Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 223, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GROUND STATE SOLUTIONS FOR NON-LOCAL FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. In this article, we study a time-independent fractional Schrödinger equation with non-local (regional) diffusion

$$(-\Delta)^\alpha_\rho u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $\alpha \in (0,1)$, $N > 2\alpha$. We establish the existence of a non-negative ground state solution by variational methods.

1. Introduction and statement of main results

Consider the fractional Schrödinger equation

$$\begin{gathered} (-\Delta)^\alpha_\rho u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in H^\alpha(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $\alpha \in (0,1)$, $N > 2\alpha$, $(-\Delta)^\alpha_\rho$ denotes a non-local (regional) fractional Laplacian operator with a range of scope determined by the positive function $\rho \in C(\mathbb{R}^N, \mathbb{R}^+)$.

The fractional Schrödinger equation was firstly introduced by Laskin [1]. This equation was of particular interest in fractional quantum mechanics in the study of particles on stochastic fields modelled by Lévy processes that give rise to equations with the fractional Laplacian operator. Recently, the study on problems of fractional Schrödinger equations has attracted much attention. Some existence and nonexistence of Dirichlet problem involving the fractional Laplacian on bounded domains have been established, see [2, 3] and their references. Using the equivalent definition of the fractional operator, some authors introduced a variational principle and studied the existence and multiplicity of solutions in $\mathbb{R}^N$. Cheng [4] considered the equation

$$(-\Delta)^\alpha u + V(x)u = |u|^{p-1}u, \quad u \in H^\alpha(\mathbb{R}^N)$$

with unbounded potential V, he obtained the existence of ground state solution by a Lagrange multiplier method and the Nehari manifold method. Dipierro, Palatucci and Valdinoci in [5] proved existence and symmetry results for the solutions. In [6],

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2010 *Mathematics Subject Classification.* 35J60, 35J20.
Key words and phrases. Nonlinear Schrödinger equation; fractional Laplacian; non-local operator; ground state solution.

Submitted June 15, 2015. Published August 28, 2015.

the authors studied the same equation with a more general right-hand side $f(x,u)$. Secchi [7] provided a generalization of the main result to equations of the form

$$(-\Delta)^\alpha u + V(x)u = f(x,u), \quad u \in H^\alpha(\mathbb{R}^N). \tag{1.2}$$

The existence of positive solutions was obtained using the Nehari manifold method. Chang [8] proved the existence of a positive ground state solution of (1.2) when $f(x,t)$ is asymptotically linear with respect to t at infinity.

There is another definition of regional fractional Laplacian in [10] and [11], the authors introduced the regional fractional Laplacian operator $\Delta_G^{\alpha/2}$ in an arbitrary open set G of $\mathbb{R}^N$, which is not the one used here. More recently, Felmer and Torres [12] studied the regional fractional Laplacian equation

$$\varepsilon^{2\alpha}(-\Delta)^\alpha_\rho u + u = f(u), \quad u \in H^\alpha(\mathbb{R}^N),$$

where $\alpha \in (0,1)$, $N > 2\alpha$. The operator $(-\Delta)^\alpha_\rho$ was defined by

$$\int_{\mathbb{R}^N} (-\Delta)^\alpha_\rho u(x)v(x)dx = \int_{\mathbb{R}^N}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{N+2\alpha}}\,dz\,dx$$

for all $u, v \in H^\alpha(\mathbb{R}^N)$. They showed the existence of a ground state and analyzed the behavior of the semi-classical solutions as $\varepsilon \to 0$. Following some ideas in [12], Felmer and Torres [13] proved that the ground state level is achieved by a radially symmetry solution.

Up to now, no results for non-autonomous regional fractional Laplacian equations with potential have appeared in the literature. In this note, we investigate the existence of a non-negative ground state solution for such equations.

Throughout this article, we assume the following conditions:

- (A1) $\rho \in C(\mathbb{R}^N, \mathbb{R}^+)$, there exists a constant $\rho_0 > 0$ such that $\rho(x) \geq \rho_0$.
- (A2) $V \in C(\mathbb{R}^N, \mathbb{R}^+)$, $\inf_{x\in\mathbb{R}^N} V(x) \geq c > 0$, there exists $r_0 > 0$ such that, for any $M > 0$,
$$\lim_{|y|\to\infty} \operatorname{meas}\big(\{x \in \mathbb{R}^N : |x-y| \leq r_0, V(x) \leq M\}\big) = 0.$$
- (A3) $f \in C(\mathbb{R}^N \times \mathbb{R}^+, R)$ and $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$.
- (A4) $\lim_{t\to+\infty} \frac{f(x,t)}{t^{2^*_\alpha - 1}} = 0$ uniformly in $x \in \mathbb{R}^N$, where $2^*_\alpha = \frac{2N}{N-2\alpha}$ is the fractional critical exponent.
- (A5) $\lim_{t\to+\infty} \frac{F(x,t)}{t^2} = +\infty$ uniformly in $x \in \mathbb{R}^N$.

The main results of this article are as follows:

Theorem 1.1. *Assume* (A1)–(A5) *and*

- (A6) *there exists $T_1 > 0$ such that $\mu F(x,t) \leq f(x,t)t$ for $t > T_1$, where $\mu > 2$ is a constant.*

Then (1.1) *has a non-negative ground state solution.*

Theorem 1.2. *Assume* (A1)–(A5) *and*

- (A7) $\frac{f(x,t)}{t}$ *is increasing on* $(0,\infty)$.

Then (1.1) *has a non-negative ground state solution.*

Theorem 1.3. *Assume that* (A1)–(A5) *and*

- (A8) *There exist $b > 0$ and $\nu > 2$ such that* $\limsup_{t\to+\infty} \frac{F(x,t)}{t^\nu} \leq b$;
- (A9) $\liminf_{t\to+\infty} \frac{f(x,t)t - 2F(x,t)}{t^\sigma} \geq \eta > 0$, *where* $\sigma > \frac{N}{2\alpha}(\nu - 2)$.

Then (1.1) *has a non-negative ground state solution.*

Remark 1.4. Note that we impose the subcritical growth condition by a general condition (A4). Also note that (A6), (A7) and (A9) almost cover all types of superquadratic conditions.

