2) $u_x(u_y)$ is continuous up to $AA_0 \cup BC$, (BC);
(3) $u(x,y)$ is a regular solution of equation (2.1) in the domains Ω_1 and Ω_2;
(4) $u(x,y)$ satisfies the boundary value conditions (2.2) and

$$u(x,y)\big|_{BC} = \psi_3(x), \quad \frac{1}{2} \leqslant x \leqslant 1, \tag{2.5}$$

$$\frac{\partial u(x,y)}{\partial n}\big|_{BC} = \psi_4(x), \quad \frac{1}{2} \leqslant x \leqslant 1, \tag{2.6}$$

where n is an inner normal, $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\psi_3(x)$ and $\psi_4(x)$ are given real-valued functions, moreover $\varphi_2(0) = \psi_3(0)$.

3. Main results

From condition (1) problems 2.1 and 2.2 it follows that

$$u(x,+0) = u(x,-0) = \tau(x), \tag{3.1}$$

$$u_y(x,+0) = u_y(x,-0) = \nu(x), \tag{3.2}$$

$$u_x(x,+0) = u_x(x,-0) = \tau'(x), \tag{3.3}$$

where $\tau(x)$ and $\nu(x)$, are still unknown functions. Assuming that

$$u(x,y)=\begin{cases}u_1(x,y), & (x,y)\in\bar{\Omega}_1,\\ u_2(x,y), & (x,y)\in\bar{\Omega}_2,\end{cases}$$

equation (2.1) can be represented by two systems:

$$\begin{gathered}L_1u_1+\sum_{i=1}^{n}d_iD_{0x}^{\alpha_i}u_1(x,0)=\upsilon_1(x,y),\quad (x,y)\in\bar{\Omega}_1,\\ a\upsilon_{1x}+c\upsilon_1=0,\end{gathered}\tag{3.4}$$

$$\begin{gathered}L_2u_2+\sum_{i=1}^{n}e_iD_{0\xi}^{\beta_i}u_2(\xi,0)=\upsilon_2(x,y),\quad (x,y)\in\bar{\Omega}_2,\\ a\upsilon_{2x}+c\upsilon_2=0,\end{gathered}\tag{3.5}$$

where $\upsilon_1(x,y)$, $\upsilon_2(x,y)$ are continuous differentiable functions.

Theorem 3.1. *If $b_1(x,y)<0$, $c_1(x,y)\le 0$ and $a_i(x,y)\ge 0$ for all $(x,y)\in\Omega_i$,*

$$\varphi_i(y)\in C^1[0,h],\ (i=1,2),\quad \varphi_3(y)\in C[0,h]\cap C^1(0,h),\tag{3.6}$$

$$\psi_1(x)\in C^1[0,1/2]\cap C^3(0,1/2),\quad \psi_2(x)\in C[0,1/2]\cap C^2(0,1/2),\tag{3.7}$$

then there exists a unique solution to the problem 2.1 in the domain Ω.

Theorem 3.2. *If $b_1(x,y)<0$, $c_1(x,y)\le 0$ and $a_i(x,y)\ge 0$ for all $(x,y)\in\Omega_i$, condition* (3.6) *is satisfied and*

$$\psi_3(x)\in C^1[1/2,1]\cap C^3(1/2,1),\quad \psi_4(x)\in C[1/2,1]\cap C^2(1/2,1),\tag{3.8}$$

then there exists a unique solution to the problem 2.2 in the domain Ω.

Proof of Theorem 3.1. Bearing in mind [5] that system (3.5) is reduced to the form

$$L_2u_2+\sum_{i=1}^{n}e_iD_{0\xi}^{\beta_i}u_2(\xi,0)=w_2(y)\exp\big(-\frac{c}{a}x\big).\tag{3.9}$$

Hence going over to the characteristic coordinates $\xi=x+y$, $\eta=x-y$, we obtain

$$\begin{aligned}&u_{2\xi\eta}+a_3(\xi,\eta)u_{2\xi}+b_3(\xi,\eta)u_{2\eta}+c_3(\xi,\eta)u_2\\ &=E_i(\xi,\eta)D_{0\xi}^{\beta_i}\tau+\frac{1}{4}\omega_2\big(\frac{\xi-\eta}{2}\big)\exp\big(-\frac{c}{2a}(\xi+\eta)\big),\end{aligned}\tag{3.10}$$

where $a_3(\xi,\eta)$, $b_3(\xi,\eta)$, $c_3(\xi,\eta)$ depend on the coefficients of equation (3.9),

$$E_i(\xi,\eta)=\frac{1}{4}e_i\big(\frac{\xi+\eta}{2},\frac{\eta-\xi}{2}\big),$$

with recurring index $i=1,2,\dots,n$ implied summation. The boundary value conditions (2.3) and (2.4) is reduced to the form

$$u_2(\xi,\eta)\big|_{\xi=0}=\psi_1\big(\frac{\eta}{2}\big),\quad 0\leqslant\eta\leqslant 1,\tag{3.11}$$

and

$$\frac{\partial u_2(\xi,\eta)}{\partial\xi}\big|_{\xi=0}=\frac{1}{\sqrt{2}}\psi_2\big(\frac{\eta}{2}\big),\quad 0<\eta<1.\tag{3.12}$$