When $\rho \equiv +\infty$, the regional fractional operator becomes a common fractional operator, our results are also new under the assumptions on f. Moreover, we can find a positive ground state solution by the strong maximum principle [14].

2. Preliminaries

Firstly we give some basic notation. The fractional Sobolev space of order α on $\mathbb{R}^N$ is defined by

$$H^\alpha(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy < \infty\Big\},$$

endowed with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N}|u|^2dx\Big)^{1/2}.$$

In this article, we consider the space

$$\begin{aligned} E := \Big\{u \in L^2(\mathbb{R}^N) : &\int_{\mathbb{R}^N}\int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{N+2\alpha}}\,dz\,dx \\ &+ \int_{\mathbb{R}^N} V(x)|u(x)|^2dx < \infty\Big\}, \end{aligned}$$

equipped with the inner product

$$\begin{aligned} \langle u, v\rangle = &\int_{\mathbb{R}^N}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{N+2\alpha}}\,dz\,dx \\ &+ \int_{\mathbb{R}^N} V(x)u(x)v(x)dx \end{aligned}$$

and the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N}\int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{N+2\alpha}}\,dz\,dx + \int_{\mathbb{R}^N} V(x)|u(x)|^2dx\Big)^{1/2}.$$

We denote by $\|\cdot\|_p$ the usual L^p-norm. We define

$$u^\pm(x) = \max\{\pm u(x), 0\}.$$

Obviously, $u \in E$ implies that u^+, $u^- \in E$.

Let $I : E \to R$ be the functional defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u^+)dx \tag{2.1}$$

with $F(x,t)$ being the primitive of $f(x,t)$.

Next, we give some lemmas that play important roles in proving our main results.

Lemma 2.1. *Suppose* (A1) *and* (A2) *hold. Then there exists a constant* $C_0 > 0$ *such that*

$$\|u\|_{2^*_\alpha}^2 \le C_0\|u\|^2.$$

Proof. From [9, Theorem 6.5], there exists $C > 0$ such that

$$\|u\|_{2_\alpha^*}^2 \le C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x+z) - u(x)|^2}{|z|^{N+2\alpha}} \, dz \, dx.$$

As in the proof of [12, Proposition 2.1], we have

$$\begin{aligned}
&C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x+z) - u(x)|^2}{|z|^{N+2\alpha}} \, dz \, dx \\
&= C\Big(\int_{\mathbb{R}^N} \int_{B(0,\rho_0)} \frac{|u(x+z) - u(x)|^2}{|z|^{N+2\alpha}} \, dz \, dx \\
&\quad + \int_{\mathbb{R}^N} \int_{B^c(0,\rho_0)} \frac{|u(x+z) - u(x)|^2}{|z|^{N+2\alpha}} \, dz \, dx\Big) \\
&\le C \int_{\mathbb{R}^N} \int_{B(0,\rho_0)} \frac{|u(x+z) - u(x)|^2}{|z|^{N+2\alpha}} \, dz \, dx + \frac{2C|S^{n-1}|}{\alpha\rho_0^{2\alpha}} \|u\|_2^2 \\
&\le C \int_{\mathbb{R}^N} \int_{B(0,\rho_0)} \frac{|u(x+z) - u(x)|^2}{|z|^{N+2\alpha}} \, dz \, dx + \frac{2C|S^{n-1}|}{c\alpha\rho_0^{2\alpha}} \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx \\
&\le C_0 \|u\|^2,
\end{aligned}$$

where $C_0 = \max\{C, \frac{2C|S^{n-1}|}{c\alpha\rho_0^{2\alpha}}\}$. This completes the proof. □

Lemma 2.2. *Suppose* (A1)–(A2) *hold, and that there exists a constant* $K > 0$ *such that*

$$\|u\|_q \le K\|u\|,$$

where $2 \le q \le 2_\alpha^*$. *Then the embedding* $E \hookrightarrow L^q(\mathbb{R}^N)$ *is continuous for* $2 \le q \le 2_\alpha^*$ *and* $E \hookrightarrow L^s_{\rm loc}(\mathbb{R}^N)$ *is compact for* $2 \le s < 2_\alpha^*$.

Proof. When $q = 2$, by the definition of $\|\cdot\|$, there exists $C_1 > 0$ such that

$$\|u\|_2 \le C_1\|u\|.$$

When $q = 2_\alpha^*$, using Lemma 2.1, we obtain

$$\|u\|_{2_\alpha^*} \le C_0^{1/2}\|u\|.$$

When $2 < q < 2_\alpha^*$, according to the Hölder inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\|u\|_q^q &\le \Big(\int_{\mathbb{R}^N} |u|^{2_\alpha^*} dx \Big)^{\frac{q-2}{2_\alpha^*-2}} \Big(\int_{\mathbb{R}^N} |u|^2 dx \Big)^{\frac{2_\alpha^*-q}{2_\alpha^*-2}} \\
&\le C_0^{\frac{(q-2)2_\alpha^*}{2(2_\alpha^*-2)}} \|u\|^{\frac{(q-2)2_\alpha^*}{2_\alpha^*-2}} C_1^{\frac{2(2_\alpha^*-q)}{2_\alpha^*-2}} \|u\|^{\frac{2(2_\alpha^*-q)}{2_\alpha^*-2}} \\
&\le C_2^q \|u\|^q,
\end{aligned}$$

where

$$C_2 = \Big(C_0^{\frac{(q-2)2_\alpha^*}{2(2_\alpha^*-2)}} C_1^{\frac{2(2_\alpha^*-q)}{2_\alpha^*-2}} \Big)^{1/q}.$$

Let $K = \max\{C_0^{1/2}, C_1, C_2\}$, we have

$$\|u\|_q \le K\|u\|$$

for $2 \le q \le 2_\alpha^*$. Therefore, the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $2 \le q \le 2_\alpha^*$ and $E \hookrightarrow L^s_{\rm loc}(\mathbb{R}^N)$ is compact for $2 \le s < 2_\alpha^*$. □

Lemma 2.3. *Suppose* (A1)–(A2) *hold. Then* $E \hookrightarrow L^q(\mathbb{R}^N)$ *is compact for any* $2 \le q < 2^*_\alpha$.

Proof. First, we show that $E \hookrightarrow L^2(\mathbb{R}^N)$ is compact. Let $\{u_n\} \subset E$ with $\{u_n\}$ bounded in E. Then $u_n \rightharpoonup u$ weakly in E, by Lemma 2.2, which implies that $u_n \to u$ strongly in $L^2_{\rm loc}(\mathbb{R}^N)$.

Set $\delta_n = \|u_n\|_2$, we may suppose that there is a subsequence of $\{u_n\}$ and $\delta \in R$ such that $\delta_n \to \delta$. For any bounded domain Ω in $\mathbb{R}^N$, we have

$$\int_\Omega |u|^2 dx = \lim_{n\to\infty} \int_\Omega |u_n|^2 dx \le \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx = \delta^2,$$

so $\|u\|_2 \le \delta$.

Let $\Omega_1 := \{x \in B^c_R | V(x) \ge M\}$, $\Omega_2 := \{x \in B^c_R | V(x) < M\}$. On the basis of Ω_1, we obtain

$$\int_{\Omega_1} |u_n|^2 dx \le \int_{\mathbb{R}^N} \frac{V(x)}{M} |u_n|^2 dx \le \frac{\|u_n\|^2}{M}.$$

Let $\{y_i\}$ be a sequence satisfying $\mathbb{R}^N \subset \bigcup_{i=1}^\infty B(y_i, r_0)$ and each point x is contained in at most 2^N such balls $B(y_i, r_0)$. By the Hölder inequality, we choose a positive constant $s \in (1, \frac{2^*_\alpha}{2})$ such that

$$\begin{aligned}\int_{\Omega_2} |u_n|^2 dx &\le \sum_{i=1}^\infty \int_{\Omega_2 \cap B(y_i,r_0)} |u_n|^2 dx \\ &\le \sum_{i=1}^\infty \Big(\int_{\Omega_2 \cap B(y_i,r_0)} |u_n|^{2s} dx \Big)^{1/s} \Big(\int_{\Omega_2 \cap B(y_i,r_0)} 1 dx \Big)^{1-\frac{1}{s}}.\end{aligned}$$

Define

$$\begin{aligned}\|u_n\|^2_{B(y_i,r_0)} = &\int_{B(y_i,r_0)} \int_{B(0,\rho(x)) \cap B(y_i,r_0)} \frac{|u(x+z) - u(x)|^2}{|z|^{N+2\alpha}} \, dz \, dx \\ &+ \int_{B(y_i,r_0)} V(x) |u(x)|^2 dx.\end{aligned}$$

Using proof similar to the one of Lemmas 2.1 and 2.2, there exists $C > 0$ such that

$$\Big(\int_{\Omega_2 \cap B(y_i,r_0)} |u_n|^{2s} dx \Big)^{\frac{1}{s}} \le \Big(\int_{B(y_i,r_0)} |u_n|^{2s} dx \Big)^{\frac{1}{s}} \le C \|u_n\|^2_{B(y_i,r_0)}.$$

Now, we can estimate

$$\int_{\Omega_2} |u_n|^2 dx \le \sum_{i=1}^\infty C \|u_n\|^2_{B(y_i,r_0)} \operatorname{meas}(\Omega_2 \cap B(y_i, r_0))^{1-\frac{1}{s}} \le 2^N C \varepsilon_R \|u_n\|^2,$$

where $\varepsilon_R = \sup_{y_i} \operatorname{meas}(\Omega_2 \cap B(y_i, r_0))^{1-\frac{1}{s}}$. Note that $\{u_n\}$ is bounded in E, for M large enough, we have $\frac{\|u_n\|^2}{M} \to 0$. Since (A2) holds, for R large enough, we obtain $\varepsilon_R \to 0$. It is easy to check that, given any $\varepsilon > 0$, for sufficiently large R and M,

$$\int_{B^c_R} |u_n|^2 dx = \int_{\Omega_1} |u_n|^2 dx + \int_{\Omega_2} |u_n|^2 dx \le \varepsilon$$

and

$$\|u\|^2_2 = \int_{B_R} |u|^2 dx + \int_{B^c_R} |u|^2 dx$$

$$
\begin{aligned}
&\geq \lim_{n\to\infty} \int_{B_R} |u_n|^2 dx \\
&= \lim_{n\to\infty} \Big(\int_{R_N} |u_n|^2 dx - \int_{B_R^c} |u_n|^2 dx \Big) \\
&\geq \delta^2 - \varepsilon.
\end{aligned}
$$

It means that $\delta \leq \|u\|_2$. Therefore, $\delta = \|u\|_2$ and it implies that $E \hookrightarrow L^2(\mathbb{R}^N)$ is compact.

Next, we prove that $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $2 < q < 2_\alpha^*$. Let $r \in (0,1)$ be such that $q = 2r + 2_\alpha^*(1-r)$. Using the Hölder inequality again, we have

$$
\|u_n - u\|_q^q \leq \|u_n - u\|_2^{2r} \|u_n - u\|_{2_\alpha^*}^{2_\alpha^*(1-r)}.
$$

Since $\{u_n - u\}$ is bounded in E, then $u_n - u \rightharpoonup 0$, by Lemma 2.2, there exists $C > 0$ such that $\|u_n - u\|_{2_\alpha^*}^{2_\alpha^*(1-r)} \leq C$. Thus,

$$
\|u_n - u\|_q^q \to 0.
$$

Here, we can make a result of the proof of Lemma 2.3. □

Lemma 2.4. *Under the assumptions of Theorem 1.1 hold, the functional I satisfies the $(Ce)_c$ condition for $c > 0$.*

Proof. Assume that $\{u_n\} \subset E$ is a $(Ce)_c$ sequence for $c > 0$,

$$
I(u_n) \to c, \quad (1 + \|u_n\|) I'(u_n) \to 0 \quad \text{as} n \to \infty.
$$

For any $\varphi \in E$, we obtain that

$$
\begin{aligned}
\langle I'(u_n), \varphi \rangle = &\int_{\mathbb{R}^N} \int_{B(0,\rho(x))} \frac{|u_n(x+z) - u_n(x)||\varphi(x+z) - \varphi(x)|}{|z|^{N+2\alpha}} \, dz \, dx \\
&+ \int_{\mathbb{R}^N} V(x) |u_n(x)||\varphi(x)| dx - \int_{\mathbb{R}^N} f(x, u_n^+) \varphi(x) dx \to 0
\end{aligned}
\tag{2.2}
$$

and

$$
\langle I'(u_n), u_n \rangle = \frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} f(x, u_n^+) u_n^+ dx \to 0. \tag{2.3}
$$