The solution of the equation (3.10), with boundary conditions (3.11) and

$$(u_{2\xi}-u_{2\eta})\big|_{\eta=\xi}=\nu(\xi),\quad 0<\xi<1,\tag{3.13}$$

(problem Cauchy-Goursat), is represented analogously as [14]

$$
\begin{aligned}
u_2(\xi,\eta) = F(\xi,\eta) &+ \frac{1}{4}\int_0^{\xi} dt \int_t^{\eta} T(t,\tau;\xi,\eta)\exp\big(-\frac{c}{2a}(t+\tau)\big)\omega_2\big(\frac{t-\tau}{2}\big)d\tau \\
&+ \int_0^{\xi} T_0(\xi,\eta;t)\nu(t)dt \\
&+ \frac{1}{4}\int_0^{\xi} dt \int_t^{\xi} S(t,\tau;\xi,\eta)\exp\big(-\frac{c}{2a}(t+\tau)\big)\omega_2\big(\frac{t-\tau}{2}\big)d\tau \\
&+ \int_0^{\xi} dt \int_t^{\xi} E_i(t,\tau)D_{0t}^{\beta_i}\tau(t)S(t,\tau;\xi,\eta)d\tau \\
&+ \int_0^{\xi} dt \int_t^{\eta} E_i(t,\tau)D_{0t}^{\beta_i}\tau(t)T(t,\tau;\xi,\eta)d\tau,
\end{aligned} \tag{3.14}
$$

where

$$
\begin{aligned}
F(\xi,\eta) &= \psi_1(\frac{\eta}{2}) + \psi_1(\frac{\xi}{2}) - \psi_1(0) \\
&\quad + \int_0^{\xi} dt \int_t^{\xi} K(t,\tau)S(t,\tau;\xi,\eta)d\tau + \int_0^{\xi} dt \int_t^{\eta} K(t,\tau)T(t,\tau;\xi,\eta)d\tau,
\end{aligned}
$$

$$
K(\xi,\eta) = -\frac{1}{2}a_3(\xi,\eta)\psi_1'(\frac{\xi}{2}) - \frac{1}{2}b_3(\xi,\eta)\psi_1'(\frac{\eta}{2}) - c_3(\xi,\eta)\Big(\psi_1(\frac{\xi}{2}) + \psi_1(\frac{\eta}{2}) - \psi_1(0)\Big),
$$

$$
\begin{aligned}
T_0(\xi,\eta;t) = 1 &- \int_t^{\xi} a_3\,(t,\tau)\,S(t,\tau;\xi,\eta)d\tau - \int_t^{\xi} a_3\,(t,\tau)\,T(t,\tau;\xi,\eta)d\tau \\
&- \int_t^{\xi} ds \int_s^{\xi} c_3(s,\tau)S(s,\tau;\xi,\eta)d\tau \\
&- \int_0^{\xi} ds \int_t^{\eta} c_3(s,\tau)T(s,\tau;\xi,\eta)d\tau,
\end{aligned}
$$

where $S(t,\tau;\xi,\eta)$ and $T(t,\tau;\xi,\eta)$ are expressed via coefficients a_3, b_3, c_3 and continuous in $\bar{\Omega}_2\times\bar{\Omega}_2$ functions S_ξ, S_η, T_η are continuous in $\bar{\Omega}_2\times\bar{\Omega}_2$, and function T_ξ it can have discontinuities of the first kind on compact subsets of this domains. More properties of these functions are established in [14].