First we claim that $\{u_n\}$ is bounded in E. In fact, if not, we may assume by the contradiction that there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) with $\|u_n\| \to +\infty$, and we set

$$
w_n = \frac{u_n}{\|u_n\|}.
$$

Clearly, $\{w_n\}$ is bounded in E. Going if necessary to a subsequence of $\{w_n\}$, we can assume that

$$
\begin{gathered}
w_n \rightharpoonup w \quad \text{weakly in } E, \\
w_n \to w \quad \text{strongly in } L^q(\mathbb{R}^N) \quad (2 \leq q < 2_\alpha^*), \\
w_n \to w \quad \text{a.e. } x \in \mathbb{R}^N.
\end{gathered}
$$

Similarly, we denote

$$
w_n^+ = \frac{u_n^+}{\|u_n\|},
$$

then

$$
w_n^+ \rightharpoonup w^+ \quad \text{weakly in } E,
$$

$$w_n^+ \to w^+ \quad \text{strongly in } L^q(\mathbb{R}^N) \quad (2 \le q < 2_\alpha^*),$$
$$w_n^+ \to w^+ \quad \text{a.e. } x \in \mathbb{R}^N.$$

Next we claim that $w \neq 0$. Otherwise, if $w \equiv 0$, we know that $w_n^+ \to 0$ strongly in $L^2(\mathbb{R}^N)$. By (f_1), for each $\varepsilon > 0$, there exists $T_0 > 0$ such that

$$|f(x,t)t| < \varepsilon t^2,$$

for $0 < t < T_0$. Through the continuity of f, there exists $C > 0$ such that

$$|f(x,t)t| \le C \le \frac{C}{T_0^2} t^2$$

for $T_0 \le t \le T_1$. Hence, we have

$$|f(x,t)t| \le \frac{C}{T_0^2} t^2 \tag{2.4}$$

for $0 < t \le T_1$ and

$$|F(x,t)| \le \frac{C}{2T_0^2} t^2 \tag{2.5}$$

for $0 < t \le T_1$. By (A6), it is easy to see that

$$\frac{1}{\mu} f(x,t)t - F(x,t) \ge 0$$

for $t > T_1$. Combining (2.4) and (2.5), we have

$$\frac{1}{\mu} f(x,t)t - F(x,t) \ge -\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\frac{C}{T_0^2} t^2 \tag{2.6}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. Under the definition of $\{u_n\}$, we see that

$$\begin{aligned}
&I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 - \int_{\mathbb{R}^N} \Big(F(x,u_n^+) - \frac{1}{\mu} f(x,u_n^+)u_n^+\Big)dx \\
&= c - o_n(1).
\end{aligned}$$

It follows from the preceding step and (2.6) that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle}{\|u_n\|^2} \\
&= \frac{1}{2} - \frac{1}{\mu} - \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n^+) - \frac{1}{\mu} f(x,u_n^+)u_n^+}{\|u_n\|^2} dx \\
&\ge \frac{1}{2} - \frac{1}{\mu} - \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\frac{C}{T_0^2} \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{(u_n^+)^2}{\|u_n\|^2} dx \\
&\ge \frac{1}{2} - \frac{1}{\mu} - \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\frac{C}{T_0^2} \lim_{n\to\infty} \|w_n^+\|_2^2 \\
&\ge \frac{1}{2} - \frac{1}{\mu}.
\end{aligned}$$

Note that $\mu > 2$, this is a contradiction, so $w \neq 0$.

Since $\{u_n\}$ is $(Ce)_c$ sequence, by (2.2), we know that

$$\langle I'(u_n), u_n^-\rangle \le \|I'(u_n)\|\|u_n^-\| \le \|I'(u_n)\|\|u_n\| \to 0.$$

Then $w_n^- = \frac{u_n^-}{\|u_n\|} \to 0$, which implies that $w^- = 0$ for a.e. $x \in \mathbb{R}^N$. We set

$$h(s) = s^\mu F(x, \frac{t}{s}),$$

then

$$h'(s) = \mu s^{\mu-1} F\big(x, \frac{t}{s}\big) - s^\mu f\big(x, \frac{t}{s}\big)\frac{t}{s^2} = s^{\mu-1}\Big(\mu F\big(x, \frac{t}{s}\big) - f\big(x, \frac{t}{s}\big)\frac{t}{s}\Big).$$

By (A5), we can find a $T_2 > 0$ such that

$$\inf_{t \geq T_2,\ x \in \mathbb{R}^N} F(x,t) > 0.$$

Denote $T = \max\{T_1, T_2\}$. For $s \in [1, \frac{t}{T}]$, by (f_4), we have

$$\mu F(x, \frac{t}{s}) - f(x, \frac{t}{s})\frac{t}{s} \leq 0.$$

Hence, $h'(s) \leq 0$ and $h(1) \geq h(\frac{t}{T})$. This implies

$$F(x,t) \geq \frac{1}{T_1^\mu} F(x,T) t^\mu \tag{2.7}$$

for all $(x,t) \in (\mathbb{R}^N, \mathbb{R}^+)$, where

$$c_0 = \frac{1}{T^\mu} \inf_{t=T, x \in \mathbb{R}^N} F(x,t). \tag{2.8}$$

In accordance with (2.7) that

$$F(x,t) \geq c_0 t^\mu$$

for all $(x,t) \in (\mathbb{R}^N, \mathbb{R}^+)$. Then, one has

$$f(x,t)t \geq \mu c_0 t^\mu$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. Related to (2.3), we obtain

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^\mu} = \frac{1}{\|u_n\|^{\mu-2}} - \int_{\mathbb{R}^N} \frac{f(x, u_n^+)u_n^+}{\|u_n\|^\mu} dx = o_n(1).$$

Therefore, we know

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{f(x,u_n^+)u_n^+}{\|u_n\|^\mu} dx \\ &\geq \lim_{n\to\infty} \mu c_0 \|w_n^+\|_\mu^\mu \\ &\geq \mu c_0 \|w^+\|_\mu^\mu > 0. \end{aligned}$$

However, this is a contradiction. So our assumption $\|u_n\| \to +\infty$ is false, and that is to say, $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ is bounded in E, there exists $C_1 > 0$ such that

$$\|u_n\| \leq C_1.$$

Moreover, there exists $C_2 > 0$ such that

$$\|u_n - u\|_{2_\alpha^*} \leq C_2.$$

By the reflexivity of E, there exists a subsequence of $\{u_n\}$ (which we also denote by $\{u_n\}$) and $u \in E$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } E, \\ u_n \to u \quad \text{strongly in } L^q(\mathbb{R}^N) \quad (2 \leq q < 2_\alpha^*). \end{gathered} \tag{2.9}$$

At the same time,
$$\langle I'(u_n), u_n - u\rangle \to 0, \quad \langle I'(u), u_n - u\rangle \to 0.$$
Hence, we have
$$\begin{aligned} &\langle I'(u_n) - I'(u), u_n - u\rangle \\ &= \|u_n - u\|^2 - \int_\Omega \big(f(x,u_n^+) - f(x,u^+)\big)(u_n^+ - u^+)dx \to 0. \end{aligned} \tag{2.10}$$
By (A3), (A4) and $f \in C(\mathbb{R}^N \times \mathbb{R}^+, R)$, for $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
$$f(x,t) \le C_\varepsilon t + \varepsilon t^{2_\alpha^*-1} \tag{2.11}$$
for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. By (2.11) and the Hölder inequality, we have
$$\begin{aligned} \Big|\int_{\mathbb{R}^N} f(x,u_n^+)(u_n^+ - u^+)dx\Big| &\le \int_{\mathbb{R}^N} |f(x,u_n^+)||u_n - u|dx \\ &\le \int_{\mathbb{R}^N} \Big(C_\varepsilon |u_n^+||u_n - u| + \varepsilon |u_n^+|^{2_\alpha^*-1}|u_n - u|\Big)dx \\ &\le C_\varepsilon \|u_n\|_2 \|u_n - u\|_2 + \varepsilon \|u_n\|_{2_\alpha^*}^{2_\alpha^*-1}\|u_n - u\|_{2_\alpha^*} \\ &\le C_\varepsilon K C_1 \|u_n - u\|_2 + \varepsilon K^{2_\alpha^*-1} C_1^{2_\alpha^*-1} C_2. \end{aligned}$$
Related to (2.9), we obtain
$$\int_{\mathbb{R}^N} f(x,u_n^+)(u_n^+ - u^+)dx \to 0.$$
Consequently,
$$\int_{\mathbb{R}^N} \big(f(x,u_n^+) - f(x,u^+)\big)(u_n^+ - u^+)dx \to 0.$$
By (2.10), we have $\|u_n - u\|^2 \to 0$. So we derive that $u_n \to u$ strongly in E, we conclude that the $(Ce)_c$ condition is satisfied. □

Lemma 2.5. *Under the assumptions of Theorem 1.2, the functional I satisfies the $(Ce)_c$ condition for $c > 0$.*

Proof. Assume that $\{u_n\} \subset E$ is a $(Ce)_c$ sequence for $c > 0$,
$$I(u_n) \to c, \quad (1 + \|u_n\|)I'(u_n) \to 0 \quad (n \to \infty).$$
We first claim that the sequence $\{u_n\}$ is bounded in E. Otherwise, there is a subsequence, again denoted by $\{u_n\}$, such that $\|u_n\| \to +\infty$ as $n \to \infty$. Set
$$w_n = \frac{u_n}{\|u_n\|}, \quad w_n^+ = \frac{u_n^+}{\|u_n\|}.$$
Clearly, $\{w_n\}$ is bounded in E. Going if necessary to a subsequence of $\{w_n\}$, we can assume that
$$\begin{gathered} w_n \rightharpoonup w \quad \text{weakly in } E, \\ w_n \to w \quad \text{strongly in } L^q(\mathbb{R}^N) \quad (2 \le q < 2_\alpha^*), \\ w_n \to w \quad \text{a.e. } x \in \mathbb{R}^N. \end{gathered}$$
Then we claim that $w \neq 0$. Otherwise, if $w \equiv 0$, we obtain $w_n \to 0$ strongly in $L^q(\mathbb{R}^N)$. Since $I(tu_n)(t \in [0,1])$ is continuous, there exist $t_n \in [0,1]$ such that
$$I(t_n u_n) = \max_{t \in [0,1]} I(tu_n).$$

For $t_n \in (0,1)$, we have

$$\langle I'(t_n u_n), t_n u_n\rangle = \|t_n u_n\|^2 - \int_{\mathbb{R}^N} f(x, t_n u_n^+) t_n u_n^+ dx = t_n \frac{d}{dt}|_{t=t_n} I(tu_n) = 0.$$

It is obvious for $t_n = 0$, we obtain

$$\langle I'(t_n u_n), t_n u_n\rangle = 0.$$

When $t_n = 1$, we obtain

$$\langle I'(t_n u_n), t_n u_n\rangle = \langle I'(u_n), u_n\rangle \to 0.$$

Hence, we have

$$\langle I'(t_n u_n), t_n u_n\rangle = \|t_n u_n\|^2 - \int_{\mathbb{R}^N} f(x, t_n u_n^+) t_n u_n^+ dx \to 0 \tag{2.12}$$

for $t_n \in [0,1]$. Set

$$v_n = 2m^{1/2} w_n$$

for each $m > 0$. By (2.11), we have

$$F(x,t) \le \frac{C_\varepsilon}{2} t^2 + \frac{\varepsilon}{2^*_\alpha} t^{2^*_\alpha} \tag{2.13}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. Since $v_n \to 0$ strongly in $L^2(\mathbb{R}^N)$ and v_n is bounded in E, we come to a conclusion that

$$\int_{\mathbb{R}^N} F(x, v_n^+) dx \to 0.$$

It is apparently showed that $\frac{2m^{1/2}}{\|u_n\|} \in (0,1)$ for n large enough, so

$$I(t_n u_n) \ge I(v_n) = 2m - \int_{\mathbb{R}^N} F(x, v_n^+) \ge m.$$

Then we have $I(t_n u_n) \to +\infty$. Denote $H(x,t) = f(x,t)t - 2F(x,t)$, it follows

$$\int_{\mathbb{R}^N} H(x, u_n^+) dx = 2I(u_n) - \langle I'(u_n), u_n\rangle \to 2c. \tag{2.14}$$