Substituting (3.14) in (3.12), taking into account that $\nu(0) = u_{2\eta}(0,0) = u_{1\eta}(0,0) = \varphi_1'(0)$ and $\varphi_1'(0) = \frac{1}{2}\big(\sqrt{2}\psi_2(0) - \psi_1'(0)\big)$, we obtain

$$
\begin{aligned}
&\int_0^{\eta} T(0,\tau;0,\eta)\exp\Big(-\frac{c}{2a}\tau\Big)\,\omega_2\Big(-\frac{\tau}{2}\Big)\,d\tau \\
&= 2\sqrt{2}\psi_2(\frac{\eta}{2}) - 2\psi_1'(0) - 4\int_0^{\eta} K(0,\tau)T(0,\tau;0,\eta)d\tau \\
&\quad - 4\varphi_1'(0)\Big(1 - \int_0^{\eta} a_3(0,\tau)T(0,\tau;0,\eta)d\tau\Big).
\end{aligned} \tag{3.15}
$$

From here with regard (3.7), differentiating (3.15) with respect to η, reduction in this integral equation of the second kind

$$
\omega_2\big(-\frac{\eta}{2}\big) - \int_0^{\eta} T_\eta(0,\tau;0,\eta)\exp\Big(\frac{c}{2a}(\eta-\tau)\Big)\,\omega_2\Big(-\frac{\tau}{2}\Big)\,d\tau = g\,(\eta), \tag{3.16}
$$

$$
\begin{aligned}
g(\eta) = \Big(\sqrt{2}\,\psi_2'(\frac{\eta}{2}) - 4K(0,\eta) - 4\int_0^\eta K(0,\tau)T_\eta(0,\tau;0,\eta)d\tau \\
+ 4\,\varphi_1'(0)\Big(a_3(0,\eta) + \int_0^\eta a_3(0,\tau)T_\eta(0,\tau;0,\eta)\,d\tau\Big)\Big)\exp\Big(\frac{c}{2a}\eta\Big).
\end{aligned}
$$

From (3.7), we conclude that the kernel and $g(\eta)$ are continuous. Then it leads to a unique solutions in the class of continuous functions. Solving this, we obtain $\omega_2\big(-\frac{\eta}{2}\big)$ in $-\frac{1}{2}\leqslant -\frac{\eta}{2}\leqslant 0$. Therefore in instead of $\omega_2\big(-\frac{\eta}{2}\big)$ we can take $\omega_2\left(\frac{\xi-\eta}{2}\right)$. Substituting in (3.14) the expression $\omega_2\left(\frac{\xi-\eta}{2}\right)$ we find the solution $u_2(\xi,\eta)$ in the form

$$
\begin{aligned}
u_2(\xi,\eta) = M(\xi,\eta) + \int_0^\xi T_0(\xi,\eta;t)\nu(t)dt \\
+ \int_0^\xi dt\int_t^\xi E_i(t,\tau)D_{0t}^{\beta_i}\tau(t)S(t,\tau;\xi,\eta)d\tau \\
+ \int_0^\xi dt\int_t^\eta E_i(t,\tau)D_{0t}^{\beta_i}\tau(t)T(t,\tau;\xi,\eta)d\tau,
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
M(\xi,\eta) = F(\xi,\eta) + \frac{1}{4}\int_0^\xi dt\int_t^\xi S(t,\tau;\xi,\eta)\exp\Big(-\frac{c}{2a}(t+\tau)\Big)\omega_2\big(\frac{t-\tau}{2}\big)d\tau \\
+ \frac{1}{4}\int_0^\xi dt\int_t^\eta T(t,\tau;\xi,\eta)\exp\Big(-\frac{c}{2a}(t+\tau)\Big)\omega_2\big(\frac{t-\tau}{2}\big)d\tau,
\end{aligned}
$$

depend on a given function.

In $\eta=\xi=x$, setting $M(x)=M(x,x)$, $T_0(x,t)=T_0(x,x;t), \tau(x)=u_2(x,x)$, from (3.14) we obtain

$$
\begin{aligned}
\tau(x) = M(x) + \int_0^x T_0(x,t)\nu(t)dt + \int_0^x D_{0t}^{\beta_i}\tau(t)dt\int_t^x E_i(t,\tau)S(t,\tau;x,x)d\tau \\
+ \int_0^x D_{0t}^{\beta_i}\tau(t)dt\int_t^x E_i(t,\tau)T(t,\tau;x,x)d\tau.
\end{aligned}
$$

Differentiating the last relation, obtain integral equation second kind relative to $\nu(x)$:

$$
\nu(x) + \int_0^x T_{ox}'(x,t)\nu(t)ds = \tau'(x) - \int_0^x L(x,t)D_{0t}^{\beta_i}\tau(t)dt - M'(x), \tag{3.18}
$$

where

$$
\begin{aligned}
L(x,t) = E_i(t,x)\left(S(t,x;x,x) - T(t,x;x,x)\right) \\
+ \int_t^x E_i(t,\tau)\left(S'(t,\tau;x,x) + T'(t,\tau;x,x)\right)d\tau.
\end{aligned}
$$