Assume $0 \le s_1 \le s_2$, by (f_5), we have

$$\begin{aligned}
&H(x,s_2) - H(x,s_1)\\
&= 2\Big[\frac{1}{2}\left(f(x,s_2)s_2 - f(x,s_1)s_1\right) - \left(F(x,s_2) - F(x,s_1)\right)\Big]\\
&= 2\int_{s_1}^{s_2} \Big(\frac{f(x,s_2)}{s_2} - \frac{f(x,v)}{v}\Big) v dv + 2\int_{T_2}^{s_1} \Big(\frac{f(x,s_2)}{s_2} - \frac{f(x,s_1)}{s_1}\Big) v dv\\
&\quad + T_2^2 \Big(\frac{f(x,s_2)}{s_2} - \frac{f(x,s_1)}{s_1}\Big) \ge 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} H(x, u_n^+) dx &\ge \int_{\mathbb{R}^N} H(x, t_n u_n^+) dx\\
&= 2I(t_n u_n) - \langle I'(t_n u_n), t_n u_n\rangle \to +\infty.
\end{aligned}$$

This contradicts (2.14), so $w \ne 0$. Since $\{u_n\}$ is $(Ce)_c$ sequence, we know that

$$\langle I'(u_n), u_n^-\rangle \le \|I'(u_n)\| \|u_n^-\| \le \|I'(u_n)\| \|u_n\| \to 0.$$

Thus, $w_n^- = \frac{u_n^-}{\|u_n\|} \to 0$, which implies that $w^- = 0$ for a.e. $x \in \mathbb{R}^N$. Then we set

$$\Omega_0 = \{x \in \mathbb{R}^N | w(x) = 0\}, \quad \Omega_1 = \{x \in \mathbb{R}^N | w(x) > 0\}.$$

By (A5) and the Fatou Lemma, we have

$$\liminf_{n\to\infty} \int_{\Omega_1} \frac{F(x, u_n^+)}{u_n^2} w_n^2 dx \geq \int_{\Omega_1} \liminf_{n\to\infty} \frac{F(x, u_n^+)}{u_n^2} w_n^2 dx \to +\infty.$$

In the meantime, by (A5), there exists $T_0 > 0$, such that

$$\frac{F(x,t)}{t^2} \geq 1$$

for $t > T_0$. By the continuity of F, there exists $C_1 > 0$ such that

$$|F(x,t)| \leq C_1$$

for $0 < t \leq T_0$. Combining the preceding two inequalities, there exists $C_2 > 0$ such that

$$\begin{aligned}
\int_{\Omega_0} \frac{F(x, u_n^+)}{u_n^2} w_n^2 dx &= \int_{\Omega_0(0<u_n(x)\leq T_0)} \frac{F(x, u_n)}{\|u_n\|^2} dx + \int_{\Omega_0(u_n(x)>T_0)} \frac{F(x, u_n)}{u_n^2} w_n^2 dx \\
&\geq -\frac{C_3}{\|u_n\|^2} \operatorname{meas}\big(\Omega_0(0 < u_n(x) \\
&\leq T_0)\big) + \int_{\Omega_0} w_n^2 dx > -C_2.
\end{aligned}$$

Dividing (2.1) with $\|u_n\|^2$, we have

$$\frac{I(u_n)}{\|u_n\|^2} = \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, u_n^+)}{u_n^2} (w_n^+)^2 dx = o_n(1).$$

Hence, we obtain

$$\begin{aligned}
\frac{1}{2} - o_n(1) &= \int_{\mathbb{R}^N} \frac{F(x, u_n^+)}{u_n^2} w_n^2 dx \\
&= \int_{\Omega_0} \frac{F(x, u_n^+)}{u_n^2} w_n^2 dx + \int_{\Omega_1} \frac{F(x, u_n^+)}{u_n^2} w_n^2 dx \to +\infty.
\end{aligned}$$

It is easy to see that it is a contradiction. So $\{u_n\}$ is bounded in E.

By the standard processes similar to Lemma 2.4, we know that $u_n \to u$ strongly in E. This completes the proof. $\square$

Lemma 2.6. *Under the assumptions of Theorem 1.3, the functional I satisfies the $(Ce)_c$ condition for $c > 0$.*

Proof. Assume that $\{u_n\} \subset E$ is a $(Ce)_c$ sequence for $c > 0$,

$$I(u_n) \to c, \quad (1 + \|u_n\|) I'(u_n) \to 0 \text{ as } n \to \infty.$$

We argue that the sequence $\{u_n\}$ is bounded in E. Otherwise, there is a subsequence, again denoted by $\{u_n\}$, such that $\|u_n\| \to +\infty$ as $n \to \infty$. By (f_6), we see that, there exists $R_1 > 0$ such that

$$f(x,t)t \leq \nu b t^\nu \tag{2.15}$$

for $t \geq R_1$. By (A9), for $0 < \delta < \eta$, there exists $R_2 > 0$ such that

$$f(x,t)t - 2F(x,t) \geq (\eta - \delta) t^\sigma \tag{2.16}$$

for $t \geq R_2$. Set $T = \max\{R_1, R_2\}$, let

$$\Omega_1 = \{x \in \mathbb{R}^N : u_n(x) \geq T\}, \quad \Omega_2 = \{x \in \mathbb{R}^N : 0 < u_n(x) < T\}.$$

Moreover, by the continuity of f and F, there exists $C_1 > 0$ such that

$$\Big| \int_{\Omega_2} (f(x, u_n)u_n - 2F(x, u_n))\, dx \Big| \leq C_1.$$

Since $\{u_n\} \subset E$ is a $(Ce)_c$ sequence, we obtain

$$\begin{aligned} 2c + o_n(1) &= \int_{\mathbb{R}^N} \big(f(x, u_n^+)u_n^+ - 2F(x, u_n^+)\big)\, dx \\ &\geq \int_{\Omega_1} (f(x, u_n)u_n - 2F(x, u_n))dx - C_1. \end{aligned}$$

Thus,

$$\int_{\Omega_1} (f(x, u_n)u_n - 2F(x, u_n))dx \leq 2c + C_1 + o_n(1).$$

It follows from (2.16) that there exists $C_2 > 0$ such that

$$\int_{\Omega_1} u_n^\sigma dx \leq C_2.$$

Using the continuity of f again, there exists $C_3 > 0$ such that

$$\Big| \int_{\Omega_2} f(x, u_n)u_n dx \Big| \leq C_3.$$

Through the definition of $\{u_n\}$, it reaches

$$\begin{aligned} \|u_n\|^2 - o_n(1) &= \int_{\mathbb{R}^N} f(x, u_n^+)u_n^+ dx \\ &= \int_{\Omega_1} f(x, u_n)u_n dx + \int_{\Omega_2} f(x, u_n)u_n dx \\ &\leq \int_{\Omega_1} f(x, u_n)u_n dx + C_3. \end{aligned}$$

From (2.15), we obtain

$$\int_{\Omega_1} f(x, u_n)u_n dx \leq \nu b \int_{\Omega_1} u_n^\nu dx.$$

Then, we will consider two cases.

Case 1. $\sigma \geq \nu$, there exists $t \in (0, 1)$ such that $\frac{1}{\nu} = \frac{1-t}{\sigma} + \frac{t}{2}$. By the Hölder inequality,

$$\begin{aligned} \int_{\Omega_1} u_n^\nu dx &\leq \Big(\int_{\Omega_1} u_n^\sigma dx\Big)^{\frac{(1-t)\nu}{\sigma}} \Big(\int_{\Omega_1} u_n^2 dx\Big)^{\frac{t\nu}{2}} \\ &\leq C_2^{\frac{(1-t)\nu}{\sigma}} K^{t\nu} \|u_n\|^{t\nu}. \end{aligned}$$

Case 2. $\sigma < \nu$, there exists $t \in (0, 1)$ such that $\frac{1}{\nu} = \frac{1-t}{\sigma} + \frac{(N-2\alpha)t}{2N}$. By the Hölder inequality,

$$\int_{\Omega_1} u_n^\nu dx \leq \Big(\int_{\Omega_1} u_n^\sigma dx\Big)^{\frac{(1-t)\nu}{\sigma}} \Big(\int_{\Omega_1} u_n^{\frac{2N}{N-2\alpha}} dx\Big)^{\frac{t\nu(N-2\alpha)}{2N}}$$

$$\leq C_2^{\frac{(1-t)\nu}{\sigma}} K^{t\nu} \|u_n\|^{t\nu}.$$

Hence, one has

$$\|u_n\|^2 - o_n(1) \leq C_3 + \nu b C_2^{\frac{(1-t)\nu}{\sigma}} K^{t\nu} \|u_n\|^{t\nu}.$$

It is easily observed that $\sigma > \frac{N}{2\alpha}(\nu - 2)$ is equivalent to $t\nu < 2$, it is easy to conclude that the assumption is false. So $\{u_n\}$ is bounded in E.

Since (A3) and (A4) hold, we know that $u_n \to u$ strongly in E by the standard processes similar to Lemma 2.4. So I satisfies the $(Ce)_c$ condition. □

Lemma 2.7. *Assume that* (A1)–(A4) *hold. Then I satisfies the the following conditions:*

1. *There exist $\theta > 0$, $\xi > 0$ such that $I(u) \geq \xi > 0$ for all $u \in E$ with $\|u\| = \theta$;*
2. *There exists $e \in E$ with $\|e\| > \theta$ such that $I(e) \leq 0$.*

Proof. (1) By (A3), (A4) and $f \in C(\mathbb{R}^N \times \mathbb{R}^+, R)$, for $0 < \varepsilon < \frac{1}{2K^2}$, there exists $C_\varepsilon > 0$ such that

$$f(x,t)t \leq \varepsilon t^2 + C_\varepsilon t^{2_\alpha^*} \tag{2.17}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. This implies that

$$\begin{aligned} F(x,t) &= \int_0^1 f(x,st)t\,ds \\ &\leq \int_0^1 \left(\varepsilon s t^2 + C_\varepsilon s^{2_\alpha^*-1} t^{2_\alpha^*}\right) ds \\ &\leq \frac{\varepsilon}{2} t^2 + \frac{C_\varepsilon}{2_\alpha^*} t^{2_\alpha^*} \end{aligned} \tag{2.18}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. Pay attention to the definition of I given in (2.1), it follows from (2.18) and Lemma 2.2 that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u^+)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \frac{\varepsilon}{2}(u^+)^2 dx - \int_{\mathbb{R}^N} \frac{C_\varepsilon}{2_\alpha^*}(u^+)^{2_\alpha^*} dx \\ &\geq \frac{1}{2}\left(1 - \varepsilon K^2\right)\|u\|^2 - \frac{C_\varepsilon K^{2_\alpha^*}}{2_\alpha^*}\|u\|^{2_\alpha^*} \\ &\geq \Big(\frac{1}{4} - \frac{C_\varepsilon K^{2_\alpha^*}}{2_\alpha^*}\|u\|^{2_\alpha^*-2}\Big)\|u\|^2. \end{aligned} \tag{2.19}$$

Set

$$\theta = \Big(\frac{2_\alpha^*}{8 C_\varepsilon K^{2_\alpha^*}}\Big)^{\frac{1}{2_\alpha^*-2}}.$$

Taking $\|u\| = \theta$, it follows from (2.19) that

$$I(u) \geq \frac{1}{8}\theta^2 > 0,$$

then (1) is proved.

(2) By (A5), for any $M > 0$, there exists a $T_0 > 0$ such that

$$F(x,t) \geq M t^2 > 0$$

for all $t \geq T_0$. By the continuity of F, there exists $C > 0$ such that

$$F(x,t) \geq Mt^2 - C$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. We choose $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$, $\|\varphi\| = 1$ and $\mathrm{supp}(\varphi) \subset B(0,R)$ for some $R > 0$. we have that for $M \int_{B(0,R)} \varphi^2 dx > 1/2$.