The right-hand side equation (3.18) is continuous and kernel can be discontinuous of the first kind. Therefore $\nu(x)$:

$$
\begin{aligned}
\nu(x) = \tau'(x) - \int_0^x L(x,t)D_{0t}^{\beta_i}\tau(t)dt - M'(x) \\
+ \int_0^x \Gamma_0(x,t)\Big(M'(t) - \tau'(t) + \int_0^t L(t,s)D_{0s}^{\beta_i}\tau(s)ds\Big)dt,
\end{aligned}
\tag{3.19}
$$

where $\Gamma_0(x,t)$ is the resolvent of the kernel $T'_{0x}(x,t)$. This is the first functional relation between the function $\tau(x)$ and $\nu(x)$ transferred from the Ω_2. Present we need obtain second functional relation between this functions. To this end equation (2.1) at $y>0$ rewrite in the form

$$L_1u_1 \equiv u_{1xx} + a_1u_{1x} + b_1u_{1y} + c_1u_1 + \sum_{i=1}^{n} d_i D_{0x}^{\alpha_i} u_1(x,0) = w_1(y)\exp\left(-\frac{c}{a}x\right),$$

where $w_1(y)$ is arbitrary continuous functions. Hence, considering property of the problem 2.1, in $b_1=-1$, passage to the limit, we obtain second functional relation between the function $\tau(x)$ and $\nu(x)$ transferred from the Ω_1:

$$\tau''(x) + a_1(x,0)\tau'(x) + c_1(x,0)\tau(x) - \sum_{j=1}^{n} d_j D_{0x}^{\alpha_j}\tau(x) - \nu(x) = \omega_1(0)\exp\left(-\frac{c}{a}x\right). \tag{3.20}$$

Substituting (3.19) in (3.20), results

$$\begin{aligned}
&\tau''(x) + p(x)\tau'(x) + q(x)\tau(x) - \sum_{j=1}^{n} d_j D_{0x}^{\alpha_j}\tau(x) \\
&+ \int_0^x \Gamma_0(x,t)\tau'(t)dt + \int_0^x \Gamma_1(x,t) D_{0t}^{\beta_i}\tau(t)dt \\
&= \omega_1(0)\exp\left(-\frac{c}{a}x\right) + m(x),
\end{aligned} \tag{3.21}$$

where

$$\begin{gathered}
p(x) = a_1(x,0) - 1, \quad q(x) = c_1(x,0), \\
\Gamma_1(x,t) = L(x,t) - \int_t^x \Gamma_0(x,s)\, L(s,t)\, ds, \\
m(x) = \int_0^x \Gamma_0(x,t) M'(t)dt - M'(x).
\end{gathered}$$

Solve (3.21) under the initial condition

$$\tau(0) = \varphi_1(0) = \psi_1(0), \quad \tau'(0) = \sqrt{2}\psi_2(0) - \varphi_1'(0).$$

Introduce new unknown function $\tau''(x) = z(x)$. Then with regards the next conditions we have

$$\begin{gathered}
\tau'(x) = \int_0^x z(t)dt + \sqrt{2}\psi_2(0) - \varphi_1'(0), \\
\tau(x) = \int_0^x (x-t)\, z(t)dt + \left(\sqrt{2}\psi_2(0) - \varphi_1'(0)\right)x + \psi_1(0).
\end{gathered}$$

Bearing mind this, we rewrite equation (3.21) in form

$$z(x) + \int_0^x Q(x,t;\alpha_j,\beta_i) z(t)dt = \omega_1(0)\exp\left(-\frac{c}{a}x\right) + M(x), \tag{3.22}$$

where

$$Q(x,t;\alpha_j,\beta_i) = p(x) + q(x)\,(x-t) - Q_1(x,t;\alpha_j) + Q_2(x,t;\beta_i) + \int_t^x \Gamma_0(x,s)ds,$$

$$Q_1(x,t;\alpha_j)=\begin{cases}\sum_{j=1}^n \frac{d_jB(2;-\alpha_j)}{\Gamma(-\alpha_j)}\left(x-t\right)^{1-\alpha_j}, & \alpha_j<0,\\ \sum_{j=1}^n \frac{d_j(2-\alpha_j)B(2;1-\alpha_j)}{\Gamma(1-\alpha_j)}\left(x-t\right)^{1-\alpha_j}, & 0<\alpha_j<1,\end{cases}$$

$$Q_2(x,t;\beta_i)=\begin{cases}\frac{B(2;-\beta_i)}{\Gamma(-\beta_i)}\int_t^x \Gamma_1\left(x,s\right)\left(s-t\right)^{1-\beta_i}ds, & \beta_i<0,\\ \frac{B(2;1-\beta_i)}{\Gamma(1-\beta_i)}\int_t^x \Gamma_1\left(x,s\right)\left(s-t\right)^{2-\beta_i}ds, & 0<\beta_i<1.\end{cases}$$

where B is the Beta function and $\Gamma(z)$ is the Gamma function.