For $t > 0$ large enough, it follows from (A5) that $F(x,t\varphi) \geq 0$. Hence,

$$\begin{aligned} I(t\varphi) &= \frac{t^2}{2}\|\varphi\|^2 - \int_{\mathbb{R}^N} F(x,t\varphi)dx \\ &\leq \frac{t^2}{2} - \int_{B(0,R)} F(x,t\varphi)dx \\ &\leq t^2\Big(\frac{1}{2} - M\int_{B(0,R)} \varphi^2 dx\Big) + C|B(0,R)|. \end{aligned}$$

Choosing $\|e\| = \|t\varphi\| > \rho$, we have $I(e) < 0$, then (2) is proved. $\square$

3. Proof of main results

Proof of Theorem 1.1. By Lemmas 2.4 and 2.7, it is easy to obtain a nontrivial critical point u_0 of I by the mountain pass theorem, which implies that

$$c = \inf_{\gamma\in\Gamma} \max_{t\in(0,1)} I(\gamma(t)) = I(u_0),$$

where $\Gamma = \{\gamma \in C([0,1],E), \gamma(0) = 0, \gamma(1) = e\}$ and $I(u_0) \geq \theta > 0$. Let

$$\mathcal{N} = \{u \in E\backslash\{0\} : I'(u) = 0\},$$

since $u_0 \in \mathcal{N}$, $\mathcal{N} \neq \emptyset$. Then we claim that I is bound from below on $\mathcal{N}$, moreover, there exists $d > 0$ such that $I(u) > d$ for every $u \in \mathcal{N}$. If not, there exists a sequence $\{u_n\} \subset \mathcal{N}$ such that

$$I(u_n) < \frac{1}{n}, \quad \forall n \in \mathbb{N}^+.$$

Using (2.18), we know that

$$\frac{1}{n} > I(u_n) \geq \frac{1}{2}\left(1 - \varepsilon K^2\right)\|u_n\|^2 - \frac{C_\varepsilon K^{2_\alpha^*}}{2_\alpha^*}\|u_n\|^{2_\alpha^*}.$$

Since $u_n \in \mathcal{N}$, $I'(u_n) = 0$, it follows from (2.17) that

$$\|u_n\|^2 = \int_{\mathbb{R}^N} f(x,u_n^+)u_n^+ dx \leq \varepsilon K^2\|u_n\|^2 + C_\varepsilon K^{2_\alpha^*}\|u_n\|^{2_\alpha^*}. \tag{3.1}$$

Thus,

$$\frac{1}{n} > \big(\frac{1}{2} - \frac{1}{2_\alpha^*}\big)(1 - \varepsilon K^2)\|u_n\|^2.$$

By the definition of $\varepsilon < \frac{1}{2K^2}$, we obtain

$$\frac{1}{n} > \big(1 - \frac{2}{2_\alpha^*}\big)\|u_n\|^2.$$

Since $2 < 2_\alpha^*$, it is easy to see that $\|u_n\| \to 0$. From (3.1) and $\varepsilon < \frac{1}{2K^2}$, we know

$$\|u_n\|^2 < \frac{1}{2}\|u_n\|^2 + C_\varepsilon K^{2_\alpha^*}\|u_n\|^{2_\alpha^*},$$

so there exists C such that $\|u_n\| > C$. It is a contradiction. Hence, there exists $c_1 > 0$ such that

$$c_1 = \inf_{u\in\mathcal{N}} I(u).$$

Clearly, $c_1 \leq c$. Let $\{v_n\} \subset \mathcal{N}$ be a minimizing sequence for c_1, so $\{v_n\}$ is a $(Ce)_{c_1}$ sequence. By Lemma 2.4, $\{v_n\}$ is bounded in E and it has a convergence subsequence $\{v_n\}$ such that $v_n \to u_2$ in E. Since $u_2 \in \mathcal{N}$, we know that

$$\langle I(u_2), u_2^-\rangle = -\|u_2^-\|^2 = 0,$$

therefore, u_2 is a non-negative ground state solution. □

Proof of Theorem 1.2. Since the $(Ce)_c$ condition is satisfied by Lemma 2.5 and the mountain geometrical structure is proved by Lemma 2.7. So we can get a ground state solution by using the same method as that of Theorem 1.1. □

Proof of Theorem 1.3. Since the $(Ce)_c$ condition is satisfied by Lemma 2.6 and the mountain geometrical structure is proved by Lemma 2.7. So we can get a ground state solution by using the same method as that of Theorem 1.1. □

Acknowledgments. This research was supported by National Natural Science Foundation of China (No. 11471267). The authors want to thank the anonymous referees for their valuable comments.

References

[1] N. Laskin; *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. 268 (2000), 298–305.
[2] J. Tan; *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. 42 (2011), 21–41.
[3] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez; *On some critical problems for the fractional Laplacian operator*, J. Differential Equations 252 (2012), 6133–162.
[4] M. Cheng; *Gound state for the fractional Schrödinger equation with unbounded potential*, J. Math. Phys. 53 (2012), 043507.
[5] S. Dipierro, G. Palatucci, E. Valdinoci; *Existence and symmetry results for a Schröinger type problem involving the fractional Laplacian*, Le Matematiche. LXVIII (2013), 201–216.
[6] P. Felmer, A. Quaas, J. Tan; *Positive solutions of nonlinear Schrödinguer equation with the fractional laplacian*, Proc. Roy. Soc. Edinburgh Sect. A 142 (2012), 1237–1262.
[7] S. Secchi; *Ground state solutions for nonlinear fractional Schröinger equations in $\mathbb{R}^N$*, J. Math. Phys. 54 (2013), 031501.
[8] X. J. Chang; *Gound state solutions of asymptotically linear fractional Schrödinger equations*, J. Math. Phys. 54 (2013), 061504.
[9] E. Di Nezza, G. Patalluci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), 521–573.
[10] Q. Y. Guan, Z. N. Ma; *The reflected-symmetric stable processes and regional fractional Laplacian*, Probab. Theory Relat. Fields 134 (2006), 649–694.
[11] Q. Y. Guan; *Integration by parts formula for regional fractional Laplacian*, Commun. Math. Phys. 266 (2006), 289–329.
[12] P. Felmer, C. Torres; *Non-linear Schrödinger equation with non-local regional diffusion*, Calc. Var., to appear.
[13] P. Felmer, C. Torres; *Radial symmetry of ground states for a regional fractional nonlinear schrödinger equation*, Commun. Pure Appl. Anal. 13 (2014), 2395–2406.
[14] X. Cabré, J. Dávila; *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles and Hamiltonian estimates*, Ann. Inst. Henri Poincare Nonliear Anal. 31 (2014), 23–53.

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