The Kernel and the right-hand side of (3.22) are continuous. Therefore, $z(x)\in C[0,1]$. Solving we find $z(x)$:

$$\begin{aligned}z(x)&=M(x)+\int_0^x R(x,t;\alpha_j,\beta_i)M(t)dt\\&\quad+\omega_1(0)\Big(\exp\Big(-\frac{c}{a}x\Big)+\int_0^x R(x,t;\alpha_j,\beta_i)\exp\Big(-\frac{c}{a}t\Big)\,dt\Big),\end{aligned}$$

where $R(x,t;\alpha_j,\beta_i)$ is a resolvent of the kernel $Q(x,t;\alpha_j,\beta_i)$. Taking into account the last equality, we obtain

$$\begin{aligned}\tau(x)&=\omega_1(0)\int_0^x (x-t)\Big(\exp\Big(-\frac{c}{a}t\Big)\\&\quad+\int_0^t R\left(t,s;\alpha_j,\beta_i\right)\exp\Big(-\frac{c}{a}s\Big)\,ds\Big)dt+M_1(x),\end{aligned}\tag{3.23}$$

where

$$\begin{aligned}M_1(x)&=\int_0^x (x-t)\Big(M(t)+\int_0^t R\left(t,s;\alpha_j,\beta_i\right)M\left(s\right)ds\Big)dt\\&\quad+\Big(\sqrt{2}\psi_2(0)-\varphi_1'(0)\Big)\,x+\psi_1(x).\end{aligned}$$

Hence, by the condition $\tau(1)=\varphi_2(0)$, $w_1(0)$ are determined uniquely. Thus, from function $\tau(x)$ using relation (3.20) we uniquely define function $\nu(x)$. Set value function $\tau(x)$ and $\nu(x)$ in (3.14), we obtain function $u_2(\xi,\eta)$ in domain Ω_2. For determination function $u_1(x,y)$ in domain Ω_1 reduce to problem 2.2 and

$$u_1(x,0)=\tau(x),$$

for the equation

$$\big(a\frac{\partial}{\partial x}+c\big)\left(u_{1xx}+a_1(x,y)u_{1x}+b_1(x,y)u_{1y}+c_1(x,y)u_1\right)=F(x,y),\tag{3.24}$$

where $F(x,y)=\big(a\frac{\partial}{\partial x}+c\big)\sum_{i=1}^n d_iD_{0x}^{\alpha_i}\tau(x)$ is a well-known function. Unique solvability this problem was proved in [5, §2, chapter 4]. We can conclude from these that, there exists a regular solution of problem in Ω_1. Therefore, we can conclude from these that, there exists a regular solution of problem 2.1. □

Proof of Theorem 3.2. The proof for Problem 2.2 is analogous to the proof for Problem 2.1. We omit it. □

Remark 3.3. For problems 2.1 and 2.2 it is possible examine with general discontinuous gluing conditions. In this case 1, problems 2.1 and 2.2, change in the following way: function $u(x,y)$ is continuous in each closed domains $\bar{\Omega}_1$ and $\bar{\Omega}_2$, conditions (2), (3) and (4) it remains invariant. Indeed, the following conditions are fulfilled:

$$u(x,+0)=\alpha_1(x)\,u(x,-0)+\gamma_1(x),\quad 0<x<1,$$
$$u_y(x,+0)=\beta_1(x)\,u_y(x,-0)+\alpha_2(x)\,u_y(x,-0)+\gamma_2(x),\quad 0<x<1,$$

where $\alpha_1,\gamma_1\in C^3$, $\alpha_2,\beta_1,\gamma_2\in C^2$ are given functions, and $\alpha_1\beta_1\neq 0$ for $0<x<1$. For problems 2.1 and 2.2, in this case there exist also unique solutions.

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Bozor Islomov
National University of Uzbekistan, 100174, Tashkent, Uzbekistan
E-mail address: `islomovbozor@yandex.ru`
`http://math.nuu.uz/en/islomov-bozor-islomovich`

Umida Ismoilovna Baltaeva
Mathematical Institute University of Göttingen, 37073 Göttingen, Germany
E-mail address: `umida_baltayeva@mail.ru`